Virtual topological invariants of moduli spaces

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We want to compute

generating functions of virtual Euler numbers of moduli spaces of sheaves on algebraic surfaces

You should understand all the words not in red
The red ones I will try to explain.
Much of the lecture will be devoted to that
To make it more elementary, I will tell you many lies.
Then I state the results.
If there is time I give a very brief outline of some of the methods

If $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ are interesting numbers, the formal power series

$$f(x) = \sum_n a_n x^n$$

is their generating function. If we know $f(x)$ we know all the a_n.
Analytical properties of $f(x)$ reflect relations between the a_n

Example

$p(n)$ number of partitions of a number n,
i.e. number of ways to write it as sum of numbers e.g.

$$3 = 3, 3 = 2 + 1, 3 = 1 + 1 + 1, \quad p(3) = 3$$

Exercise: Generating function is

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1 - x^n}.$$

Let P be a regular polyeder, (tetraeder, cube, octaeder, ...).
$V = \#$vertices, $\quad E = \#$edges, $\quad S = \#$sides,
Euler (1751) made the remarkable observation that

$$V - E + S = 2$$

e.g.

$$\text{tetraeder: } 4 - 6 + 4 = 2, \quad \text{cube}: \ 8 - 12 + 6 = 2$$

If X is a reasonable topological space, the Euler number $e(X)$ is defined as follows: divide X into spaces homeomorphic to simplices. Then

$$e(X) = \sum_{i \geq 0} (-1)^i \# i\text{-simplicies.}$$

This does not depend on how you divide X!
Note: A regular polyeder is topologically a 2-sphere S^2, so Euler's observation says $e(S^2) = 2$

The Euler number can be expressed in homology groups. There are several equivalent definitions of homology and cohomology, which we will not review

X oriented compact manifold of dimension n has

homology groups $H_i(X, \mathbb{Z}), i = 0, \ldots, n$
cohomology groups $H^i(X, \mathbb{Z}), i = 0, \ldots, n$

The Betti numbers of X are their ranks

$$b_i(X) = \operatorname{rk} H_i(X, \mathbb{Z})$$

The Euler number is their alternating sum

$$e(X) = \sum_{i=0}^{n} (-1)^i b_i(X)$$

Let $Y \subset X$ oriented closed submanifold of dimension d
it has fundamental class $[Y] \in H_d(X, \mathbb{Z})$
In particular $[X] \in H_n(X, \mathbb{Z})$ fundamental class of X
$H_0(X, \mathbb{Z}) = [pt]\mathbb{Z} \simeq \mathbb{Z}$, for $[pt]$ the class of a point in X
Cohomology is a graded ring via cup product

$$H^i(X, \mathbb{Z}) \times H^k(X, \mathbb{Z}) \to H^{i+k}(X, \mathbb{Z}), (\alpha, \beta) \to \alpha \cdot \beta$$

and homology is a module via the cap product

$$H^i(X, \mathbb{Z}) \times H_k(X, \mathbb{Z}) \times \to H_{k-i}(X, \mathbb{Z}), (\alpha, \beta) \to \alpha \cap \beta$$

Poincaré duality:

$$PD = \cdot \cap [X] : H^i(X,\mathbb{Z}) \to H_{n-i}(X,\mathbb{Z}), \alpha \to [X] \cap \alpha$$

is an isomorphism, in particular $H^n(X,\mathbb{Z}) \simeq \mathbb{Z}$
Poincar'e duality identifies cup product on cohomology with intersection on homology

$$PD^{-1}([Y_1]) \cdot PD^{-1}([Y_2]) = \sum_i \pm [Z_i]$$

Here assume that Y_1 and Y_2 intersect transversally along disjoint submanifolds Z_i The sign depends on the orientations
In particular we have
$\int_{[X]} : H^n(X,\mathbb{Z}) \to \mathbb{Z}, \alpha \mapsto \int_{[X]} \alpha := \alpha \cap [X] \in H_0(X,\mathbb{Z}) = \mathbb{Z}$.

Let X complex manifold (or nonsingular variety over $\mathbb{C}$)
A (holomorphic) **vector bundle of rank** r is $\pi : E \to X$

1. E complex manifold, π holomorphic map
2. fibres $E(p) = \pi^{-1}(p)$ are complex vector spaces of dim r
3. E is locally trivial: every $p \in X$ has nbhd U, s.th $\pi^{-1}(U) \simeq U \times \mathbb{C}^r$, by an isomorphism linear on the fibres.

Example:

1. $TX \to X$ tangent bundle, fibre over p is tangent space T_pX
2. The **trivial** vector bundle of rank r is $X \times \mathbb{C}^r \to X$

A **section** of a vector bundle $E \to X$ is a holomorphic map $s : X \to E$ with $s(p) \in E(p)$ for all $p \in X$.

A vector bundle of rank r on X has **Chern classes**

$$c(E) = 1 + c_1(E) + \ldots + c_r(E), \quad c_i(E) \in H^{2i}(X, \mathbb{Z})$$

They measure how far E is from being trivial, i.e. $c(X \times \mathbb{C}^r) = 1$
Whitney Product formula:

$$c(E \oplus F) = c(E) \cdot c(F)$$

Assume E has "good" (transversal) sections $s_1, \ldots, s_r : X \to E$. Then

$$PD(c_{r-k}(E)) = \left[\{p \in X \mid s_1(p), \ldots, s_{k+1}(p) \text{ linearly dep. in } E(p)\}\right]$$

In particular $PD(c_r(E)) = \left[\text{zero set of } s_1\right]$

Let X complex manifold of dimension n
Chern classes of X: $c_i(X) = c_i(TX)$
They give invariants of X, for instance Euler number of X

$$e(X) = \#\text{zeros of general holomorphic vector field}$$

In other words

$$\int_{[X]} c_n(X) = e(X).$$

More generally we have the Chern numbers

$$\int_{[X]} c_{n_1}(X) \cdots c_{n_k}(X) \in \mathbb{Z}$$

if $n_1 + \ldots + n_k = n$.

A **coherent** (torsion free) **sheaf** F of rank r on X, is a vector bundle with singularities i.e. some fibres of dimension $> r$

Identify vector bundles E with locally free sheaves
i.e in neighbourhood U of any $p \in X$ the sections $E(U)$ of E over U are isomorphic to r-tuples of holom. functions on U

For a torsion free sheaf generalize this

Example

Let $Z = \{p_1, \ldots, p_k\}$ be a finite subset of X
The ideal sheaf I_Z of Z is the torsion free rank 1 sheaf on X given by

$$I_Z(U) = \{\text{holomorphic functions on } U \text{ vanishing on } Z \cap U\}$$

Chern classes are also defined for torsion free sheaves
For instance if $dim(X) = 2$

$$c_1(I_Z) = 0, \quad PD(c_2(I_Z)) = [Z] = k[pt]$$

Let S be a smooth projective surface over $\mathbb{C}$

Assume $H^1(S,\mathbb{Z})=0$, and $p_g(S)>0$

$p_g(S)$ is dim of the space of global holom. 2-forms on S
The canonical class of S is $K_S\in H^2(S,\mathbb{Z})$ given by
$PD(K_S)=[Z(\omega)]$ (also $K_S=-c_1(S)$)
The most important numbers associated to S are

1. The holomorphic Euler characteristic
$\chi(\mathcal{O}_S)=1+p_g(S)\geq 2$
2. $K_S^2:=\int_{[S]}K_S^2$

e.g. $H^1(\mathbb{P}^2,\mathbb{Z})=0$, $p_g(\mathbb{P}^2)=0$, $K_{\mathbb{P}^2}=-3H$, $\chi(\mathcal{O}_S)=1$, $K_S^2=9$
S surface of degree 4 in $\mathbb{P}^3$: $H^1(S,\mathbb{Z})=0$, $p_g(S)=1$, $K_S=0$,
$\chi(\mathcal{O}_S)=2$, $K_S^2=0$
S surface of degree 5 in $\mathbb{P}^3$: $H^1(S,\mathbb{Z})=0$, $\chi(\mathcal{O}_S)=5$, $K_S^2=5$

A **moduli space** is an algebraic variety that parametrizes isomorphism classes of objects we are interested in

Example

The Hilbert scheme $S^{[n]}$ of n points on S parametrizes ideal sheaves I_Z of n (possibly nonreduced) points on S
It is a smooth projective variety of dimension $2n$

Euler numbers of Hilbert schemes:

Let $e(M)$ be the topological Euler number of M

Write $\overline{\eta}(x) := \prod_{n>0}(1 - x^n)$

Theorem

$$\sum_{n\geq 0} e(S^{[n]})x^n = \frac{1}{\overline{\eta}(x)^{e(S)}}$$

Using physics arguments, Vafa and Witten ('94) proposed generating function for Euler numbers of moduli spaces of rank 2 sheaves

$$M_S^H(2, c_1, c_2) = \begin{array}{c}\text{moduli space of rank 2 } H\text{-semistable sheaves} \\ \text{on } S \text{ with Chern classes } c_1,\ c_2\end{array}$$

Here $H \in H^2(S, \mathbb{Z})$ is an "ample" class (class of hyperplane section), and H-stable means that subsheaves cannot be too large, and this is somehow measured using H.

$M_S^H(c_1, c_2) = M_S^H(2, c_1, c_2)$ is projective, usually singular, has expected dimension

$$vd = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S).$$

Here write $c_2 := \int_{[S]} c_2 \in \mathbb{Z}$, $c_1^2 := \int_{[S]} c_1^2 \in \mathbb{Z}$

We have $M_S^H(1, 0, n) = S^{[n]}$, because every rank 2 torsion-free sheaf on S with $c_1 = 0$, $c_2 = n$ (i.e. $PD(c_2) = n[pt]$) is the ideal sheaf of n points with multiplicities.

In whole talk assume stable=semistable (condition on c_1).
Assume for simplicity in whole talk: there is a holomorphic 2 form ω on S whose zero set $Z(\omega)$ is irreducible

$$\mathrm{vd} = \mathrm{vd}_{M_S^H(c_1,c_2)} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$$

Vafa-Witten conjecture

$$\overline{\eta}(x) := \prod_{n>0}(1 - x^n), \quad \theta_3^0(x) = \sum_{n\in\mathbb{Z}} x^{n^2}$$

$$\psi_S(x) := 8\left(\frac{1}{2\overline{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)} \left(\frac{2\overline{\eta}(x^4)^2}{\theta_3^0(x)}\right)^{K_S^2}$$

Then $e(M_S^H(c_1,c_2)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}[\psi_S(x)]$.

Want to interpret, check and refine this formula.

$M = M_S^H(c_1, c_2)$ usually very singular
of dimension different from $\mathrm{vd} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$

the Vafa-Witten conjecture is usually wrong as stated,
But I do not think this is what Vafa-Witten meant

M is virtually smooth, i.e. it wants to be smooth

One should more think of the Euler number
that the moduli space would have
it was smooth of dimension vd

What does this mean?

Virtual dimension vd

tangent space T_pM: infinitesimal directions in M near p

Obstruction space O_p: obstructions to moving a finite distance in M along curve with given tangent vector

M has expected dimension vd means $\dim(T_pM) - \dim O_p = \mathrm{vd}$

Theorem

(Kuranishi) There exists an analytic map $\mu : T_pM \to O_p$, such that an analytic neighbourhood of p in M is isomorphic to $\mu^{-1}(0)$.

So locally near p in the analytic topology M is given by $\dim O_p$ (analytic) equations in a vector space of dimension $\dim(T_pM)$, so if these equations were transversal then M would be nonsingular of dimension vd

M is **virtually smooth** of dimension vd
This means there is a map of complex vector bundles $\phi : E_0 \to E_1$ over M, with $\mathrm{rk}\, E_0 - \mathrm{rk}\, E_1 = \mathrm{vd}$ such that at every point

$$T_pM = ker(\phi : E_0(p) \to E_1(p)),\ O_p = coker(\phi : E_0(p) \to E_1(p))$$

Use this structure to define **virtual fundamental class** $[M]^{\mathrm{vir}} \in H_{2\mathrm{vd}}(M, \mathbb{Z})$. $\Longrightarrow$ virtual invariants of varieties
This is used to define most invariants in modern enumerative geometry
e.g. Gromov-Witten, Donaldson invariants, Donaldson Thomas invariants

Virtual Tangent bundle $T^{\mathrm{vir}}M := E_0 - E_1$, with Chern classes

$$c(T^{\mathrm{vir}}M) = 1 + c_1^{\mathrm{vir}}(M) + c_2^{\mathrm{vir}}(M) \ldots := c(E_0)/c(E_1)$$
$$= 1 + (c_1(E_0) - c_1(E_1)) + \left(c_2(E_0) - c_1(E_0)c_1(E_1) - c_2(E_1) + c_1(E_1)^2\right) + \ldots$$

Recall if X is smooth, then $e(X) = \int_{[X]} c_{\dim(X)}(X)$.

Definition

Virtual Euler number:

$$e^{\mathrm{vir}}(M) := \int_{[M]^{\mathrm{vir}}} c_{\mathrm{vd}}(T^{\mathrm{vir}}(M))$$

Conjecture

The Vafa-Witten formula holds with $e(M_S^H(c_1, c_2))$ replaced by $e^{\mathrm{vir}}(M_S^H(c_1, c_2))$.

Refinement: the χ_{-y} genus
Remember: Betti numbers $b_i(X) = \mathrm{rk}\, H^i(X, \mathbb{Z})$
Algebraic var. : Hodge numbers $h^{p,q}(X)$, $\quad p, q = 0, \ldots \dim_{\mathbb{C}}(X)$
and $b_i = h^{0,i} + \ldots + h^{i,0}$
χ_{-y}**-genus:** $\chi_{-y}(X) = y^{-\dim(X)/2} \sum_{p,q} (-1)^{p+q} y^p h^{p,q}(X)$

e.g. S hypersurface of degree 4 in $\mathbb{P}^3$ (K3 surface)
then Hodge numbers

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1, \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$$b_0 = b_4 = 1$$
$$b_1 = b_3 = 0$$
$$b_2 = 22$$

$$\chi_{-y}(S) = 2y^{-1} + 20y + 2y$$

Note that $\chi_{-y}(X)|_{y=1} = e(X)$
Can define **Virtual** χ_{-y}**-genus,** $\chi_{-y}^{\mathrm{vir}}(M) \in \mathbb{Z}[y^{\pm 1/2}]$
Again $\chi_{-y}^{\mathrm{vir}}(M)|_{y=1} = e^{\mathrm{vir}}(M)$, so refinement of $e^{\mathrm{vir}}(M)$

Conjecture for virtual χ_{-y}-genus:

$$\theta_3(x,y) := \sum_{n\in\mathbb{Z}} x^{n^2} y^n, \quad \overline{\eta}(x) = \prod_{n>0}(1-x^n)$$

$$\psi_S(x,y) := 8\left(\frac{1}{2\prod_{n>0}(1-x^{2n})^{10}(1-x^{2n}y)(1-x^{2n}/y)}\right)^{\chi(\mathcal{O}_S)} \cdot \left(\frac{2\overline{\eta}(x^4)^2}{\theta_3(x,y^{1/2})}\right)^{K_S^2}$$

Conjecture

$\chi^{\mathrm{vir}}_{-y}(M_S^H(c_1,c_2)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}[\psi_S(x,y)].$

Specializes to our version of VW conjecture for $y=1$.

Final generalization: the cobordism class:
Two complex manifolds M, N have the same cobordism class
$\{M\} = \{N\}$
if they have the same Chern numbers:

$$\int_{[M]} c_{i_1}(M) \cdots c_{i_k}(M) = \int_{[N]} c_{i_1}(N) \cdots c_{i_k}(N) \quad \forall_{k, i_1, \dots, i_k}$$

Cobordism classes of complex manifolds generate a ring
$R = \sum_n R_n$ (graded by dimension)
$\{M\}\{N\} = \{M \times N\}, \quad \{M\} + \{N\} = \{M \sqcup N\}$
In fact

$$R \otimes \mathbb{Q} = \mathbb{Q}[\{\mathbb{P}^1\}, \{\mathbb{P}^2\}, \{\mathbb{P}^3\}, \dots]$$

Ellingsrud-G-Lehn showed $\{S^{[n]}\}$ depends only on $\{S\}$ (equivalent: Chern numbers of $S^{[n]}$ depend only on K_S^2, $c_2(S)$)

For $M = M_S^H(c_1, c_2)$ let $\{M\}^{\mathrm{vir}}$ be the virtual cobordism class given by the

$$\int_{[M]^{\mathrm{vir}}} c_{i_1}(T_M^{\mathrm{vir}}) \cdots c_{i_k}(T_M^{\mathrm{vir}}).$$

Conjecture

There is a power series $P(x) = 1 + \sum_{n>0} P_n x^n$, with $P_n \in R_n$, s.th.

$$\{M_S^H(c_1, c_2)\}^{\mathrm{vir}} = \mathrm{Coeff}_{x^{\mathrm{vd}}}\left[8\left(\frac{1}{4}\sum_{n\geq 0}\{K3^{[n]}\}x^{2n}\right)^{\chi(\mathcal{O}_S)/2}(2P(x))^{K_S^2}\right].$$

We developed an algorithm to compute $e^{\mathrm{vir}}(M)$, $\chi^{\mathrm{vir}}_{-y}(M)$, $\{M\}^{\mathrm{vir}}$ for $M = M_S^H(c_1, c_2)$ for any S, c_1, c_2

Confirms conjectures for K3 surfaces, their blowups, elliptic surfaces, double covers of $\mathbb{P}^2$, complete intersections, etc, up to large vd

Example: Quintic hypersurface S_5 generic surface of degree 5 in $\mathbb{P}^3$, $K_{S_5} = H$ hyperplane class

$K_{S_5}^2 = \chi(\mathcal{O}_{S_5}) = 5$.

$$\sum_{c_2} e^{\mathrm{vir}}(M_S^H(H, c_2))x^{4c_2-20} = 8 + 52720x^4 + 48754480x^8 + \ldots + O(x^{28})$$

$$\sum_{c_2} \chi^{\mathrm{vir}}_{-y}(M_S^H(H, c_2))x^{4c_2-20} = 8$$

$$+ (1280y^2 + 11440y + 27280 + 11440y^{-1} + 1280^{-2})x^4 + \ldots + O(x^{12}).$$

Now $M := M_{S_5}^H(H, 6)$,

$$c_4^{\mathrm{vir}} = 52720,\ c_1^{\mathrm{vir}} c_3^{\mathrm{vir}} = 93280,\ (c_2^{\mathrm{vir}})^2 = 145200,$$
$$(c_1^{\mathrm{vir}})^2 c_2^{\mathrm{vir}} = 157760,\ (c_1^{\mathrm{vir}})^4 = 185600.$$

Now: state version of the Vafa-Witten formula for moduli space $M_S^H(3, c_1, c_2)$ of rank 3 sheaves.
(There is a physics prediction for rank ≥ 3 by Labastida and Lozano, but we believe it is wrong)
Have formulas both for $\chi_{-y}^{\mathrm{vir}}(M)$ and $e^{\mathrm{vir}}(M)$. For simplicity state only for $e^{\mathrm{vir}}(M)$.

The formula again depends on the expected dimension

$$\mathrm{vd} = \mathrm{vd}(M_S^H(3, c_1, c_2)) = 6c_2 - 2c_1^2 - 8\chi(\mathcal{O}_S).$$

Again assume S algebraic surface with $b_1(S) = 0$ and $p_g(S) > 0$. For simplicity assume S contains an irreducible canonical curve (zero set of a holomorphic 2 form).

$$\Theta_{A,0}(x) = \sum_{(n,m)\in\mathbb{Z}^2} x^{2(n^2-nm+m^2)}, \quad \Theta_{A,1}(x) = \sum_{(n,m)\in\mathbb{Z}^2} \epsilon^{n+m} x^{2(n^2-nm+m^2)}$$

Theta functions, here $\epsilon = e^{2\pi i/3}$. Define modular function

$$z(x) := \frac{\Theta_{A,0}(x)}{\Theta_{A,1}(x)} = 1 + 9x^2 + 27x^4 + 81x^6 + O(x^8),$$

Define $z_1(x)$, $z_2(x) = z_1(-x)$ as the solutions of the equation

$$w^2 - 4z(x)^2 w + 4z(x) = 0.$$

Recall $\overline{\eta}(x) = \prod_{n>0}(1 - x^n)$, and define

$$\Psi_{S,c_1}(x) = 9 \left(\frac{1}{3\overline{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)} \left(\frac{3\overline{\eta}(x^6)^3}{\Theta_{A,1}(x)}\right)^{K_S^2}$$
$$\cdot \left(z_1(x)^{K_S^2} + z_2(x)^{K_S^2} + (-1)^{\chi(\mathcal{O}_S)}(\epsilon^{c_1 K_S} + \epsilon^{-c_1 K_S})\right).$$

Conjecture

$$e^{\mathrm{vir}}(M_S^H(3, c_1, c_2)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}\left[\Psi_{S,c_1}(x)\right].$$

Main tool: Mochizuki's formula:

Recall: The Hilbert scheme $S^{[n]}$ of n points on S parametrizes ideal sheaves I_Z of n points (counted with multiplicities) on S
It is a smooth projective variety of dimension $2n$

Write $M = M_S^H(2, c_1, c_2)$
Mochizuki's formula computes virtual Chern numbers

$$\int_{[M]^{\mathrm{vir}}} c_{i_1}^{\mathrm{vir}}(M) \cdots c_{i_m}^{\mathrm{vir}}(M)$$

as a sum over contributions $F_S(a, c_1, c_2)$ for each Seiberg-Witten class a which are expressions in Chern numbers of natural vector bundles on Hilbert schemes $S^{[n]}$

$$\int_{[S^{[n]}]} c_{i_1}(E_1) \cdots c_{i_k}(E_k)$$

We develop a method for computing $F_S(a, c_1, c_2)$

Seiberg-Witten invariants:
invariants of differentiable 4-manifolds
S projective algebraic surface: $H^2(S,\mathbb{Z}) \ni a \mapsto SW(a) \in \mathbb{Z}$
a is called SW class if $SW(a) \neq 0$.

If $b_1(S) = 0$, $p_g(S) > 0$ and S has an irreducible canonical curve, then SW cl. of S are $0, K_S$ with

$$SW(0) = 1, \quad SW(K_S) = (-1)^{\chi(\mathcal{O}_S)}$$

This is the reason for our assumption that S has an irreducible canonical curve, otherwise our formula is more complicated containing SW classes.

Now the argument is in 3 steps:
(1) Universality:

Proposition

The contribution $F_S(a, c_1, c_2)$ for a SW-class a depends only on the numbers $c_1^2, c_1 a, a^2, c_1 K_S, K_S^2, aK_S, \chi(\mathcal{O}_S)$

(2) Reduction to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$: Therefore Mochizuki's formula is determined by its value for 7 triples (S, c_1, a) s.th. corresponding 7-tuples $(c_1^2, c_1 a, a^2, c_1 K_S, K_S^2, aK_S, \chi(\mathcal{O}_S))$ are linearly independent. We can choose $S = \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.
(3) Localization: In this case S has action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints. Gives action of T on $S^{[n]}$ still with finitely many fixpoints described by partitions, compute by localization i.e. Chern numbers of bundles are computed in terms action on fibres at fixpoints
This computes $F_S(a_1, a_2, s, q)$ in terms of combinatorics of partitions.

Let X smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \ldots, p_l$
E equivariant vector bundle of rank r on X
Fibre $E(p_i)$ of X has basis of eigenvect. for T-action
$E(p_i) = \bigoplus_{k=1}^r \mathbb{C} v_i$, with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$,
weight $w(v_i) := n_i \epsilon_1 + n_2 \epsilon_2$.

$$c^T(E(p_i)) = (1 + w(v_1)) \cdots (1 + w(v_r)) \in \mathbb{Z}[\epsilon_1, \epsilon_2].$$

Let $E \in K_T^0(X)$, let $P([c_i(E)]_i)$ be a polynomial in the Chern classes of E, of degree $d = \dim(X)$.

Theorem (Bott residue formula)

$$\int_{[X]} P([c_i(E)]_i) = \sum_{k=1}^{l} \frac{P([c_i^T(E(p_k)]_i)}{c_d^T(T_X(p_k))}$$

(independ. of ϵ_1, ϵ_2)