# Refined Verlinde formulas for moduli spaces of sheaves on surfaces

Lothar Göttsche,
partially joint works in progress with Martijn Kool, William Runako, Don Zagier

Enumeration and Moduli, Oslo 8.12.2018



**Verlinde formula:** generating formula for the dimension of the spaces of sections (conformal blocks)

$$H^0(M_C^H(r,d), L^{\otimes k})$$

of line bundles $L^{\otimes k}$ on moduli spaces of rank $r$ degree $d$ vector bundles on a nonsingular projective curve $C$.
Want to study analogue for algebraic surfaces, and we also want to refine it, $H^0(M, L)$ (or $\chi(M, L)$) by finer invariants, like

1. twisted $\chi_y$-genus $\chi_{-y}(M, L)$,
2. twisted elliptic genus $Ell_M(y, q; L)$.

Let $S$ projective algebraic surface
$S^{[n]}$ =Hilbert scheme of finite subschemes of length $n$ on $S$
$S^{(n)} = S^n/\mathfrak{S}_n$ symmetric power
$S^{[n]}$ is smooth projective of dimension $2n$,
$\pi : S^{[n]} \to S^{(n)}; Z \mapsto supp(Z)$ crepant resolution
**Universal subscheme**

$$Z_n(S) = \{(x, Z) \mid x \in Z\} \subset S \times S^{[n]}$$

with projections $p : Z_n(S) \to S^{[n]}$ , $q : Z_n(S) \to S^{[n]}$
**Tautological sheaves** of rank $n$: For vector bundle $V$ of rank $r$ on $S$ have $V^{[n]} = p_* q^*(V)$ or rank $rn$
in part. $\mathcal{O}_S^{[n]}(Z) = H^0(\mathcal{O}_Z)$
Have $\mathrm{Pic}(S^{[n]}) = \mu(\mathrm{Pic}(S)) \oplus \mathbb{Z}E$, with $E = \det(\mathcal{O}_S^{[n]})$, and
$\mu(L) = \pi^*\sigma_*(\otimes_{i=1}^n pr_i^* L)^{\mathfrak{S}_n}$, with $\sigma : S^n \to S^{(n)}$ proj.



Important tool for us:

**Theorem (Ellingsrud-G-Lehn)**

*Let $P(x_1, \ldots, x_{2n}, y_1, \ldots, y_n)$ polynomial. Put*

$$P[S^{[n]}, L] := \int_{S^{[n]}} P(c_1(S^{[n]}), \ldots c_{2n}(S^{[n]}), c_1(L^{[n]}), \ldots, c_n(L^{[n]}))$$

*There is a polynomial $\widetilde{P}(x, y, z, w)$, such that for all surfaces $S$, all line bundles $L$ on $S$ we have*

$$P[S, L] = \widetilde{P}(K_S^2, \chi(\mathcal{O}_S), LK_S, K_S^2).$$

Usually have sequence of polynomials
$P_n(x_1, \ldots x_{2n}, y_1, \ldots, y_n)$, $n \geq 0$, "nicely organized", then

$$\sum_{n\geq 0} P_n[S^{[n]}, L]x^n = A_1(x)^{L^2} A_2(x)^{LK_S} A_3(x)^{K_S^2} A_4(x)^{\chi(O_S)}$$

for universal power series $A_1, \ldots, A_4$

Long time ago used this to prove version of Verlinde formula

**Theorem (Ellingsrud-G-Lehn)**

*For every $r \in \mathbb{Z}$ have power series $A_r(t)$, $B_r(t)$ s.th*

$$\sum_{n\geq 0} \chi(S^{[n]}, \mu(L)\otimes E^r)x^n = \frac{1}{(1-t)^{\chi(L)}}\left(\frac{1}{(1-t)^{r^2-1}(1+(r^2-1)t)}\right)^{\chi(\mathcal{O}_S)/2} \cdot A_r(t)^{K_S L} B_r(t)^{K_S^2}$$

*with* $x = \frac{t}{(1-t)^{r^2}}$. *We have* $A_r = \frac{B_{-r}}{B_r}$
*Furthermore* $A_r = B_r = 1$ *for* $r = 0, \pm 1$
*In part.* $\chi(S^{[n]}, \mu(L)) = \binom{\chi(L)+n-1}{n}$, $\chi(S^{[n]}, \mu(L)\otimes E^{\pm 1}) = \binom{\chi(L)}{n}$

With Don-Zagier try to determine the $A_r$, $B_r$. Until now conjectural formula for $A_r$, $B_r$ for $r = \pm 2, \pm 3$, and for $A_{\pm 4}$



With Don-Zagier try to determine the $A_r$, $B_r$. Until now conjectural formula for $A_r$, $B_r$ for $r = \pm 2, \pm 3$, and for $A_{\pm 4}$
e.g.

$$B_2(t) = 1 + u^3, \quad B_{-2}(t) = 1 - u^2, \quad \text{for } t = \frac{u}{1-u+u^2}$$

(found earlier in different form by Marian-Oprea-Pandharipande in context of Lehn conjecture)

$$B_{-3}(t) = \frac{(1+u^2)(1+u+u^2)(1+4u+u^2)(1-u)^2}{(1+u)^3}$$

$$B_3(t) = \frac{(1+4u+u^2)(1+u^2)^4}{(1+u)^4}, \quad \text{for } t = \frac{u(1+u+u^2)}{(1+u)^2}$$

Conversely our formula for $A_{\pm 3}$, $B_{\pm 3}$ gives conjectural formula for

$$\sum_{n\geq 0} \int_{S^{[n]}} c_{2n}(V^{[n]})x^n, \quad rk(V) = -2, 4$$

**Refinement:** Replace $\chi(L)$ by finer invariant

$$\chi_{-y}(X,L) = y^{-\dim(X)/2}\sum_{p=0}^{\dim(X)}(-y)^p\chi(X,\Omega^p\otimes L)$$

All we do works for further refinement: twisted elliptic genus $Ell_{(X,L)}(y,z)$. Note $\chi_{-y}(X)=\chi_{-y}(X,\mathcal{O}_X)$. We know

$$\sum_{n\geq 0}\chi_{-y}(S^{[n]})x^n = \frac{\prod_{n>0}(1-x^n)^{K_S^2}}{\left((1-x^n)^{10}(1-x^ny)(1-x^n/y)\right)^{\chi(\mathcal{O}_S)}}$$

Put $g(x,y) := \prod_{n\geq 1}\left(\frac{(1-x^n)^2}{(1-x^ny)(1-x^n/y)}\right)^{n^2}$.

**Theorem**

$$\sum_{n\geq 0}\chi_{-y}(S^{[n]},L_n)x^n = g(x,y)^{L^2/2}\prod_{n\geq 1}\left(\left(\frac{1-x^n/y}{1-x^ny}\right)^n\right)^{LK_S/2} \cdot\sum_{n\geq 0}\chi_{-y}(S^{[n]})x^n.$$

Introduction | Hilbert schemes of points | Moduli of rank 2 sheaves | Check of conjectures
Refined Verlinde formula

$g(x,y) := \prod_{n\geq 1}\left(\frac{(1-x^n)^2}{(1-x^ny)(1-x^n/y)}\right)^{n^2}$.

**Theorem**

*If $K_S^2 = LK_S = 0$, then putting $x := tg(t,y)^{r^2}$, we have*

$$\sum_{n\geq 0}\chi_{-y}(S^{[n]},L_n\otimes E^r)x^n = g(t,y)^{L^2/2}\left(\frac{g(t,y)^{r^2}}{1+r^2\frac{t\frac{d}{dt}g(t,y)}{g(t,y)}}\right)^{\chi(\mathcal{O}_S)/2} \cdot\sum_{n\geq 0}\chi_{-y}(S^{[n]})t^n$$

Similar formulas for twisted elliptic genus generalizing the DMVV formla

In Borisov-Libgober proof of DMVV formula for ell. gen. $Ell(S^{[n]})$, they introduce orbifold elliptic class $ELL_{orb}(X/G) \in H^*(X)$ for the group $G$ acting on $X$, and the elliptic class $ELL(Y) \in H^*(Y)$ for nonsingular $Y$, such that

1. $Ell(Y) = \int_Y ELL(Y)$,
2. if $\pi : X \to X/G$ quotient, $\eta : Y \to X/G$ crepant resolution, then $\eta_* ELL(Y) = \pi_* ELL_{orb}(X, G)$.

They apply this to $Y = S^{[n]}$, $X = S^n$, $G = \mathfrak{S}_n$. As $\mu(L)$ is a line bundle pulled back from $S^{(n)}$, we can compute

$$Ell_{(S^{[n]},\mu(L))} = \int_{S^{[n]}} ELL(S^{[n]}) ch(\mu(L)).$$

The second theorem is reduced to the first by restricting to the case of K3 surfaces and using the Beauville Bogomolov quadratic form. This is a quadratic form $q$ on $H^2(X)$ for any hyperkähler manifold $X$. For any polynomial $p(c_i(X))$ in the Chern classes of $X$, $\int_X p(c_i(X)) \exp(\alpha)$ is a polynomial (depending on $p$) in $q(\alpha)$. This implies that the formula for $\mu(L)$ determines the one for $\mu(L) \otimes E^{\otimes r}$



$S$ projective complex surface, $H$ ample line bundle on $S$

$$M_S^H(c_1, c_2) = \begin{array}{c}\text{moduli space of rank 2 } H\text{-semistable sheaves}\\ \text{on } S \text{ with Chern classes } c_1, c_2\end{array}$$

$\mathcal{E}$ semistable $\iff \forall_{n \gg 0} \frac{h^0(S, \mathcal{F} \otimes H^{\otimes n})}{\mathrm{rk}(\mathcal{F})} \le \frac{h^0(S, \mathcal{E} \otimes H^{\otimes n})}{\mathrm{rk}(\mathcal{E})}$ for all $\mathcal{F}$ subsheaf of $\mathcal{E}$.

$M = M_S^H(c_1, c_2)$ is usually singular, has *expected dimension*

$$vd = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S).$$

$vd$ is the dimension $M$ should have, more about that later

Here write $c_2 := \int_{[S]} c_2 \in \mathbb{Z}$, $c_1^2 := \int_{[S]} c_1^2 \in \mathbb{Z}$

We assume always that $p_g(S) = h^0(S, K_S) > 0$,

$b_1(S) = \dim H^1(S) = 0$

$M = M_S^H(c_1, c_2)$ usually very singular
might have dimension different from $\mathrm{vd} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$

But $M$ has a perfect obstruction theory of virtual dimension $\mathrm{vd}$
Can define virtual analogues of all invariants of smooth projective varieties

At every point $[F] \in M$, tangent space $T_{[F]} = Ext^1(F, F)_0$
obstruction space $O_{[F]} = Ext^2(F, F)_0$

**Perfect obstruction theory:**
Complex $E_\bullet = [E_0 \to E_1]$ of vb on $M$, s.th. $\forall F \in M$:
$T_{[F]} \simeq ker(E_0(F) \to E_1(F))$, $O_F \hookrightarrow coker(E_0(F) \to E_1(F))$
i.e $E_\bullet$ captures tangents and obstructions via vector bundles

Define: $T_M^{\mathrm{vir}} := [E_0] - [E_1] \in K^0(M)$,
$\mathrm{vd} := \mathrm{rk}\, T_M^{\mathrm{vir}} = \mathrm{rk}(E_0) - \mathrm{rk}(E_1)$
virtual fundamental class $[M]^{\mathrm{vir}} \in H_{2\mathrm{vd}}(M)$
virtual structure sheaf $\mathcal{O}_M^{\mathrm{vir}} \in K_0(M)$
**Virtual holomorphic Euler characteristic** For $V \in K^0(M)$, put
$\chi^{\mathrm{vir}}(M, V) := \chi(M, \mathcal{O}_M^{\mathrm{vir}} \otimes V)$



Now again $S$ surface with $p_g(S) = 0$, $b_1(S) = 0$
Let $L \in \mathrm{Pic}(S)$. Assume $Lc_1$ is even
Assume for simplicity $\exists$ universal sheaf $\mathbb{E}$ on $S \times M_S^H(c_1, c_2)$
Put $\mu(L) = -ch_2(\mathbb{E} \otimes \det(\mathbb{E})^{-1/2})/c_1(L) \in H^2(M)$
There is a line bundle $\mu(L) \in Pic(S)$ with $c_1(\mu(L)) = \mu(L)$
(Donaldson line bundle), $\chi^{\mathrm{vir}}(M_S^H(c_1, c_2), \mu(L))$ are $K$-theoretic Donaldson invariants

**Conjecture**

*Assume $|K_S|$ contains an irreducible curve. Then*

$$\chi^{\mathrm{vir}}(M_S^H(c_1, c_2), \mu(L)) = 2^{3+K_S^2-\chi(\mathcal{O}_S)} \mathrm{Coeff}_{x^{vd}} \left[ \frac{(1+x)^{K_S(L-K_S)}}{(1-x^2)^{\chi(L)}} \right]$$

Again $L \in \mathrm{Pic}(S)$ with $Lc_1$ is even, $\mathbb{E}$ universal sheaf

$$\mu(L) = -ch_2(\mathbb{E}\otimes\det(\mathbb{E})^{-1/2})/c_1(L), \quad E = 2ch_3(\mathbb{E}\otimes\det(\mathbb{E})^{-1/2})/1$$

If $L$ is sufficiently ample on $S$, then $\mu(L)+E$ is ample

**Conjecture**

*Assume $|K_S|$ contains an irreducible curve. Then*

$$\chi^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(L)+E) = 2^{3+K_S^2-\chi(O_S)}\mathrm{Coeff}_{x^{vd}}\left[\frac{(1-x^2)^{\chi(L)}}{(1-x)^{LK_S}}\right]$$



**Twisted Virtual $\chi_{-y}$-genus.** Put $\Omega_M^{\mathrm{vir}} := (T_M^{\mathrm{vir}})^\vee$.

$$\chi_{-y}^{\mathrm{vir}}(M,L) := y^{-\mathrm{vd}/2}\sum_p(-y)^p\chi^{\mathrm{vir}}(M,\Lambda^p\Omega_M^{\mathrm{vir}}\otimes L), \quad \chi_{-y}^{\mathrm{vir}}(M) = \chi_{-y}^{\mathrm{vir}}(M,\mathcal{O})$$

The Vafa-Witten conjecture is a formula for the Euler numbers $e(M)$. We refine it to a conjecture $\chi_{-y}^{\mathrm{vir}}(M)$. Keep assuming that $|K_S|$ contains irreducible curve.

$$\theta_3(x,y) := \sum_{n\in\mathbb{Z}} x^{n^2}y^n, \quad \overline{\eta}(x) := \prod_{n>0}(1-x^n)$$

$$\psi_S(x,y) := 8\left(\frac{1}{2\prod_{n>0}(1-x^{2n})^{10}(1-x^{2n}y)(1-x^{2n}/y)}\right)^{\chi(\mathcal{O}_S)} \cdot \left(\frac{2\overline{\eta}(x^4)^2}{\theta_3(x,y^{1/2})}\right)^{K_S^2}$$

**Conjecture**

$\chi_{-y}^{\mathrm{vir}}(M_S^H(c_1,c_2)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}[\psi_S(x,y)]$.

## Conjecture

*Assume $Lc_1$ is even.*

$$\chi^{\mathrm{vir}}_{-y}(M^H_S(c_1,c_2),\mu(L))$$
$$= \mathrm{Coeff}_{x^{vd}}\Bigg[\psi_S(x,y)\left(\prod_{n=1}^{\infty}\left(\frac{(1-x^{2n})^2}{(1-x^{2n}y)(1-x^{2n}y^{-1})}\right)^{n^2}\right)^{\frac{L^2}{2}}$$
$$\left(\prod_{n=1}^{\infty}\left(\frac{1-x^{2n}y^{-1}}{1-x^{2n}y}\right)^{n}\prod_{\substack{n>0\\ \text{odd}}}\left(\frac{(1+x^ny^{-\frac{1}{2}})(1-x^ny^{\frac{1}{2}})}{(1-x^ny^{-\frac{1}{2}})(1+x^ny^{\frac{1}{2}})}\right)^{n}\right)^{\frac{LK_S}{2}}\Bigg]$$



**Main tool: Mochizuki's formula**:
Compute intersection numbers on $M = M^H_S(c_1,c_2)$ in terms of intersection numbers on Hilbert scheme of points.

On $S\times M$ have $\mathcal{E}$ universal sheaf
i.e. if $[E]\in M$ corresponds to a sheaf $E$ on $S$ then $\mathcal{E}|_{S\times[E]} = E$.
For $\alpha\in H^k(S)$, put

$$\tau_i(\alpha) := \pi_{M_*}(c_i(\mathcal{E})\pi_S^*(\alpha)) \in H^{2i-4+k}(M)$$

Let $P(\mathcal{E})$ be any polynomial in the $\tau_i(\alpha)$
Mochizuki's formula expresses $\int_{[M]^{\mathrm{vir}}} P(\mathcal{E})$ in terms of intersec. numbers on $S^{[n_1]}\times S^{[n_2]}$, and Seiberg-Witten invariants.

$\chi^{\mathrm{vir}}(M,\mu(L))$, $\chi^{\mathrm{vir}}_{-y}(M,\mu(L))$ can all be expressed as $\int_{[M]^{\mathrm{vir}}} P(\mathcal{E})$, so can reduce computation to Hilbert schemes.

For $\chi^{\mathrm{vir}}(M,L)$, $\chi^{\mathrm{vir}}_{-y}(M,L)$ use **virtual Riemann-Roch formula**

**Theorem (Fantechi-G.)**

*For $V \in K^0(M)$ have*

$$\chi^{\mathrm{vir}}(M,V) = \int_{[M]^{\mathrm{vir}}} \mathrm{ch}(V)\mathrm{td}(T_M^{\mathrm{vir}}).$$



**Seiberg-Witten invariants:**
differentiable invariants of differentiable 4-manifolds
$S$ projective algebraic surface: $H^2(S,\mathbb{Z}) \ni a \mapsto SW(a) \in \mathbb{Z}$
$a$ is called SW class if $SW(a) \neq 0$.

If $b_1(S)=0$, $p_g(S)>0$ and $|K_S|$ contains smooth connected curve, then SW cl. of $S$ are $0, K_S$ with

$$SW(0)=1, \quad SW(K_S)=(-1)^{\chi(\mathcal{O}_S)}$$

This is the reason for the simplifying assumption that $|K_S|$ contains smooth connected curve, otherwise the formulas are more complicated.

$$S^{[n_1]} \times S^{[n_2]} = \{\text{pairs } (Z_1, Z_2) \text{ of subsch. of deg. } (n_1, n_2) \text{ on } S\}$$

Work on $S \times S^{[n_1]} \times S^{[n_2]}$, projection $p$ to $S^{[n_1]} \times S^{[n_2]}$

Two universal sheaves: Let $a \in \mathrm{Pic}(S)$

1. $\mathcal{I}_i(a)$ sheaf on $S \times S^{[n_1]} \times S^{[n_2]}$ with $\mathcal{I}_i(a)|_{S \times (Z_1, Z_2)} = I_{Z_i} \otimes a$
2. $\mathcal{O}_i(a)$, vector bundle of rank $n_i$ on $S^{[n_1]} \times S^{[n_2]}$, with fibre $\mathcal{O}_i(a)(Z_1, Z_2) = H^0(\mathcal{O}_{Z_i} \otimes a)$

For a vector bundle $E$ of rank $r$ and variable $s$ put

$$c_i(E \otimes s) = \sum_{k=0}^{i} \binom{r-i}{k} s^{i-k} c_k(E), \quad Eu(E) = c_r(E)$$



For sheaves $\mathcal{E}_1, \mathcal{E}_2$ on $S \times S^{[n_1]} \times S^{[n_2]}$ put

$$Q(\mathcal{E}_1, \mathcal{E}_2) = Eu(-RHom_p(\mathcal{E}_1, \mathcal{E}_2) - RHom_p(\mathcal{E}_2, \mathcal{E}_1))$$

For $a_1, a_2 \in \mathrm{Pic}(S)$ put

$$\Psi(a_1, a_2, n_1, n_2, s) = \frac{P(\mathcal{I}_1(a_1) \otimes s^{-1} \oplus \mathcal{I}_2(a_2) \otimes s) Eu(\mathcal{O}_1(a_1)) Eu(\mathcal{O}_2(a_2) \otimes s^2)}{Q(\mathcal{I}_1(a_1) \otimes s^{-1}, \mathcal{I}_2(a_2) \otimes s) \cdot (2s)^{n_1+n_2-\chi(\mathcal{O}_S)}}$$

$$A(a_1, a_2, c_2, s) = \sum_{n_1+n_2=c_2-a_1a_2} \int_{S^{[n_1]} \times S^{[n_2]}} \Psi(a_1, a_2, n_1, n_2, s) \in \mathbb{Q}[s, s^{-1}]$$

**Theorem (Mochizuki)**

*Assume $\chi(E) > 0$ for $E \in M_H^S(c_1, c_2)$. Then*

$$\int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}} P(\mathcal{E}) = \sum_{\substack{c_1=a_1+a_2 \\ a_1H<a_2H}} SW(a_1) \mathrm{Coeff}_{s^0} A(a_1, a_2, c_2, s)$$

i.e. replace simple formula on a space where we cannot compute anything, by terrible formula on simpler space

**Universality** Take now for $P(\mathcal{E}) = ch(\mu(L))\mathrm{td}(T_M^{\mathrm{vir}})$ (works the same for the others). Put

$$Z_S(a_1, a_2, s, q) = \sum_{n_1, n_2 \geq 0} \int_{S^{[n_1]} \times S^{[n_2]}} A(a_1, a_2, a_1 a_2 + n_1 + n_2, s) q^{n_1+n_2}$$

**Proposition**

*There exist univ. functions $A_1(s,q), \ldots, A_{11}(s,q) \in \mathbb{Q}[s, s^{-1}][[q]]$ s.th. $\forall_{S, a_1, a_2, L}$*

$$Z_S(a_1, a_2, L, s, q) = F_0(a_1, a_2, L, s) A_1^{a_1^2} A_2^{a_1 a_2} A_3^{a_2^2} A_4^{a_1 K_S} A_5^{a_2 K_S} A_6^{K_S^2} A_7^{\chi(\mathcal{O}_S)} \cdot A_8^{L^2} A_9^{LK_S} A_{10}^{La_1} A_{11}^{La_2},$$

*(where $F_0(a_1, a_2, L, s)$ is some explicit elementary function).*

**Proof:** Modification of the cobordism argument for Hilbert schemes of points



$A_1(s,q), \ldots A_{11}(s,q)$ are determ. by value of $Z_S(a_1, a_2, L, s, q)$ for 11 triples $(S, a_1, a_2, L)$ ($S$ surface, $a_1, a_2, L \in \mathrm{Pic}(S)$) s.th. corresponding 11-tuples

$$(a_1^2, a_1 a_2, a_2^2, a_1 K_S, a_1 K_S, K_S^2, \chi(O_S)), L^2, LK_S, La_1, La_1)$$

are linearly independent. We take

$(\mathbb{P}^2, \mathcal{O}, \mathcal{O}, \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}, \mathcal{O}), (\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}, \mathcal{O}), (\mathbb{P}^2, \mathcal{O}, \mathcal{O}(1), \mathcal{O}),$
$(\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}(1), \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,0), \mathcal{O}, \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}(1,0), \mathcal{O})$
$(\mathbb{P}^2, \mathcal{O}, \mathcal{O}, \mathcal{O}(1)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}, \mathcal{O}(1,0)), (\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}, \mathcal{O}(1)),$
$(\mathbb{P}^2, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(1)),$

In this case $S$ is a smooth toric, i.e. have an action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints,
Action of $T$ lifts to action on $S^{[n]}$ still with finitely many fixpoints described by partitions, compute by equivariant localization.
This computes $Z_S(a_1, a_2, L, s, q)$ in terms of combinatorics of partitions.

Let $X$ be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \ldots, p_e$
Let $E$ be equivariant vector bundle of rank $r$ on $X$.

Fibre $E(p_i)$ of $X$ at fixp. $p_i$ has basis of eigenvect. for $T$-action $E(p_i) = \bigoplus_{k=1}^r \mathbb{C}v_i$, with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$

Equivariant chern class of fibre at fixpoint:

$$c^T(E(p_i)) = (1+c_1^T(E(p_i))+\ldots+c_r^T(E(p_i)) = \prod_{i=1}^r (1+n_i\epsilon_1+m_i\epsilon_2) \in \mathbb{Z}[\epsilon_1, \epsilon_2]$$

Let $P(c(E)))$polynomial in Chern classes of $E$, of degree $d = \dim(X)$

**Theorem (Bott residue formula)**

$$\int_{[X]} P(c(E)) = \sum_{k=1}^e \frac{P(c^T(E(p_k)))}{c_d^T(T_X(p_k))}$$

*(does not depend on $\epsilon_1, \epsilon_2$)*



For simplicity $S = \mathbb{P}^2$. $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{P}^2$ by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

Fixpoints are $p_0 = (1, 0, 0)$, $p_1 = (0, 1, 0)$, $p_2 = (0, 0, 1)$.
Local (equivariant) coordinates near $p_0$ are $x = \frac{X_1}{X_0}, y = \frac{X_2}{X_0}$,
$T$ action $(t_1, t_2)(x, y) = (t_1 x, t_2 y)$, similar for the $p_1, p_2$
$Z \in (\mathbb{P}^2)^{[n]}$ is $T$-invariant $\Longrightarrow Z = Z_0 \sqcup Z_1 \sqcup Z_2 \quad supp(Z_i) = p_i$.
$\Longrightarrow$ Reduce to case $supp(Z) = p_i$, e.g. $p_0$
Easy: $Z$ is $T$-invariant $\iff I_Z \in k[x, y]$ is gen. by monomials
Can write

$$I_Z = (y^{n_0}, xy^{n_1}, ...., x^r y^{n_r}, x^{r+1}) \quad (n_0, \ldots, n_r) \text{ partition of } n$$

Fixpoints on $(\mathbb{P}^2)^{[n]}$ are in bijections with triples $(P_0, P_1, P_2)$ of partitions of 3 numbers adding up to $n$.

Need to compute things like $c(\mathcal{O}^{[n]})$
$\mathcal{O}^{[n]}$ vector bundle on $(\mathbb{P}^2)^{[n]}$ with fibre $\mathcal{O}^{[n]}(Z) = H^0(\mathcal{O}_Z)$
If $Z = Z_0 \sqcup Z_1 \sqcup Z_2$, $supp(Z_i) = p_i$, then

$$\mathcal{O}^{[n]}(Z) = \mathcal{O}^{[n_0]}(Z_0) \oplus \mathcal{O}^{[n_1]}(Z_1) \oplus \mathcal{O}^{[n_2]}(Z_2)$$
$$c^T(\mathcal{O}^{[n]}(Z)) = c^T(\mathcal{O}^{[n_0]}(Z_0))c^T(\mathcal{O}^{[n_1]}(Z_1))c^T(\mathcal{O}^{[n_2]}(Z_2))$$

Let e.g. $Z = Z_0$, $I_Z = (y^4, xy^2, x^2y, x^3)$
Then the fibre $\mathcal{O}^{[n]}(Z) = H^0(\mathcal{O}_Z) = \mathbb{C}[x,y]/(y^4, xy^2, x^2y, x^3)$
Thus basis of eigenvectors of fibre for $T$ action is

$$\begin{array}{cccc} 1 & y & y^2 & y^3 \\ x & xy & & \\ x^2 & & & \end{array} \quad \text{with eigenvalues} \quad \begin{array}{cccc} 1 & t_2 & t_2^2 & t_2^3 \\ t_1 & t_1 t_2 & & \\ t_1^2 & & & \end{array}$$

Thus

$$c^T(\mathcal{O}^{[n]}(Z)) = (1+\epsilon_2)(1+2\epsilon_2)(1+3\epsilon_2)(1+\epsilon_1)(1+\epsilon_1+\epsilon_2)(1+2\epsilon_1).$$