Virtual topological invariants of moduli spaces

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We want to compute

generating functions of virtual Euler numbers of moduli spaces of sheaves on algebraic surfaces

You should understand all the words not in red
The red ones I will try to explain.
Much of the lecture will be devoted to that
To make it more elementary, I will tell you many lies.
Then I state the results.
If there is time I give a very brief outline of some of the methods
If \((a_n)_{n \in \mathbb{Z}_\geq 0}\) are interesting numbers, the formal power series

\[
f(x) = \sum_{n} a_n x^n
\]

is their generating function. If we know \(f(x)\) we know all the \(a_n\). Analytical properties of \(f(x)\) reflect relations between the \(a_n\).

**Example**

\(p(n)\) number of partitions of a number \(n\), i.e. number of ways to write it as sum of numbers e.g.

\[
3 = 3, \ 3 = 2 + 1, \ 3 = 1 + 1 + 1, \quad p(3) = 3
\]

Exercise: Generating function is

\[
\sum_{n=0}^{\infty} p(n) x^n = \prod_{n=1}^{\infty} \frac{1}{1 - x^n}.
\]
Let $P$ be a regular polyeder, (tetraeder, cube, octaeder, ...).
$V = \#\text{vertices}, \ E = \#\text{edges}, \ S = \#\text{sides},$
Euler (1751) made the remarkable observation that

$$V - E + S = 2$$

e.g.

tetraeder: $4 - 6 + 4 = 2$, cube: $8 - 12 + 6 = 2$

If $X$ is a reasonable topological space, the Euler number $e(X)$ is defined as follows: divide $X$ into spaces homeomorphic to simplices. Then

$$e(X) = \sum_{i \geq 0} (-1)^i \#i\text{-simplicies.}$$

This does not depend on how you divide $X$!

**Note:** A regular polyeder is topologically a 2-sphere $S^2$, so Euler’s observation says $e(S^2) = 2$
The Euler number can be expressed in homology groups. There are several equivalent definitions of homology and cohomology, which we will not review.

X oriented compact manifold of dimension $n$ has homology groups $H_i(X, \mathbb{Z}), i = 0, \ldots, n$

cohomology groups $H^i(X, \mathbb{Z}), i = 0, \ldots, n$

The Betti numbers of $X$ are their ranks

$$b_i(X) = \text{rk } H_i(X, \mathbb{Z})$$

The Euler number is their alternating sum

$$e(X) = \sum_{i=0}^{n} (-1)^i b_i(X)$$
Let $Y \subset X$ oriented closed submanifold of dimension $d$

it has fundamental class $[Y] \in H_d(X, \mathbb{Z})$

In particular $[X] \in H_n(X, \mathbb{Z})$ fundamental class of $X$

$H_0(X, \mathbb{Z}) = [pt] \mathbb{Z} \cong \mathbb{Z}$, for $[pt]$ the class of a point in $X$

Cohomology is a graded ring via cup product

$$H^i(X, \mathbb{Z}) \times H^k(X, \mathbb{Z}) \to H^{i+k}(X, \mathbb{Z}), (\alpha, \beta) \to \alpha \cdot \beta$$

and homology is a module via the cap product

$$H^i(X, \mathbb{Z}) \times H_k(X, \mathbb{Z}) \to H_{k-i}(X, \mathbb{Z}), (\alpha, \beta) \to \alpha \cap \beta$$
Poincaré duality:

\[ PD = \cdot \cap [X] : H^i(X, \mathbb{Z}) \to H_{n-i}(X, \mathbb{Z}), \alpha \to [X] \cap \alpha \]

is an isomorphism, in particular \( H^n(X, \mathbb{Z}) \cong \mathbb{Z} \)

Poincaré duality identifies cup product on cohomology with intersection on homology

\[ PD^{-1}([Y_1]) \cdot PD^{-1}([Y_2]) = \sum_i \pm [Z_i] \]

Here assume that \( Y_1 \) and \( Y_2 \) intersect transversally along disjoint submanifolds \( Z_i \). The sign depends on the orientations.

In particular we have

\[ \int [X] : H^n(X, \mathbb{Z}) \to \mathbb{Z}, \alpha \mapsto \int [X] \alpha := \alpha \cap [X] \in H_0(X, \mathbb{Z}) = \mathbb{Z}. \]
Let $X$ complex manifold (or nonsingular variety over $\mathbb{C}$)

A (holomorphic) **vector bundle of rank** $r$ is $\pi : E \to X$

1. $E$ complex manifold, $\pi$ holomorphic map
2. fibres $E(p) = \pi^{-1}(p)$ are complex vector spaces of dim $r$
3. $E$ is locally trivial: every $p \in X$ has nbhd $U$, s.th $\pi^{-1}(U) \simeq U \times \mathbb{C}^r$, by an isomorphism linear on the fibres.

**Example:**

1. $TX \to X$ tangent bundle, fibre over $p$ is tangent space $T_pX$
2. The **trivial** vector bundle of rank $r$ is $X \times \mathbb{C}^r \to X$

A section of a vector bundle $E \to X$ is a holomorphic map $s : X \to E$ with $s(p) \in E(p)$ for all $p \in X$. 
A vector bundle of rank $r$ on $X$ has **Chern classes**

$$c(E) = 1 + c_1(E) + \ldots + c_r(E), \quad c_i(E) \in H^{2i}(X, \mathbb{Z})$$

They measure how far $E$ is from being trivial, i.e. $c(X \times \mathbb{C}^r) = 1$

**Whitney Product formula:**

$$c(E \oplus F) = c(E) \cdot c(F)$$

Assume $E$ has "good" (transversal) sections $s_1, \ldots, s_r : X \to E$. Then

$$PD(c_{r-k}(E)) = \left[ \{ p \in X \mid s_1(p), \ldots, s_{k+1}(p) \text{ linearly dep. in } E(p) \} \right]$$

In particular $PD(c_r(E)) = \left[ \text{zero set of } s_1 \right]$
Let $X$ complex manifold of dimension $n$
Chern classes of $X$: $c_i(X) = c_i(TX)$
They give invariants of $X$, for instance Euler number of $X$
$$e(X) = \# \text{zeros of general holomorphic vector field}$$

In other words
$$\int_{[X]} c_n(X) = e(X).$$

More generally we have the Chern numbers
$$\int_{[X]} c_{n_1}(X) \cdots c_{n_k}(X) \in \mathbb{Z}$$
if $n_1 + \ldots + n_k = n$. 
A **coherent** (torsion free) **sheaf** $F$ of rank $r$ on $X$, is a vector bundle with singularities i.e. some fibres of dimension $\geq r$

Identify vector bundles $E$ with locally free sheaves i.e in neighbourhood $U$ of any $p \in X$ the sections $E(U)$ of $E$ over $U$ are isomorphic to $r$-tuples of holomorphic functions on $U$

For a torsion free sheaf generalize this

**Example**

Let $Z = \{p_1, \ldots, p_k\}$ be a finite subset of $X$

The ideal sheaf $I_Z$ of $Z$ is the torsion free rank 1 sheaf on $X$ given by

$$I_Z(U) = \{\text{holomorphic functions on } U \text{ vanishing on } Z \cap U\}$$

Chern classes are also defined for torsion free sheaves.

For instance if $\dim(X) = 2$

$$c_1(I_Z) = 0, \quad PD(c_2(I_Z)) = [Z] = k[pt]$$
Let $S$ be a smooth projective surface over $\mathbb{C}$

Assume $H^1(S, \mathbb{Z}) = 0$, and $p_g(S) > 0$

$p_g(S)$ is dim of the space of global holom. 2-forms on $S$

The canonical class of $S$ is $K_S \in H^2(S, \mathbb{Z})$ given by

$PD(K_S) = [Z(\omega)]$ (also $K_S = -c_1(S)$)

The most important numbers associated to $S$ are

1. The holomorphic Euler characteristic

$$\chi(\mathcal{O}_S) = 1 + p_g(S) \geq 2$$

2. $K^2_S := \int_{[S]} K_S^2$

E.g. $H^1(\mathbb{P}^2, \mathbb{Z}) = 0$, $p_g(\mathbb{P}^2) = 0$, $K_{\mathbb{P}^2} = -3H$, $\chi(\mathcal{O}_{\mathbb{P}^2}) = 1$, $K^2_{\mathbb{P}^2} = 9$

$S$ surface of degree 4 in $\mathbb{P}^3$: $H^1(S, \mathbb{Z}) = 0$, $p_g(S) = 1$, $K_S = 0$, $\chi(\mathcal{O}_S) = 2$, $K^2_S = 0$

$S$ surface of degree 5 in $\mathbb{P}^3$: $H^1(S, \mathbb{Z}) = 0$, $\chi(\mathcal{O}_S) = 5$, $K^2_S = 5$
A **moduli space** is an algebraic variety that parametrizes isomorphism classes of objects we are interested in.

**Example**

The Hilbert scheme $S^{[n]}$ of $n$ points on $S$ parametrizes ideal sheaves $I_Z$ of $n$ (possibly nonreduced) points on $S$. It is a smooth projective variety of dimension $2n$.

**Euler numbers of Hilbert schemes:**

Let $e(M)$ be the topological Euler number of $M$.

Write $\bar{\eta}(x) := \prod_{n>0} (1 - x^n)$.

**Theorem**

$$\sum_{n \geq 0} e(S^{[n]}) x^n = \frac{1}{\bar{\eta}(x)^e(S)}$$
Using physics arguments, Vafa and Witten ('94) proposed generating function for Euler numbers of moduli spaces of rank 2 sheaves

\[ M^H_S(2, c_1, c_2) = \text{moduli space of rank 2 } H\text{-semistable sheaves on } S \text{ with Chern classes } c_1, c_2 \]

Here \( H \in H^2(S, \mathbb{Z}) \) is an "ample" class (class of hyperplane section), and \( H\)-stable means that subsheaves cannot be too large, and this is somehow measured using \( H \).

\[ M^H_S(c_1, c_2) = M^H_S(2, c_1, c_2) \] is projective, usually singular, has expected dimension

\[ vd = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S). \]

Here write \( c_2 := \int_{[S]} c_2 \in \mathbb{Z}, c_1^2 := \int_{[S]} c_1^2 \in \mathbb{Z} \)

We have \( M^H_S(1, 0, n) = S^{[n]} \), because every rank 2 torsion-free sheaf on \( S \) with \( c_1 = 0, c_2 = n \) (i.e. \( PD(c_2) = n[pt] \)) is the ideal sheaf of \( n \) points with multiplicities.
In whole talk assume stable=semistable (condition on $c_1$). Assume for simplicity in whole talk: there is a holomorphic 2 form $\omega$ on $S$ whose zero set $Z(\omega)$ is irreducible

$$vd = vd_{M^H_S(c_1,c_2)} = 4c_2 - c_1^2 - 3\chi(O_S)$$

Vafa-Witten conjecture

$$\bar{\eta}(x) := \prod_{n>0} (1 - x^n), \quad \theta_3^0(x) = \sum_{n \in \mathbb{Z}} x^{n^2}$$

$$\psi_S(x) := 8 \left( \frac{1}{2\bar{\eta}(x^2)^{12}} \right)^{\chi(O_S)} \left( \frac{2\bar{\eta}(x^4)^2}{\theta_3^0(x)} \right)^{K_S^2}$$

Then $e(M^H_S(c_1,c_2)) = \text{Coeff}_{x^{vd}}[\psi_S(x)]$.

Want to interpret, check and refine this formula.
$M = M^H_S(c_1, c_2)$ usually very singular
of dimension different from $\nu d = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$
the Vafa-Witten conjecture is usually wrong as stated,
But I do not think this is what Vafa-Witten meant
$M$ is virtually smooth, i.e. it wants to be smooth
One should more think of the Euler number
that the moduli space would have
it was smooth of dimension $\nu d$
What does this mean?

**Virtual dimension** $\nu d$

tangent space $T_p M$: infinitesimal directions in $M$ near $p$

Obstruction space $O_p$: obstructions to moving a finite distance in $M$ along curve with given tangent vector

$M$ has expected dimension $\nu d$ means $\dim(T_p M) - \dim O_p = \nu d$

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**Theorem**

*(Kuranishi)* There exists an analytic map $\mu : T_p M \to O_p$, such that an analytic neighbourhood of $p$ in $M$ is isomorphic to $\mu^{-1}(0)$.

So locally near $p$ in the analytic topology $M$ is given by $\dim O_p$ (analytic) equations in a vector space of dimension $\dim(T_p M)$, so if these equations were transversal then $M$ would be nonsingular of dimension $\nu d$
**M** is **virtually smooth** of dimension \(vd\)

This means there is a map of complex vector bundles \(\phi : E_0 \to E_1\) over \(M\), with \(\text{rk } E_0 - \text{rk } E_1 = vd\) such that at every point:

\[
T_pM = \ker(\phi : E_0(p) \to E_1(p)), \quad O_p = \text{coker}(\phi : E_0(p) \to E_1(p))
\]

Use this structure to define **virtual fundamental class**

\([M]^{\text{vir}} \in H_{2vd}(M, \mathbb{Z}). \quad \longrightarrow \text{ virtual invariants of varieties} \]

This is used to define most invariants in modern enumerative geometry
e.g. Gromov-Witten, Donaldson invariants, Donaldson Thomas invariants
Virtual Tangent bundle $T^\text{vir} M := E_0 - E_1$, with Chern classes

\[ c(T^\text{vir} M) = 1 + c_1^\text{vir}(M) + c_2^\text{vir}(M) \ldots = c(E_0)/c(E_1) \]

\[ = 1 + (c_1(E_0) - c_1(E_1)) + (c_2(E_0) - c_1(E_0)c_1(E_1) - c_2(E_1) + c_1(E_1)^2) + \ldots \]

Recall if $X$ is smooth, then $e(X) = \int_X c_{\dim(X)}(X)$.

**Definition**

Virtual Euler number:

\[ e^\text{vir}(M) := \int_{[M]^\text{vir}} c_{vd}(T^\text{vir}(M)) \]

**Conjecture**

The Vafa-Witten formula holds with $e(M^H_S(c_1, c_2))$ replaced by $e^\text{vir}(M^H_S(c_1, c_2))$. 

Virtual Euler number
Refinement to $\chi_y$-genus

**Refinement**: the $\chi_y$ genus

Remember: Betti numbers $b_i(X) = \text{rk } H^i(X, \mathbb{Z})$

Algebraic var. : Hodge numbers $h^{p,q}(X)$, $p, q = 0, \ldots \text{dim}_\mathbb{C}(X)$

and $b_i = h^{0,i} + \ldots + h^{i,0}$

$\mathbf{\chi}_y$-genus: $\chi_y(X) = y^{-\text{dim}(X)/2} \sum_{p,q} (-1)^{p+q} y^p h^{p,q}(X)$

\[\begin{array}{ccc}
1 & & b_0 = b_4 = 1 \\
0 & 0 & b_1 = b_3 = 0 \\
1 & 20 & 1, \\
0 & 0 & b_2 = 22 \\
1 & & \\
\end{array}\]

Then Hodge numbers

$\chi_y(S) = 2y^{-1} + 20y + 2y$

Note that $\chi_y(X)|_{y=1} = e(X)$

Can define **Virtual** $\mathbf{\chi}_y$-genus, $\chi^\text{vir}(M) \in \mathbb{Z}[y^{\pm 1/2}]$

Again $\chi^\text{vir}(M)|_{y=1} = e^\text{vir}(M)$, so refinement of $e^\text{vir}(M)$
Conjecture for virtual $\chi_{-y}$-genus:

$$\theta_3(x, y) := \sum_{n \in \mathbb{Z}} x^{n^2} y^n, \quad \overline{\eta}(x) = \prod_{n > 0} (1 - x^n)$$

$$\psi_S(x, y) := 8 \left( \frac{1}{2 \prod_{n > 0} (1 - x^{2n})^{10} (1 - x^{2n}y)(1 - x^{2n}/y)} \right)^{\chi(O_S)} \cdot \left( \frac{2\overline{\eta}(x^4)^2}{\theta_3(x, y^{1/2})} \right)^{K_S^2}$$

Conjecture

$$\chi_{-y}^{\text{vir}}(M_S^H(c_1, c_2)) = \text{Coeff}_{x^{vd}}[\psi_S(x, y)].$$

Specializes to our version of VW conjecture for $y = 1$. 
Final generalization: the cobordism class:

Two complex manifolds $M$, $N$ have the same cobordism class
$\{M\} = \{N\}$
if they have the same Chern numbers:

$$\int_{[M]} c_{i_1}(M) \cdots c_{i_k}(M) = \int_{[N]} c_{i_1}(N) \cdots c_{i_k}(N) \quad \forall k, i_1, \ldots, i_k$$

Cobordism classes of complex manifolds generate a ring
$R = \sum_n R_n$ (graded by dimension)
$\{M\}\{N\} = \{M \times N\}$, $\{M\} + \{N\} = \{M \sqcup N\}$
In fact

$$R \otimes \mathbb{Q} = \mathbb{Q}[\{\mathbb{P}^1\}, \{\mathbb{P}^2\}, \{\mathbb{P}^3\}, \ldots]$$
Ellingsrud-G-Lehn showed $\{ S[n] \}$ depends only on $\{ S \}$ (equivalent: Chern numbers of $S[n]$ depend only on $K^2_S, c_2(S)$).

For $M = M^H_S(c_1, c_2)$ let $\{ M \}^{\text{vir}}$ be the virtual cobordism class given by the

$$\int_{[M]^{\text{vir}}} c_{i_1}(T^{\text{vir}}_M) \cdots c_{i_k}(T^{\text{vir}}_M).$$

**Conjecture**

There is a power series $P(x) = 1 + \sum_{n > 0} P_n x^n$, with $P_n \in R_n$, s.th.

$$\{ M^H_S(c_1, c_2) \}^{\text{vir}} = \text{Coeff}_{x^{\text{vd}}} \left[ 8 \left( \frac{1}{4} \sum_{n \geq 0} \{ K3[n] \} x^{2n} \right)^{\chi(O_S)/2} (2P(x))^{K^2_S} \right].$$
We developed an algorithm to compute $e^{\text{vir}}(M), \chi^{\text{vir}}(M), \{M\}^{\text{vir}}$ for $M = M^H_S(c_1, c_2)$ for any $S, c_1, c_2$.

Confirms conjectures for K3 surfaces, their blowups, elliptic surfaces, double covers of $\mathbb{P}^2$, complete intersections, etc, up to large $v_d$.

**Example:** **Quintic hypersurface** $S_5$ generic surface of degree 5 in $\mathbb{P}^3$, $K_{S_5} = H$ hyperplane class

\[ K^{2}_{S_5} = \chi(O_{S_5}) = 5. \]

\[ \sum_{c_2} e^{\text{vir}}(M^H_S(H, c_2)) x^{4c_2-20} = 8 + 52720 x^4 + 48754480 x^8 + \ldots + O(x^{28}) \]

\[ \sum_{c_2} \chi^{\text{vir}}(M^H_S(H, c_2)) x^{4c_2-20} = 8 + (1280y^2 + 11440y + 27280 + 11440y^{-1} + 1280^{-2})x^4 + \ldots + O(x^{12}). \]

Now $M := M^H_{S_5}(H, 6)$,

\[ c_4^{\text{vir}} = 52720, \quad c_1^{\text{vir}} c_3^{\text{vir}} = 93280, \quad (c_2^{\text{vir}})^2 = 145200, \]

\[ (c_1^{\text{vir}})^2 c_2^{\text{vir}} = 157760, \quad (c_1^{\text{vir}})^4 = 185600. \]
Now: state version of the Vafa-Witten formula for moduli space $M^H_S(3, c_1, c_2)$ of rank 3 sheaves.  
(There is a physics prediction for rank $\geq 3$ by Labastida and Lozano, but we believe it is wrong)  
Have formulas both for $\chi^\text{vir}(M)$ and $e^\text{vir}(M)$. For simplicity state only for $e^\text{vir}(M)$.  
The formula again depends on the expected dimension  
$$vd = vd(M^H_S(3, c_1, c_2) = 6c_2 - 2c_1^2 - 8\chi(O_S).$$  
Again assume $S$ algebraic surface with $b_1(S) = 0$ and $p_g(S) > 0$. For simplicity assume $S$ contains an irreducible canonical curve (zero set of a holomorphic 2 form).
The rank 3 case

$$\Theta_{A,0}(x) = \sum_{(n,m) \in \mathbb{Z}^2} x^{2(n^2 - nm + m^2)}, \quad \Theta_{A,1}(x) = \sum_{(n,m) \in \mathbb{Z}^2} \epsilon^{n+m} x^{2(n^2 - nm + m^2)}.$$ 

Theta functions, here $\epsilon = e^{2\pi i/3}$. Define modular function

$$z(x) := \frac{\Theta_{A,0}(x)}{\Theta_{A,1}(x)} = 1 + 9x^2 + 27x^4 + 81x^6 + O(x^8),$$

Define $z_1(x), z_2(x) = z_1(-x)$ as the solutions of the equation

$$w^2 - 4z(x)^2w + 4z(x) = 0.$$ 

Recall $\eta(x) = \prod_{n>0}(1 - x^n)$, and define

$$\psi_{S,c_1}(x) = 9 \left( \frac{1}{3\eta(x^2)^{12}} \right) ^{\chi(O_S)} \left( \frac{3\eta(x^6)^3}{\Theta_{A,1}(x)} \right) ^{K_S^2} \cdot \left( z_1(x)^{K_S^2} + z_2(x)^{K_S^2} + (-1)^{\chi(O_S)} (\epsilon^{c_1K_S} + \epsilon^{-c_1K_S}) \right).$$

**Conjecture**

$$e^{\text{vir}}(M_S^{H}(3, c_1, c_2)) = \text{Coeff}_{x^{vd}}[\psi_{S,c_1}(x)].$$
Main tool: Mochizuki’s formula:

Recall: The Hilbert scheme $S^{[n]}$ of $n$ points on $S$ parametrizes ideal sheaves $I_Z$ of $n$ points (counted with multiplicities) on $S$. It is a smooth projective variety of dimension $2n$.

Write $M = M_S^H(2, c_1, c_2)$.

Mochizuki’s formula computes virtual Chern numbers

$$\int_{[M]_{\text{vir}}} c_{i_1}^{\text{vir}}(M) \cdots c_{i_m}^{\text{vir}}(M)$$

as a sum over contributions $F_S(a, c_1, c_2)$ for each Seiberg-Witten class $a$ which are expressions in Chern numbers of natural vector bundles on Hilbert schemes $S^{[n]}$

$$\int_{[S^{[n]}]} c_{i_1}(E_1) \cdots c_{i_k}(E_k)$$

We develop a method for computing $F_S(a, c_1, c_2)$.
**Seiberg-Witten invariants:**
invariants of differentiable 4-manifolds
S projective algebraic surface: $H^2(S, \mathbb{Z}) \ni a \mapsto SW(a) \in \mathbb{Z}$
a is called SW class if $SW(a) \neq 0$.

If $b_1(S) = 0$, $p_g(S) > 0$ and S has an irreducible canonical curve, then SW cl. of S are 0, $K_S$ with

$$SW(0) = 1, \quad SW(K_S) = (-1)^{\chi(O_S)}$$

This is the reason for our assumption that S has an irreducible canonical curve, otherwise our formula is more complicated containing SW classes.
Now the argument is in 3 steps:

(1) **Universality:**

**Proposition**

The contribution $F_S(a, c_1, c_2)$ for a SW-class $a$ depends only on the numbers $c_1^2, c_1 a, a^2, c_1 K_S, K_S^2, aK_S, \chi(\mathcal{O}_S)$

(2) **Reduction to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$:** Therefore Mochizuki’s formula is determined by its value for 7 triples $(S, c_1, a)$ s.th. corresponding 7-tuples $(c_1^2, c_1 a, a^2, c_1 K_S, K_S^2, aK_S, \chi(\mathcal{O}_S))$ are linearly independent. We can choose $S = \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

(3) **Localization:** In this case $S$ has action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints. Gives action of $T$ on $S^n$ still with finitely many fixpoints described by partitions, compute by localization i.e. Chern numbers of bundles are computed in terms action on fibres at fixpoints

This computes $F_S(a_1, a_2, s, q)$ in terms of combinatorics of partitions.
Equivariant localization

Let $X$ smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \ldots, p_l$

$E$ equivariant vector bundle of rank $r$ on $X$

Fibre $E(p_i)$ of $X$ has basis of eigenvect. for $T$-action

$$E(p_i) = \bigoplus_{k=1}^r \mathbb{C}v_i,$$

with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$,

weight $w(v_i) := n_i \epsilon_1 + n_2 \epsilon_2$.

$$c^T(E(p_i)) = (1 + w(v_1)) \cdots (1 + w(v_r)) \in \mathbb{Z}[\epsilon_1, \epsilon_2].$$

Let $E \in K^0_T(X)$, let $P([c_i(E)]_i)$ be a polynomial in the Chern classes of $E$, of degree $d = \dim(X)$.

**Theorem (Bott residue formula)**

$$\int_{[X]} P([c_i(E)]_i) = \sum_{k=1}^l \frac{P([c^T_i(E(p_k))]_i)}{c^T_d(T_X(p_k))} (\text{indep. of } \epsilon_1, \epsilon_2)$$