Vafa-Witten formula and generalizations

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Calabi-Yau and geometry

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We work over $\mathbb{C}$.

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$\{$zero dimensional subschemes of degree n on $S\}$
(i.e. generically sets of n points on S).
2. Moduli spaces of stable sheaves $M_S^H(r, c_1, c_2)$:
$\{$rank r coherent sheaves on S with Chern classes $c_1, c_2\}$
(i.e. vector bundles with singularities).

In differential geometry can also consider moduli spaces, e.g. of asd-connections on a principal $SO(3)$-bundle over a 4-manifold X

Used to define and compute Donaldson invariants, which are C^∞ invariants of 4-manifolds

If X is a projective algebraic surface close relationship to moduli spaces $M_S^H(2, c_1, c_2)$ of stable sheaves allows to compute Donaldson invariants via algebraic geometry.

S projective complex surface, H ample line bundle on S, i.e. $S \subset \mathbb{P}^n$ and H is the hyperplane bundle (or consider S with the Fubini-Study metric induced from $\mathbb{P}^n$).
We assume always that

1. $b_1(S) = \dim H^1(S, \mathbb{Q}) = 0$
2. $p_g(S) = h^0(S, K_S) > 0$, i.e. $\exists$ nonvanishing holomorphic 2-forms on S

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$\mathcal{E}$ semistable $\iff \forall_{n \gg 0} \frac{h^0(S, \mathcal{F} \otimes H^{\otimes n})}{\mathrm{rk}(\mathcal{F})} \leq \frac{h^0(S, \mathcal{E} \otimes H^{\otimes n})}{\mathrm{rk}(\mathcal{E})}$ for all $\mathcal{F}$ subsheaf of $\mathcal{E}$.

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$M = M_S^H(r, c_1, c_2)$ is usually singular, has *expected dimension*

$$vd = 2rc_2 - (r-1)c_1^2 + (r^2-1)\chi(\mathcal{O}_S).$$

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vd is the dimension M should have, more about that later
Here write $c_2 := \int_{[S]} c_2 \in \mathbb{Z}$, $c_1^2 := \int_{[S]} c_1^2 \in \mathbb{Z}$

Rank 1 case: Hilbert scheme of points

$$S^{[n]} = \{\text{zero dimensional subschemes of length } n \text{ on } S\}$$

General pt Z of $S^{[n]}$: $Z = p_1 \sqcup \ldots \sqcup p_n$ set of n distinct pts of S

When points come together have nontrivial scheme structure, $Z = Z_1 \sqcup \ldots \sqcup Z_k$ such that $dim_{\mathbb{C}}\mathcal{O}_Z = \sum_{i=1}^k dim_{\mathbb{C}}\mathcal{O}_{Z_i} = n$.

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$S^{[n]} \to S^{(n)} = S^n/(\text{perm. of factors}), Z \mapsto supp(Z)$
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$M_S^H(1, L, c_2) = S^{[c_2]}$, via $Z \leftrightarrow I_Z \otimes \mathcal{O}(L)$. I_Z ideal sheaf of Z.

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$M_S^H(1, L, c_2) = S^{[c_2]}$
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Theorem (G'90)

$$\sum_{n\geq 0} e(S^{[n]})x^n = \frac{1}{\prod_{n>0}(1-x^n)^{e(S)}}$$

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By physics arguments, 1994 Vafa and Witten gave explicit conjectural formula for the generating function for $e(M_S^H(2, L, n))$, in terms of modular forms.

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Write $K_S^2 = \int_{[S]} K_S^2 = \int_{[S]} c_1(S)^2$,
let $\chi(\mathcal{O}_S)$ holomorphic Euler characteristic
Write in future $M_S^H(c_1, c_2) = M_S^H(2, c_1, c_2)$, and always

$$\mathrm{vd} = \mathrm{vd}_{M_S^H(c_1,c_2)} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$$

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Conjecture (Vafa-Witten conjecture)

$$e(M_S^H(c_1, c_2)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}\left[8\left(\frac{1}{2\prod_{n>0}(1-x^{2n})^{12}}\right)^{\chi(\mathcal{O}_S)} \cdot \left(\frac{2\prod_{n>0}(1-x^{4n})^2}{\sum_{n\in\mathbb{Z}} x^{n^2}}\right)^{K_S^2}\right]$$

Want to interpret, check and refine this formula

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Virtual structure is used to define most invariants in modern enumerative geometry, e.g. Gromov-Witten, Donaldson invariants, Donaldson Thomas invariants

In differential geometry, when the moduli space
(of solutions to some pde) is singular, one deforms the equation
to get a smooth moduli space
(e.g. for Donaldson invariants).
In algebraic geometry, one keeps the moduli space as is, but
adds virtual structure,
which keeps records why the moduli space is virtually smooth
This allows for better control.

At every point $[F] \in M$, tangent space $T_{[F]} = Ext^1(F, F)_0$
obstruction space $O_{[F]} = Ext^2(F, F)_0$
Kuranishi: $\exists$ analytic map $\kappa : T_{[F],0} \to O_{[F],0}$,
s.th.anal. nbhd of $[F]$ in M is isom. to $\kappa^{-1}(0)$
$\Longrightarrow$ if $O_F = 0$ or κ submersion, M is nonsingular of dim vd

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Perfect obstruction theory:
Complex $E_\bullet = [E_0 \to E_1]$ of vb on M, s.th. $\forall_{[F]\in M}$:
$T_{[F]} \simeq ker(E_0([F]) \to E_1([F]))$, $O_F \simeq coker(E_0([F]) \to E_1([F]))$
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Then define: $T_M^{\mathrm{vir}} := [E_0] - [E_1] \in K^0(M)$,
$\mathrm{vd} := \mathrm{rk}\, T_M^{\mathrm{vir}} = \mathrm{rk}(E_0) - \mathrm{rk}(E_1)$
virtual fundamental class $[M]^{\mathrm{vir}} \in H_{2\mathrm{vd}}(M,\mathbb{Z})$
virtual structure sheaf $\mathcal{O}_M^{\mathrm{vir}} \in K_0(M)$ (these last two are difficult)

Definition

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Conjecture

The Vafa-Witten formula holds with $e(M_S^H(c_1, c_2))$ replaced by $e^{\mathrm{vir}}(M_S^H(c_1, c_2))$.

holomorphic Euler characteristic:

$$\chi(X, V) := \sum_{i \geq 0} (-1)^i \dim H^i(X, V), \quad V \in K^0(X)$$

χ_{-y}-genus:

$$\chi_{-y}(X) = \sum_{p,q} (-1)^{p+q} y^p h^{p,q}(X) = \sum_p (-y)^p \chi(X, \Omega_X^p)$$

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Virtual χ_{-y}**-genus.** For $V \in K^0(M)$, put $\chi^{\mathrm{vir}}(M,V) := \chi(M, \mathcal{O}_M^{\mathrm{vir}} \otimes V)$. Let $\Omega_M^{\mathrm{vir}} := (T_M^{\mathrm{vir}})^\vee$.

$$\chi_{-y}^{\mathrm{vir}}(M) := y^{-\mathrm{vd}/2} \sum_p (-y)^p \chi^{\mathrm{vir}}(M, \Lambda^p \Omega_M^{\mathrm{vir}})$$

$\chi_{-1}^{\mathrm{vir}}(M) = e^{\mathrm{vir}}(M)$, so this is refinement of virtual Euler number

Refinement to χ_y-genus

$$\psi_S(x) := 8\left(\frac{1}{2\prod_{n>0}(1-x^{2n})^{12}}\right)^{\chi(\mathcal{O}_S)}\left(\frac{2\prod_{n>0}(1-x^{4n})^2}{\sum_{n\in\mathbb{Z}} x^{n^2}}\right)^{K_S^2}$$

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Conjecture

$e^{\mathrm{vir}}(M_S^H(c_1, c_2)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}[\psi_S(x)].$

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Conjecture for virtual χ_{-y}-genus:

$$\psi_S(x,y) := 8\left(\frac{1}{2\prod_{n>0}(1-x^{2n})^{10}(1-x^{2n}y)(1-x^{2n}/y)}\right)^{\chi(\mathcal{O}_S)} \cdot \left(\frac{2\prod_{n>0}(1-x^{4n})^2}{\sum_{n\in\mathbb{Z}} x^{n^2}y^{n/2}}\right)^{K_S^2}$$

Conjecture

$\chi_{-y}^{\mathrm{vir}}(M_S^H(c_1,c_2)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}[\psi_S(x,y)]$.

Specializes to our version of VW conjecture for $y=1$

Have conjectural generating function for virtual Elliptic genus of $M_S^H(c_1, c_2)$ in terms of Siegel modular forms
It gives generalization of the DMVV formula (Dijkgraaf-Moore-Verlinde-Verlinde '97), (Borisov-Libgober '00) for Hilbert schemes of points.
A bit too complicated to state here.

Final generalization: the cobordism class:
Two complex manifolds M, N have the same cobordism class $\{M\} = \{N\}$
if they have the same Chern numbers:

$$\int_{[M]} c_{i_1}(M) \cdots c_{i_k}(M) = \int_{[N]} c_{i_1}(N) \cdots c_{i_k}(N) \quad \forall_{k, i_1, \ldots, i_k}$$

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Cobordism classes of complex manifolds generate a ring
$R = \sum_n R_n$ (graded by dimension)
$\{M\}\{N\} = \{M \times N\}, \quad \{M\} + \{N\} = \{M \sqcup N\}$
In fact

$$R \otimes \mathbb{Q} = \mathbb{Q}[\{\mathbb{P}^1\}, \{\mathbb{P}^2\}, \{\mathbb{P}^3\}, \dots]$$

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For $M = M_S^H(c_1, c_2)$ let $\{M\}^{\mathrm{vir}}$ be the virtual cobordism class given by the

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Conjecture

There is a power series $P(x) = 1 + \sum_{n>0} P_n x^n$, with $P_n \in R_n$, s.th.

$$\{M_S^H(c_1, c_2)\}^{\mathrm{vir}} = \mathrm{Coeff}_{x^{\mathrm{vd}}}\left[8\left(\frac{1}{4}\sum_{n\geq 0}\{K3^{[n]}\}x^{2n}\right)^{\chi(\mathcal{O}_S)/2}(2P(x))^{K_S^2}\right].$$

Seiberg-Witten invariants:
invariants of differentiable 4-manifolds
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In general for alg. surfaces they are easy to compute, e.g. if $b_1(S) = 0$, $p_g(S) > 0$ and $|K_S|$ contains smooth connected curve, then SW cl. of S are $0, K_S$ with

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This is the reason for our assumption that $|K_S|$ contains smooth connected curve, otherwise our results look more complicated.

We restrict attention to the virtual Euler number
S projective surface with $b_1(S) = 0$, $p_g(S) > 0$.

Conjecture

$$e^{\mathrm{vir}}(M_S^H(c_1, c_2)) = \mathrm{Coeff}_{x^{vd}}\Bigg[4\left(\frac{1}{2\prod_{n>0}(1-x^{2n})^{12}}\right)^{\chi(\mathcal{O}_S)} \left(\frac{2\prod_{n>0}(1-x^{4n})^2}{\sum_{n\in\mathbb{Z}} x^{n^2}}\right)^{K_S^2} \sum_{a\in H^2(S,\mathbb{Z})} SW(a)(-1)^{c_1 a}\left(\frac{\sum_{n\in\mathbb{Z}} x^{n^2}}{\sum_{n\in\mathbb{Z}}(-1)^n x^{n^2}}\right)^{aK_S}\Bigg],$$

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Examples:
(1) K3 surfaces: Let S be a K3 surface,
$M=M_S^H(c_1,c_2)$ is nonsingular of dim vd and $e(M)=e(S^{[vd/2]})$ (Yoshioka)

$$\Longrightarrow e(M)=\mathrm{Coeff}_{x^{vd}}\left[\frac{1}{\prod_{n>0}(1-x^{2n})^{24}}\right]$$

Follows from our formula because $K_S^2=0$, and $SW(0)=1$ is only SW invariant.

(2) Elliptic surfaces: (Yoshioka) S ellipic surface $\chi(O_S) = d$, F fibre
$M = M_S^H(c_1, c_2)$ is nonsingular of dim vd

$$e(M) = \begin{cases} \mathrm{Coeff}_{x^{vd}}\left[\frac{1}{\prod_{n>0}(1-x^{2n})^{12d}}\right] & c_1F \equiv 1 \mod 2, \\ 0 & c_1F \equiv 0 \mod 2 \end{cases}$$

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$SW(kF) = (-1)^k\binom{d-2}{k}$, $k = 0, \ldots, d-2$

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(3) Blowup formula:(Li-Qin) Let $\widehat{S}$ the blowup of surface S. $c_1 \in H^2(S)$, E exceptional divisor. Then

$$\sum_{c_2} e(M_{\widehat{S}}^H(c_1 + aE, c_2))x^{vd} = \frac{\sum_{n\in\mathbb{Z}} x^{(2n+a)^2}}{\prod_{n>0}(1-x^{4n})^2} \sum_{c_2} e(M_S^H(c_1, c_2))x^{vd}$$

We predict the same formula with e replaced by e^{vir} on both sides, because $K_{\widehat{S}}^2 = K_S^2 - 1$ and SW invariants are $SW_{\widehat{S}}(a) = SW_{\widehat{S}}(a + E) = SW_S(a)$ for all SW classes a on S

(4) Quintic in $\mathbb{P}^3$: Let S be a nonsingular quintic in $\mathbb{P}^3$, H the hyperplane section. We show

$$\sum_{c_2} e^{\mathrm{vir}}(M_S^H(H, c_2)x^{vd} = 8 + 52720x^4 + 48754480x^8 + 17856390560x^{12} + 3626761297400x^{16} \ldots + O(x^{28})$$

conferming the conjecture

Main tool: Mochizuki's formula:
Compute intersection numbers on $M = M_S^H(c_1, c_2)$ in terms of intersection numbers on Hilbert scheme of points.

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On $S \times M$ have $\mathcal{E}$ universal sheaf
i.e. if $[E] \in M$ corresponds to a sheaf E on S then $\mathcal{E}|_{S\times[E]} = E$.
For $\alpha \in H^k(S)$, put

$$\tau_i(\alpha) := \pi_{M_*}(c_i(\mathcal{E})\pi_S^*(\alpha)) \in H^{2i-4+k}(M)$$

Main tool: Mochizuki's formula:
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Let $P(\mathcal{E})$ be any polynomial in the $\tau_i(\alpha)$
Mochizuki's formula expresses $\int_{[M]^{\mathrm{vir}}} P(\mathcal{E})$ in terms of intersec. numbers on $S^{[n_1]} \times S^{[n_2]}$, and Seiberg-Witten invariants.

$e^{\mathrm{vir}}(M)$, $\chi_{-y}^{\mathrm{vir}}(M)$ $Ell^{\mathrm{vir}}(M)$ and $\{M\}^{\mathrm{vir}}$ can all be expressed as $\int_{[M]^{\mathrm{vir}}} P(\mathcal{E})$, for suitable polyn. P, so can reduce computation to Hilbert schemes.

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For $\chi_{-y}^{\mathrm{vir}}(M)$ $Ell^{\mathrm{vir}}(M)$ use **virtual Riemann-Roch formula**

Theorem (Fantechi-G.)

For $V \in K^0(M)$ have

$$\chi^{\mathrm{vir}}(M, V) = \int_{[M]^{\mathrm{vir}}} \mathrm{ch}(V)\mathrm{td}(T_M^{\mathrm{vir}}).$$

$$S^{[n_1]} \times S^{[n_2]} = \{\text{pairs } (Z_1, Z_2) \text{ of subsch. of deg. } (n_1, n_2) \text{ on } S\}$$

Work on $S \times S^{[n_1]} \times S^{[n_2]}$, projection p to $S^{[n_1]} \times S^{[n_2]}$

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Two universal sheaves: Let $a_1, a_2 \in \mathrm{Pic}(S)$

1. $\mathcal{I}_i(a)$ sheaf on $S \times S^{[n_1]} \times S^{[n_2]}$ with $\mathcal{I}_i(a_i)|_{S\times(Z_1,Z_2)} = I_{Z_i} \otimes a_9$
2. $\mathcal{O}_i(a_i)$, vector bundle of rank n_i on $S^{[n_1]} \times S^{[n_2]}$, with fibre $\mathcal{O}_i(a_i)(Z_1, Z_2) = H^0(\mathcal{O}_{Z_i} \otimes a_i)$

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Remember, we want to compute $\int_{[M]^{\mathrm{vir}}} P(\mathcal{E})$
There is a (Laurent) polynomial $\Psi_P(a_1, a_2, n_1, n_2, s)$ associated to P in a variable s, the

$$\overline{\tau}_i(\alpha) := p_*(c_i(\mathcal{I}_1(a_1) \oplus \mathcal{I}_2(a_2))\pi_S^*(\alpha)) \in H^{2i-4+k}(S^{[n_1]} \times S^{[n_2]}), \quad \alpha \in H^k(S)$$

and the Chern classes of $\mathcal{O}_1(a_1)$, $\mathcal{O}_2(a_2)$, s.th following holds: Put

$$A_P(a_1, a_2, c_2, s) = \sum_{n_1+n_2=c_2-a_1a_2} \int_{S^{[n_1]}\times S^{[n_2]}} \Psi_P(a_1, a_2, n_1, n_2, s) \in \mathbb{Q}[s, s^{-1}]$$

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Theorem (Mochizuki)

Assume $\chi(E) > 0$ for $E \in M_H^S(c_1, c_2)$. Then

$$\int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}} P(\mathcal{E}) = \sum_{\substack{c_1=a_1+a_2 \\ a_1 H < a_2 H}} SW(a_1)\mathrm{Coeff}_{s^0} A_P(a_1, a_2, c_2, s)$$

i.e. we replace a simple formula on a space where we cannot compute anything by a terrible formula on simpler space

Take now for $P(\mathcal{E}) = c_{\mathrm{vd}}(T_M^{\mathrm{vir}})$ (works the same for the others)
Put

$$Z_S(a_1, a_2, s, q) = \sum_{n_1, n_2 \geq 0} \int_{S^{[n_1]} \times S^{[n_2]}} A(a_1, a_2, a_1 a_2 + n_1 + n_2, s) q^{n_1 + n_2}$$

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Proposition

There exist univ. functions

$$A_1(s, q), \ldots, A_7(s, q) \in \mathbb{Q}[s, s^{-1}][[q]]$$

s.th. $\forall_{S, a_1, a_2}$

$$Z_S(a_1, a_2, s, q) = F_0(a_1, a_2, s) A_1^{a_1^2} A_2^{a_1 a_2} A_3^{a_2^2} A_4^{a_1 K_S} A_5^{a_2 K_S} A_6^{K_S^2} A_7^{\chi(\mathcal{O}_S)},$$

(where $F_0(a_1, a_2, s)$ is some explicit elementary function).

Proof: Modification of an argument of Elllingsrud-G-Lehn: "Intersection numbers of universal sheaves on $S^{[n]}$ are universal polynomials in intersection numbers on S".

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Reason: Untersection numbers on $S^{[n]}$ computed inductively: $Z_n(S) := \{(x, Z) \in S \times S^{[n]} | x \in Z\}$ universal subscheme Blowup of $S \times S^{[n]}$ along $Z_n(S)$ is

$$S^{[n,n+1]} := \{(Z, W) \in S^{[n]} \times S^{[n+1]} \mid Z \in W\}$$

This allows to compute intersection numbers of $S^{[n+1]}$ in terms of inters. numbers on S and $S^{[n]}$, and conclude by induction.

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This gives:

$$\mathrm{Coeff}_{q^k s^l} Z_S(a_1, a_2, s, q) = P_{k,l}(a_1^2, a_1 a_2, a_2^2, a_1 K_S, a_1 K_S, K_S^2, \chi(O_S))$$

for some polynomial $P_{k,l}$ depending only on k, l.
For the multiplicativity use additional tricks.

$A_1(s,q),\dots A_7(s,q)$ are determined by value of $Z_S(a_1,a_2,s,q)$ for 7 triples (S,a_1,a_2) (S surface, $a_1,a_2\in \mathrm{Pic}(S)$) s.th. corresponding 7-tuples $(a_1^2, a_1a_2, a_2^2, a_1K_S, a_1K_S, K_S^2, \chi(O_S))$ are linearly independent

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We take

$$(\mathbb{P}^2, \mathcal{O}, \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}), (\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}), (\mathbb{P}^2, \mathcal{O}, \mathcal{O}(1)),$$
$$(\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}(1)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,0), \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}(1,0))$$

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In this case S is a smooth toric, i.e. have an action of $T=\mathbb{C}^*\times\mathbb{C}^*$ with finitely many fixpoints,
Action of T lifts to action on $S^{[n]}$ still with finitely many fixpoints described by partitions, compute by equivariant localization.
This computes $Z_S(a_1,a_2,s,q)$ in terms of combinatorics of partitions.

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Result: Computed $A_1, \dots A_7$

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mod q^7 for $Ell^{\mathrm{vir}}(M)$ and $\{M\}^{\mathrm{vir}}$

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This confirms conjectures for K3 surfaces, their blowups, elliptic surfaces, double covers of $\mathbb{P}^2$ and rational ruled surfaces, complete intersections, for $\mathrm{vd}(M)$ smaller than roughly $\frac{3}{2}$ times the power of q.

Let X be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \ldots, p_e$
Let E be equivariant vector bundle of rank r on X.

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Fibre $E(p_i)$ of X at fixp. p_i has basis of eigenvect. for T-action
$E(p_i) = \bigoplus_{k=1}^r \mathbb{C} v_i$, with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$

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Equivariant Chern class of fibre at fixpoint:

$$c^T(E(p_i)) = (1 + c_1^T(E(p_i)) + \ldots + c_r^T(E(p_i)) = \prod_{i=1}^r (1 + n_i\epsilon_1 + m_i\epsilon_2) \in \mathbb{Z}[\epsilon_1, \epsilon_2]$$

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Let $P(c(E)))$polynomial in Chern classes of E, of degree $d = \dim(X)$

Theorem (Bott residue formula)

$$\int_{[X]} P(c(E)) = \sum_{k=1}^e \frac{P(c^T(E(p_k)))}{c^T_{dim(X)}(T_X(p_k))}$$

(does not depend on ϵ_1, ϵ_2)

For simplicity $S = \mathbb{P}^2$. $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{P}^2$ by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

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Easy: Z is T-invariant $\iff I_Z\in k[x,y]$ is gen. by monomials
Can write

$$I_Z=(y^{n_0},xy^{n_1},....,x^ry^{n_r},x^{r+1})\quad(n_0,\dots,n_r)\text{ partition of }n$$

Fixpoints on $(\mathbb{P}^2)^{[n]}$ are in bijections with triples (P_0,P_1,P_2) of partitions of 3 numbers adding up to n.

Need to compute things like $c(\mathcal{O}^{[n]})$
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Let e.g. $Z = Z_0$, $I_Z = (y^4, xy^2, x^2y, x^3)$
Then the fibre $\mathcal{O}^{[n]}(Z) = H^0(\mathcal{O}_Z) = \mathbb{C}[x, y]/(y^4, xy^2, x^2y, x^3)$
Thus basis of eigenvectors of fibre for T action is

$$\begin{matrix} 1 & y & y^2 & y^3 \\ x & xy & & \\ x^2 & & & \end{matrix} \quad \text{with eigenvalues} \quad \begin{matrix} 1 & t_2 & t_2^2 & t_2^3 \\ t_1 & t_1t_2 & & \\ t_1^2 & & & \end{matrix}$$

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Thus

$$c^T(\mathcal{O}^{[n]}(Z)) = (1+\epsilon_2)(1+2\epsilon_2)(1+3\epsilon_2)(1+\epsilon_1)(1+\epsilon_1+\epsilon_2)(1+2\epsilon_1).$$

Now: state version of the Vafa-Witten formula for moduli space $M_S^H(3, c_1, c_2)$ of rank 3 sheaves.
(There is a *wrong* physics prediction for all ranks ≥ 3)
Have formulas both for $\chi_{-y}^{\mathrm{vir}}(M)$ and $e^{\mathrm{vir}}(M)$. For simplicity state only for $e^{\mathrm{vir}}(M)$.

The formula again depends on the expected dimension

$$\mathrm{vd} = \mathrm{vd}(M_S^H(3, c_1, c_2) = 6c_2 - 2c_1^2 - 8\chi(\mathcal{O}_S).$$

Again assume S algebraic surface with $b_1(S) = 0$ and $p_g(S) > 0$. For simplicity assume S contains an irreducible canonical curve (zero set of a holomorphic 2 form).

$$\Theta_{A,0}(x) = \sum_{(n,m)\in\mathbb{Z}^2} x^{2(n^2-nm+m^2)}, \quad \Theta_{A,1}(x) = \sum_{(n,m)\in\mathbb{Z}^2} \epsilon^{n+m} x^{2(n^2-nm+m^2)}$$

Theta functions for A_2-lattice, here $\epsilon = e^{2\pi i/3}$.

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Theta functions for A_2-lattice, here $\epsilon = e^{2\pi i/3}$. Define modular function

$$z(x) := \frac{\Theta_{A,0}(x)}{\Theta_{A,1}(x)} = 1 + 9x^2 + 27x^4 + 81x^6 + O(x^8),$$

Define $z_1(x)$, $z_2(x) = z_1(-x)$ as the solutions of the equation

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Recall $\overline{\eta}(x) = \prod_{n>0}(1-x^n)$, and define

$$\Psi_{S,c_1}(x) = 9\left(\frac{1}{3\overline{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)} \left(\frac{3\overline{\eta}(x^6)^3}{\Theta_{A,1}(x)}\right)^{K_S^2} \cdot \left(z_1(x)^{K_S^2} + z_2(x)^{K_S^2} + (-1)^{\chi(\mathcal{O}_S)}(\epsilon^{c_1 K_S} + \epsilon^{-c_1 K_S})\right).$$

$$\Theta_{A,0}(x)=\sum_{(n,m)\in\mathbb{Z}^2} x^{2(n^2-nm+m^2)},\quad \Theta_{A,1}(x)=\sum_{(n,m)\in\mathbb{Z}^2} \epsilon^{n+m}x^{2(n^2-nm+m^2)}$$

Theta functions for A_2-lattice, here $\epsilon=e^{2\pi i/3}$. Define modular function

$$z(x):=\frac{\Theta_{A,0}(x)}{\Theta_{A,1}(x)}=1+9x^2+27x^4+81x^6+O(x^8),$$

Define $z_1(x)$, $z_2(x)=z_1(-x)$ as the solutions of the equation

$$w^2-4z(x)^2w+4z(x)=0.$$

Recall $\overline{\eta}(x)=\prod_{n>0}(1-x^n)$, and define

$$\Psi_{S,c_1}(x)=9\left(\frac{1}{3\overline{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)}\left(\frac{3\overline{\eta}(x^6)^3}{\Theta_{A,1}(x)}\right)^{K_S^2}$$
$$\cdot\left(z_1(x)^{K_S^2}+z_2(x)^{K_S^2}+(-1)^{\chi(\mathcal{O}_S)}(\epsilon^{c_1K_S}+\epsilon^{-c_1K_S})\right).$$

Conjecture

$e^{\mathrm{vir}}(M_S^H(3,c_1,c_2))=\mathrm{Coeff}_{x^{\mathrm{vd}}}\left[\Psi_{S,c_1}(x)\right].$

S algebraic surface with $b_1 = 0$, $p_g > 0$
$M_S^H(c_1, c_2)$ = H-semi-stable rank 2 sheaves on S
E universal sheaf on $S \times M$. For $\alpha \in H_2(S)$, put

$$\mu(\beta) = p_{M*}(c_2(E) - c_1^2(E)/4)/\alpha \in H^2(M)$$

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Donaldson invariant:

$$D_{S,c_1}\left(\frac{\alpha^{\mathrm{vd}}}{\mathrm{vd}!}\right) = \int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}} \frac{\mu(\alpha)^{\mathrm{vd}}}{\mathrm{vd}!}$$

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Theorem (Witten conj., G.-Nakajima, Yoshioka)

$$D_{S,c_1}(\frac{\alpha^{\mathrm{vd}}}{\mathrm{vd}!}) = 2^{2+K_S^2-\chi(\mathcal{O}_S)}\mathrm{Coeff}_{z^{\mathrm{vd}}}\Big[\exp\Big(\frac{Q(\alpha)}{2}z^2\Big) \cdot \sum_{a_i \ SW cl.} SW(a_i)(-1)^{\langle c_1, a_i\rangle} \exp\big(\langle K_S - 2a_i, \alpha\rangle z\big)\Big]$$

Interpolate between VW and Don. invariants

Eisenstein series:

$$G_2(x) := \sum_{n>0}\Big(\sum_{d|n} d\Big)x^n, \quad DG_2(x) := \sum_{n>0}\Big(\sum_{d|n} nd\Big)x^n$$

Conjecture

$$\int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}} c_{\mathrm{vd}-n}(T^{\mathrm{vir}}_{M_S^H(c_1,n)})\frac{\mu(\alpha)^n}{n!} = \mathrm{Coeff}_{x^{\mathrm{vd}}z^n}\Bigg[8\left(\frac{1}{2\overline{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)}$$

$$\left(\frac{2\overline{\eta}(x^4)^2}{\theta_3(x)}\right)^{K_S^2} \exp\left(\frac{1}{2}DG_2(x^2)Q(\alpha)z^2 - 2G_2(x^2)\langle K_S,\alpha\rangle z\right)$$

$$\cdot \sum_{a_i\ SW cl.} SW(a_i)(-1)^{\langle c_1,a_i\rangle}\left(\frac{\theta_3(x)}{\theta_3(-x)}\right)^{\langle K_S,a_i\rangle} e^{\left(\frac{1}{2}(G_2(x)-G_2(-x))\langle K_S-2a_i,\alpha\rangle z\right)}\Bigg]$$

$z \to 0$: Vafa-Witten invariants, $x \to 0$, $xz \to 1$: Donaldson invariants

Elliptic genus: (Introduced by Witten, motivated by physics).
The elliptic genus is a refinement of the χ_{-y}-genus.
It associates to a smooth projective variety a Jacobi form (something like a modular form in two variables e.g. $\theta_3(x, y)$)

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For vector bundle E put

$$Ell_{q,y}(E) = y^{-\operatorname{rk}(E)/2} \bigotimes_{n\geq 1} \left(\Lambda_{-yq^{n-1}} E^\vee \otimes \Lambda_{-yq^n} E \otimes S_{q^n} E^\vee \otimes S_{q^n} E\right),$$

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$Ell(X) := \chi(X, Ell_{q,y}(T_X))$ elliptic genus.
$Ell^{\mathrm{vir}}(M) := \chi^{\mathrm{vir}}(M, Ell_{q,y}(T_M^{\mathrm{vir}}))$ virtual elliptic genus.
for $q = 0$ $Ell^{\mathrm{vir}}(M)$ specializes to $\chi^{\mathrm{vir}}_{-y}(M)$.

DMVV formula (conj. Dijkgraaf-Moore-Verlinde-Verlinde '97), (proof: Borisov-Libgober '00)
Put

$$L\Big(\sum_{m,l} c_{m,l} y^l q^m\Big) := \prod_{n>0}\prod_{m,l}(1 - x^n y^l q^m)^{c_{nm,l}}$$

Borcherds type lift, Jacobi form $\mapsto$ Siegel modular form

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Then

$$\sum_{n\geq 0} Ell(S^{[n]}) x^n = \frac{1}{L(Ell(S))} = \quad \left(\frac{1}{L(24\phi_2)} \text{ for } S = K3\right).$$

$$G_{1,0}(q,y) = -\frac{1}{2}\frac{y+1}{y-1} + \sum_{n>0}\sum_{d|n}(y^d - y^{-d})q^n, \; G_2(q) = -\frac{1}{24} + \sum_{n>0}\sum_{d|n} dq^n$$

$$G_{2,0}(q,y) = y\frac{\partial G_{1,0}(q,y)}{\partial y} - 2G_2(q) = \wp(q,y), \quad G_{3,0}(q,y) = y\frac{\partial \wp(q,y)}{\partial y}$$

$$\phi_i(q,y) := G_{i,0}(q,y)\left((y^{1/2} - y^{-1/2})\prod_{n>0}\frac{(1-q^n y)(1-q^n/y)}{(1-q^n)^2}\right)^i$$

$$L\Big(\sum_{m,l} c_{m,l} y^l q^m\Big) := \prod_{n>0}\prod_{m,l}(1 - x^n y^l q^m)^{c_{nm,l}}, \quad L_n(\phi) = L(\phi)|_{x=x^n}$$

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$$G_{2,0}(q,y)=y\frac{\partial G_{1,0}(q,y)}{\partial y}-2G_2(q)=\wp(q,y),\quad G_{3,0}(q,y)=y\frac{\partial\wp(q,y)}{\partial y}$$

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$$L\Big(\sum_{m,l}c_{m,l}y^lq^m\Big):=\prod_{n>0}\prod_{m,l}(1-x^ny^lq^m)^{c_{nm,l}},\quad L_n(\phi)=L(\phi)|_{x=x^n}$$

Conjecture

$$Ell^{\mathrm{vir}}(M_S^H(c_1,c_2))=\mathrm{Coeff}_{x^{vd}}\left[8\left(\frac{1}{2}\frac{1}{L_2(12\phi_2)}\right)^{\chi(\mathcal{O}_S)}\right.$$

$$\left.\cdot\left(\frac{2L_4(\phi_1\phi_3)L(-2\phi_1)}{L_2(-2\phi_1^{\mathrm{ev}}(q^{1/2},y)-\phi_1(q^2,y^2)+2\phi_1^2)}\right)^{K_S^2}\right].$$