Vafa-Witten formula and generalizations

Lothar Göttscbe,
joint work with Martijn Kool

Calabi-Yau and geometry

Rome, 29 May – 1 June 2019
Aim: study topological invariants of moduli spaces in algebraic geometry
We work over $\mathbb{C}$.

Moduli spaces:
A moduli space $M$ is an algebraic variety, which parametrizes in a natural way interesting objects in algebraic geometry.
Aim: study topological invariants of moduli spaces in algebraic geometry
We work over \( \mathbb{C} \).

Moduli spaces:
A moduli space \( M \) is an algebraic variety, which parametrizes in a natural way interesting objects in algebraic geometry. Examples are:

1. Hilbert schemes of points \( S^{[n]} \) on an algebraic surface:
   \{zero dimensional subschemes of degree \( n \) on \( S \}\} (i.e. generically sets of \( n \) points on \( S \)).
Aim: study topological invariants of moduli spaces in algebraic geometry

We work over $\mathbb{C}$.

**Moduli spaces:**

A moduli space $M$ is an algebraic variety, which parametrizes in a natural way interesting objects in algebraic geometry. Examples are:

1. Hilbert schemes of points $S^{[n]}$ on an algebraic surface: 
   \{zero dimensional subschemes of degree $n$ on $S$\} 
   (i.e. generically sets of $n$ points on $S$).

2. Moduli spaces of stable sheaves $M^H_S(r, c_1, c_2)$: 
   \{rank $r$ coherent sheaves on $S$ with Chern classes $c_1, c_2$\} 
   (i.e. vector bundles with singularities).
In differential geometry can also consider moduli spaces, e.g. of asd-connections on a principal $SO(3)$-bundle over a 4-manifold $X$.

Used to define and compute Donaldson invariants, which are $C^\infty$ invariants of 4-manifolds.

If $X$ is a projective algebraic surface close relationship to moduli spaces $M^H_S(2, c_1, c_2)$ of stable sheaves allows to compute Donaldson invariants via algebraic geometry.
Topological invariants of moduli spaces

$S$ projective complex surface, $H$ ample line bundle on $S$, i.e. $S \subset \mathbb{P}^n$ and $H$ is the hyperplane bundle (or consider $S$ with the Fubini-Study metric induced from $\mathbb{P}^n$). We assume always that

1. $b_1(S) = \dim H^1(S, \mathbb{Q}) = 0$
2. $p_g(S) = h^0(S, K_S) > 0$, i.e. $\exists$ nonvanishing holomorphic 2-forms on $S$
No moduli space for all coherent sheaves on $S$ exists.
Consider a nice subset: stable sheaves: not too big subsheaves.
No moduli space for all coherent sheaves on $S$ exists
Consider a nice subset: stable sheaves: not too big subsheaves

$M^H_S(r, c_1, c_2) =$ moduli space of rank $r$ $H$-semistable sheaves on $S$ with Chern classes $c_1, c_2$

$\mathcal{E}$ semistable $\iff \forall n \gg 0 \frac{h^0(S, \mathcal{F} \otimes H^\otimes n)}{\text{rk}(\mathcal{F})} \leq \frac{h^0(S, \mathcal{E} \otimes H^\otimes n)}{\text{rk}(\mathcal{E})}$ for all $\mathcal{F}$ subsheaf of $\mathcal{E}$. 
No moduli space for all coherent sheaves on $S$ exists.
Consider a nice subset: stable sheaves: not too big subsheaves

$$M^H_S(r, c_1, c_2) = \text{moduli space of rank } r \text{ } H\text{-semistable sheaves on } S \text{ with Chern classes } c_1, c_2$$

$\mathcal{E}$ semistable $\iff \forall n \gg 0 \quad \frac{h^0(S, \mathcal{F} \otimes H^\otimes n)}{\text{rk}(\mathcal{F})} \leq \frac{h^0(S, \mathcal{E} \otimes H^\otimes n)}{\text{rk}(\mathcal{E})}$ for all $\mathcal{F}$ subsheaf of $\mathcal{E}$.

$M = M^H_S(r, c_1, c_2)$ is usually singular, has expected dimension

$$vd = 2rc_2 - (r - 1)c_1^2 + (r^2 - 1)\chi(\mathcal{O}_S).$$

$vd$ is the dimension $M$ should have, more about that later
No moduli space for all coherent sheaves on $S$ exists

Consider a nice subset: stable sheaves: not too big subsheaves

$M^H_S(r, c_1, c_2) =$ moduli space of rank $r$ $H$-semistable sheaves on $S$ with Chern classes $c_1, c_2$

$\mathcal{E}$ semistable $\iff \forall n \gg 0 \quad \frac{h^0(S, \mathcal{F} \otimes H^\otimes n)}{\text{rk}(\mathcal{F})} \leq \frac{h^0(S, \mathcal{E} \otimes H^\otimes n)}{\text{rk}(\mathcal{E})}$ for all $\mathcal{F}$ subsheaf of $\mathcal{E}$.

$M = M^H_S(r, c_1, c_2)$ is usually singular, has expected dimension

$$vd = 2rc_2 - (r - 1)c_1^2 + (r^2 - 1)\chi(\mathcal{O}_S).$$

$vd$ is the dimension $M$ should have, more about that later

Here write $c_2 := \int_{[S]} c_2 \in \mathbb{Z}$, $c_1^2 := \int_{[S]} c_1^2 \in \mathbb{Z}$
**Rank 1 case: Hilbert scheme of points**

\[ S[n] = \{ \text{zero dimensional subschemes of length } n \text{ on } S \} \]

General pt \( Z \) of \( S[n] \): \( Z = p_1 \sqcup \ldots \sqcup p_n \) set of \( n \) distinct pts of \( S \)

When points come together have nontrivial scheme structure, \( Z = Z_1 \sqcup \ldots \sqcup Z_k \) such that \( \dim_{\mathbb{C}} O_Z = \sum_{i=1}^{k} \dim_{\mathbb{C}} O_{Z_i} = n \).
Rank 1 case: Hilbert scheme of points

\[ S[n] = \{ \text{zero dimensional subschemes of length } n \text{ on } S \} \]

General pt \( Z \) of \( S[n] \): \( Z = p_1 \sqcup \ldots \sqcup p_n \) set of \( n \) distinct pts of \( S \)

When points come together have nontrivial scheme structure, \( Z = Z_1 \sqcup \ldots \sqcup Z_k \) such that \( \text{dim}_\mathbb{C} \mathcal{O}_Z = \sum_{i=1}^{k} \text{dim}_\mathbb{C} \mathcal{O}_{Z_i} = n \).

\( S[n] \) is smooth projective of dimension \( 2n \).

\( S[n] \rightarrow S(n) = S^n / (\text{perm. of factors}) \), \( Z \mapsto \text{supp}(Z) \) is resolution of singularities.
Rank 1 case: Hilbert scheme of points

\[ S^{[n]} = \{ \text{zero dimensional subschemes of length } n \text{ on } S \} \]

General pt \( Z \) of \( S^{[n]} \): \( Z = p_1 \sqcup \ldots \sqcup p_n \) set of \( n \) distinct pts of \( S \)

When points come together have nontrivial scheme structure, \( Z = Z_1 \sqcup \ldots \sqcup Z_k \) such that \( \dim \mathcal{O}_Z = \sum_{i=1}^{k} \dim \mathcal{O}_{Z_i} = n \).

\( S^{[n]} \) is smooth projective of dimension \( 2n \).

\( S^{[n]} \to S^{(n)} = S^n/(\text{perm. of factors}), Z \mapsto \text{supp}(Z) \)

is resolution of singularities.

\( M^H_S(1, L, c_2) = S^{[c_2]} \), via \( Z \leftrightarrow I_Z \otimes \mathcal{O}(L) \). \( I_Z \) ideal sheaf of \( Z \).
Euler numbers of Hilbert schemes:

$$M^H_S(1, L, c_2) = S^{[c_2]}$$

Let $$e(M)$$ be the topological Euler number of $$M$$
Euler numbers of Hilbert schemes: 

\[ M^H_S(1, L, c_2) = S^{[c_2]} \]

Let \( e(M) \) be the topological Euler number of \( M \)

**Theorem (G’90)**

\[
\sum_{n \geq 0} e(S^{[n]}) x^n = \frac{1}{\prod_{n>0} (1 - x^n)^{e(S)}}
\]
Euler numbers of Hilbert schemes:
\[ M^H_S(1, L, c_2) = S^{[c_2]} \]
Let \( e(M) \) be the topological Euler number of \( M \)

**Theorem (G’90)**

\[
\sum_{n \geq 0} e(S^{[n]}) x^n = \frac{1}{\prod_{n > 0} (1 - x^n)^{e(S)}}
\]

By physics arguments, 1994 Vafa and Witten gave explicit conjectural formula for the generating function for \( e(M^H_S(2, L, n)) \), in terms of modular forms.
In whole talk assume stable=semistable (condition on $c_1$). Assume for simplicity in whole talk:

∃ smooth conn. curve in $|K_S|$ (zero set of holomorphic 2-form.)
In whole talk assume stable=semistable (condition on $c_1$).
Assume for simplicity in whole talk:
$\exists$ smooth conn. curve in $|K_S|$ (zero set of holomorphic 2-form.)
Write $K_S^2 = \int [S] K_S^2 = \int [S] c_1(S)^2$,
let $\chi(\mathcal{O}_S)$ holomorphic Euler characteristic
Write in future $M^H_S(c_1, c_2) = M^H_S(2, c_1, c_2)$, and always

$$vd = vd_{M^H_S(c_1, c_2)} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$$
In whole talk assume stable = semistable (condition on $c_1$).
Assume for simplicity in whole talk:
$\exists$ smooth conn. curve in $|K_S|$ (zero set of holomorphic 2-form.)
Write $K_S^2 = \int_S K_S^2 = \int_S c_1(S)^2$,
let $\chi(O_S)$ holomorphic Euler characteristic
Write in future $M^H_S(c_1, c_2) = M^H_S(2, c_1, c_2)$, and always
\[
vd = vd_{M^H_S(c_1,c_2)} = 4c_2 - c_1^2 - 3\chi(O_S)
\]

**Conjecture (Vafa-Witten conjecture)**

\[
e(M^H_S(c_1, c_2)) = \text{Coeff}_{x^{vd}} \left[ 8 \left( \frac{1}{2 \prod_{n>0} (1 - x^{2n})^{12}} \right)^{\chi(O_S)} \sum_{n\in\mathbb{Z}} \frac{2^{\prod_{n>0} (1 - x^{4n})^2}}{x^{n^2}} \right]
\]

Want to interpret, check and refine this formula
Virtual Euler number

\[ M = M^H_S(c_1, c_2) \] usually very singular
might have dimension different from \( \text{vd} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S) \)
Virtual Euler number and its refinements

Examples

Check of conjectures

Further results

Virtual Euler number

\[ M = M^H_S(c_1, c_2) \] usually very singular
might have dimension different from \( vd = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S) \)

But \( M \) has a virtual smooth structure of dimension \( vd \)
with this behaves like smooth projective variety of dim. \( vd \)
Can define virtual analogues of all invariants of smooth
projective varieties
Virtual Euler number

\[ M = M_S^H(c_1, c_2) \] usually very singular
might have dimension different from \( \text{vd} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S) \)

But \( M \) has a virtual smooth structure of dimension \( \text{vd} \)
with this behaves like smooth projective variety of dim. \( \text{vd} \)
Can define virtual analogues of all invariants of smooth projective varieties

**Idea:** virtual Euler number \( e^{\text{vir}}(M) \) and all other virtual invariants of \( M \) are invariant under deformation
If one can deform to a smooth moduli space \( M_s \), then e.g. \( e^{\text{vir}}(M) = e(M_s) \).
Virtual Euler number

\[ M = M^H_S(c_1, c_2) \] usually very singular
might have dimension different from \( \text{vd} = 4c_2 - c_1^2 - 3\chi(O_S) \)

But \( M \) has a virtual smooth structure of dimension \( \text{vd} \)
with this behaves like smooth projective variety of dim. \( \text{vd} \)
Can define virtual analogues of all invariants of smooth
projective varieties

**Idea:** virtual Euler number \( e^{\text{vir}}(M) \) and all other virtual
invariants of \( M \) are invariant under deformation
If one can deform to a smooth moduli space \( M_s \), then e.g.
\( e^{\text{vir}}(M) = e(M_s) \).

Virtual structure is used to define most invariants in modern
enumerative geometry, e.g. Gromov-Witten, Donaldson
invariants, Donaldson Thomas invariants
In differential geometry, when the moduli space (of solutions to some pde) is singular, one deforms the equation to get a smooth moduli space (e.g. for Donaldson invariants). In algebraic geometry, one keeps the moduli space as is, but adds virtual structure, which keeps records why the moduli space is virtually smooth. This allows for better control.
Perfect obstruction theory

At every point $[F] \in M$, tangent space $T_{[F]} = \operatorname{Ext}^1(F, F)_0$

obstruction space $O_{[F]} = \operatorname{Ext}^2(F, F)_0$

**Kuranishi:** $\exists$ analytic map $\kappa : T_{[F], 0} \rightarrow O_{[F], 0}$,

s.th.anal. nbhd of $[F]$ in $M$ is isom. to $\kappa^{-1}(0)$

$\implies$ if $O_F = 0$ or $\kappa$ submersion, $M$ is nonsingular of dim $vd$
At every point \([F] \in M\), tangent space \(T_{[F]} = \text{Ext}^1(F, F)_0\), obstruction space \(O_{[F]} = \text{Ext}^2(F, F)_0\).

Kuranishi: \(\exists\) analytic map \(\kappa : T_{[F], 0} \to O_{[F], 0}\), s.th. anal. nbhd of \([F]\) in \(M\) is isom. to \(\kappa^{-1}(0)\), \(\implies\) if \(O_F = 0\) or \(\kappa\) submersion, \(M\) is nonsingular of dim \(vd\).

**Perfect obstruction theory:**
Complex \(E_\bullet = [E_0 \to E_1]\) of vb on \(M\), s.th. \(\forall [F] \in M:\)
\(T_{[F]} \simeq \ker(E_0([F]) \to E_1([F])), O_F \simeq \text{coker}(E_0([F]) \to E_1([F]))\)
i.e \(E_\bullet\) captures all tangents and obstructions via vector bundles.
At every point \([F] \in M\), tangent space \(T_{[F]} = \text{Ext}^1(F, F)_0\) 
obstruction space \(O_{[F]} = \text{Ext}^2(F, F)_0\) 

Kuranishi: \(\exists\) analytic map \(\kappa : T_{[F], 0} \to O_{[F], 0}\), 
s.th.anal. nbhd of \([F]\) in \(M\) is isom. to \(\kappa^{-1}(0)\) 
\(\implies\) if \(O_F = 0\) or \(\kappa\) submersion, \(M\) is nonsingular of dim \(vd\)

**Perfect obstruction theory:**
Complex \(E_\bullet = [E_0 \to E_1]\) of vb on \(M\), s.th. \(\forall [F] \in M:\) 
\(T_{[F]} \cong \ker(E_0([F]) \to E_1([F])), O_F \cong \text{coker}(E_0([F]) \to E_1([F]))\) 
i.e \(E_\bullet\) captures all tangents and obstructions via vector bundles

Then define: \(T^\text{vir}_M := [E_0] - [E_1] \in K^0(M),\) 
\(vd := \text{rk} T^\text{vir}_M = \text{rk}(E_0) - \text{rk}(E_1)\) 
virtual fundamental class \([M]^\text{vir} \in H_{2vd}(M, \mathbb{Z})\) 
virtual structure sheaf \(O^\text{vir}_M \in K_0(M)\) (these last two are difficult)
Virtual Euler number

**Definition**

Virtual Euler number:

\[ e^{\text{vir}}(M) := \int_{[M]^{\text{vir}}} c_{\text{vd}}(T^{\text{vir}}(M)) \]
Virtual Euler number:

\[ e_{\text{vir}}(M) := \int_{[M]_{\text{vir}}} c_{\text{vd}}(T^\text{vir}(M)) \]

Conjecture

The Vafa-Witten formula holds with \( e(M^H_S(c_1, c_2)) \) replaced by \( e_{\text{vir}}(M^H_S(c_1, c_2)) \).
holomorphic Euler characteristic:

$$\chi(X, V) := \sum_{i \geq 0} (-1)^i \dim H^i(X, V), \quad V \in K^0(X)$$

$\chi_y$-genus:

$$\chi_y(X) = \sum_{p, q} (-1)^{p+q} y^p h^{p,q}(X) = \sum_p (-y)^p \chi(X, \Omega_X^p)$$

alternating sum of Hodge numbers
holomorphic Euler characteristic:

\[ \chi(X, V) := \sum_{i \geq 0} (-1)^i \dim H^i(X, V), \quad V \in K^0(X) \]

\(\chi_{-y}\)-genus:

\[ \chi_{-y}(X) = \sum_{p, q} (-1)^{p+q} y^p h^{p,q}(X) = \sum_p (-y)^p \chi(X, \Omega^p_X) \]

alternating sum of Hodge numbers

**Virtual** \(\chi_{-y}\)-genus. For \(V \in K^0(M)\), put

\[ \chi^{\text{vir}}(M, V) := \chi(M, \mathcal{O}^{\text{vir}}_M \otimes V). \]  
Let \(\Omega^{\text{vir}}_M := (T^{\text{vir}}_M)^\vee\).

\[ \chi^{\text{vir}}_{-y}(M) := y^{-vd/2} \sum_p (-y)^p \chi^{\text{vir}}(M, \wedge^p \Omega^{\text{vir}}_M) \]

\[ \chi^{\text{vir}}_{-1}(M) = e^{\text{vir}}(M), \]  so this is refinement of virtual Euler number
Refinement to $\chi_y$-genus

$$\psi_S(x) := 8 \left( \frac{1}{2 \prod_{n>0} (1 - x^{2n})^{12}} \right) \chi(O_S) \left( \frac{2 \prod_{n>0} (1 - x^{4n})^2}{\sum_{n \in \mathbb{Z}} x^{n^2}} \right)^{K_S^2}$$
Refinement to $\chi_y$-genus

$$\psi_S(x) := 8 \left( \frac{1}{2 \prod_{n > 0}(1 - x^{2n})^{12}} \right) \chi(O_S) \left( \frac{2 \prod_{n > 0}(1 - x^{4n})^2}{\sum_{n \in \mathbb{Z}} x^{n^2}} \right)^{K_S^2}$$

Conjecture

$$e^\text{vir}(M^H_S(c_1, c_2)) = \text{Coeff}_{x^\text{vd}}[\psi_S(x)].$$
Refinement to $\chi_y$-genus

$$\psi_S(x) := 8 \left( \frac{1}{2 \prod_{n>0} (1 - x^{2n})^{12}} \right)^{\chi(O_S)} \left( \frac{2 \prod_{n>0} (1 - x^{4n})^2}{\sum_{n \in \mathbb{Z}} x^{n^2}} \right)^{K_S^2}$$

Conjecture

$$e^{\text{vir}}(M^H_S(c_1, c_2)) = \text{Coeff}_{x^{vd}}[\psi_S(x)].$$

Conjecture for virtual $\chi_y$-genus:

$$\psi_S(x, y) := 8 \left( \frac{1}{2 \prod_{n>0} (1 - x^{2n})^{10}(1 - x^{2n}y)(1 - x^{2n}/y)} \right)^{\chi(O_S)} \left( \frac{2 \prod_{n>0} (1 - x^{4n})^2}{\sum_{n \in \mathbb{Z}} x^{n^2} y^{n/2}} \right)^{K_S^2}$$

Conjecture

$$\chi^{\text{vir}}(M^H_S(c_1, c_2)) = \text{Coeff}_{x^{vd}}[\psi_S(x, y)].$$

Specializes to our version of VW conjecture for $y = 1$
Have conjectural generating function for virtual Elliptic genus of $M^H_S(c_1, c_2)$ in terms of Siegel modular forms. It gives generalization of the DMVV formula (Dijkgraaf-Moore-Verlinde-Verlinde ’97), (Borisov-Libgober ’00) for Hilbert schemes of points. A bit too complicated to state here.
**Final generalization: the cobordism class:**
Two complex manifolds $M$, $N$ have the same cobordism class
\(\{M\} = \{N\}\)
if they have the same Chern numbers:

\[
\int_{[M]} c_{i_1}(M) \cdots c_{i_k}(M) = \int_{[N]} c_{i_1}(N) \cdots c_{i_k}(N) \quad \forall k, i_1, \ldots, i_k
\]
Final generalization: the cobordism class:
Two complex manifolds $M$, $N$ have the same cobordism class
$\{M\} = \{N\}$
if they have the same Chern numbers:

$$\int_{[M]} c_{i_1}(M) \cdots c_{i_k}(M) = \int_{[N]} c_{i_1}(N) \cdots c_{i_k}(N) \quad \forall k, i_1, \ldots, i_k$$

Cobordism classes of complex manifolds generate a ring
$R = \sum_n R_n$ (graded by dimension)
$\{M\}\{N\} = \{M \times N\}$, $\{M\} + \{N\} = \{M \sqcup N\}$
In fact

$$R \otimes \mathbb{Q} = \mathbb{Q}[[\mathbb{P}^1], \{\mathbb{P}^2\}, \{\mathbb{P}^3\}, \ldots]$$
Ellingsrud-G-Lehn showed \( \{ S[n] \} \) depends only on \( \{ S \} \) (equivalent: Chern numbers of \( S[n] \) depend only on \( K^2_S, c_2(S) \))
Ellingsrud-G-Lehn showed \( \{ S[n] \} \) depends only on \( \{ S \} \)
(equivalent: Chern numbers of \( S[n] \) depend only on \( K^2_S, c_2(S) \))

For \( M = M^H_S(c_1, c_2) \) let \( \{ M \}^{\text{vir}} \) be the virtual cobordism class
given by the

\[
\int [M]^{\text{vir}} c_{i_1}(T_M^{\text{vir}}) \cdots c_{i_k}(T_M^{\text{vir}}).
\]
Ellingsrud-G-Lehn showed \( \{S[n]\} \) depends only on \( \{S\} \) (equivalent: Chern numbers of \( S[n] \) depend only on \( K_S^2, c_2(S) \))

For \( M = M^H_S(c_1, c_2) \) let \( \{M\}^{\text{vir}} \) be the virtual cobordism class given by the

\[
\int_{[M]^{\text{vir}}} c_{i_1}(T_{M}^{\text{vir}}) \cdots c_{i_k}(T_{M}^{\text{vir}}).
\]

**Conjecture**

There is a power series

\[
P(x) = 1 + \sum_{n>0} P_n x^n, \text{ with } P_n \in R_n, \text{ s.th.}
\]

\[
\{M^H_S(c_1, c_2)\}^{\text{vir}} = \text{Coeff}_{x^{\text{vd}}} \left[ 8 \left( \frac{1}{4} \sum_{n \geq 0} \{K^3[n]\} x^{2n} \right)^{\chi(O_S)/2} \left(2P(x)\right)^{K_S^2} \right].
\]
Seiberg-Witten invariants:
invariants of differentiable 4-manifolds
$S$ projective algebraic surface $H^2(S, \mathbb{Z}) \ni a \mapsto SW(a) \in \mathbb{Z}$, $a$ is called SW class if $SW(a) \neq 0$. 
Seiberg-Witten invariants: invariants of differentiable 4-manifolds
$S$ projective algebraic surface $H^2(S, \mathbb{Z}) \ni a \mapsto SW(a) \in \mathbb{Z}$, $a$ is called SW class if $SW(a) \neq 0$.

In general for alg. surfaces they are easy to compute, e.g. if $b_1(S) = 0$, $p_g(S) > 0$ and $|K_S|$ contains smooth connected curve, then SW cl. of $S$ are 0, $K_S$ with

$$SW(0) = 1, \quad SW(K_S) = (-1)^{\chi(O_S)}$$
Seiberg-Witten invariants:

invariants of differentiable 4-manifolds

$S$ projective algebraic surface $H^2(S, \mathbb{Z}) \ni a \mapsto SW(a) \in \mathbb{Z}$, $a$ is called SW class if $SW(a) \neq 0$.

In general for alg. surfaces they are easy to compute, e.g. if $b_1(S) = 0$, $p_g(S) > 0$ and $|K_S|$ contains smooth connected curve, then SW cl. of $S$ are 0, $K_S$ with

$$SW(0) = 1, \quad SW(K_S) = (-1)^{\chi(O_S)}$$

This is the reason for our assumption that $|K_S|$ contains smooth connected curve, otherwise our results look more complicated.
We restrict attention to the virtual Euler number $S$ projective surface with $b_1(S) = 0$, $p_g(S) > 0$.

**Conjecture**

$$e^{\text{vir}}(M^H_S(c_1, c_2)) = \text{Coeff}_{x^v_d} \left[ 4 \left( \frac{1}{2 \prod_{n>0} (1 - x^{2n})^{12}} \right)^{\chi(O_S)} \left( \frac{2 \prod_{n>0} (1 - x^{4n})^2}{\sum_{n \in \mathbb{Z}} x^{n^2}} \right)^{K^2_S} \sum_{a \in H^2(S, \mathbb{Z})} SW(a)(-1)^{c_1 a} \left( \frac{\sum_{n \in \mathbb{Z}} x^{n^2}}{\sum_{n \in \mathbb{Z}} (-1)^n x^{n^2}} \right)^{aK_S} \right]$$
We restrict attention to the virtual Euler number $S$ projective surface with $b_1(S) = 0$, $p_g(S) > 0$.

**Conjecture**

$$e^{\text{vir}}(M^H_S(c_1, c_2)) = \text{Coeff}_{x^{vd}} \left[ 4 \left( \frac{1}{2 \prod_{n>0}(1 - x^{2n})^{12}} \right)^{\chi(O_S)} \left( \frac{2 \prod_{n>0}(1 - x^{4n})^2}{\sum_{n \in \mathbb{Z}} x^{n^2}} \right)^{K_S^2} \sum_{a \in H^2(S, \mathbb{Z})} \text{SW}(a)(-1)^{c_1 a} \left( \frac{\sum_{n \in \mathbb{Z}} x^{n^2}}{\sum_{n \in \mathbb{Z}} (-1)^n x^{n^2}} \right)^{aK_S} \right],$$

**Examples:**

1. **K3 surfaces:** Let $S$ be a K3 surface, $M = M^H_S(c_1, c_2)$ is nonsingular of dim $vd$ and $e(M) = e(S^{[vd/2]})$ (Yoshioka)

   $$\implies e(M) = \text{Coeff}_{x^{vd}} \left[ \frac{1}{\prod_{n>0}(1 - x^{2n})^{24}} \right],$$

   Follows from our formula because $K_S^2 = 0$, and $\text{SW}(0) = 1$ is only SW invariant.
(2) Elliptic surfaces: (Yoshioka) $S$ elliptic surface $\chi(O_S) = d$, $F$ fibre $M = M_S^H(c_1, c_2)$ is nonsingular of dim $vd$

$$e(M) = \begin{cases} \text{Coeff}_{vd} \left[ \frac{1}{\prod_{n>0} (1-x^{2n})^{12d}} \right] & c_1 F \equiv 1 \mod 2, \\ 0 & c_1 F \equiv 0 \mod 2 \end{cases}$$

Follows from our formula because $K_S^2 = 0$ and SW invariants are $SW(kF) = (-1)^k \binom{d-2}{k}$, $k = 0, \ldots, d - 2$
**Examples**

(2) **Elliptic surfaces:** (Yoshioka) $S$ elliptic surface $\chi(O_S) = d$, $F$ fibre $M = M^H_S(c_1, c_2)$ is nonsingular of dim $vd$

$$e(M) = \begin{cases} 
\text{Coeff}_{x^{vd}} \left[ \frac{1}{\prod_{n>0}(1-x^{2n})^{12d}} \right] & c_1 F \equiv 1 \mod 2, \\
0 & c_1 F \equiv 0 \mod 2
\end{cases}$$

Follows from our formula because $K^2_S = 0$ and SW invariants are $SW(kF) = (-1)^k \binom{d-2}{k}$, $k = 0, \ldots, d - 2$

(3) **Blowup formula:** (Li-Qin) Let $\hat{S}$ the blowup of surface $S$. $c_1 \in H^2(S)$, $E$ exceptional divisor. Then

$$\sum_{c_2} e(M^H_S(c_1 + aE, c_2)) x^{vd} = \sum_{n \in \mathbb{Z}} \frac{x^{(2n+a)^2}}{\prod_{n>0}(1-x^{4n})^2} \sum_{c_2} e(M^H_S(c_1, c_2)) x^{vd}$$

We predict the same formula with $e$ replaced by $e^{\text{vir}}$ on both sides, because $K^2_{\hat{S}} = K^2_S - 1$ and SW invariants are $SW_{\hat{S}}(a) = SW_{\hat{S}}(a + E) = SW_S(a)$ for all SW classes $a$ on $S$
(4) Quintic in $\mathbb{P}^3$: Let $S$ be a nonsingular quintic in $\mathbb{P}^3$, $H$ the hyperplane section. We show

$$\sum_{c_2} e^{\text{vir}}(M^H_S(H, c_2))x^{vd} = 8 + 52720x^4 + 48754480x^8$$

$$+ 17856390560x^{12} + 3626761297400x^{16} \ldots + O(x^{28})$$

confirming the conjecture
Main tool: Mochizuki’s formula:
Compute intersection numbers on $M = M^H_S(c_1, c_2)$ in terms of intersection numbers on Hilbert scheme of points.
Main tool: Mochizuki’s formula:

Compute intersection numbers on $M = M^H_S(c_1, c_2)$ in terms of intersection numbers on Hilbert scheme of points.

On $S \times M$ have $\mathcal{E}$ universal sheaf

i.e. if $[E] \in M$ corresponds to a sheaf $E$ on $S$ then $\mathcal{E}|_{S \times [E]} = E$.

For $\alpha \in H^k(S)$, put

$$\tau_i(\alpha) := \pi_{M*}(c_i(\mathcal{E})\pi^*_S(\alpha)) \in H^{2i-4+k}(M)$$
Main tool: Mochizuki’s formula:
Compute intersection numbers on \( M = M_S^H(c_1, c_2) \) in terms of intersection numbers on Hilbert scheme of points.

On \( S \times M \) have \( \mathcal{E} \) universal sheaf
i.e. if \([E] \in M\) corresponds to a sheaf \( E \) on \( S \) then \( \mathcal{E}|_{S \times [E]} = E \).

For \( \alpha \in H^k(S) \), put
\[
\tau_i(\alpha) := \pi_{M*}(c_i(\mathcal{E})\pi_S^*(\alpha)) \in H^{2i-4+k}(M)
\]

Let \( P(\mathcal{E}) \) be any polynomial in the \( \tau_i(\alpha) \)
Mochizuki’s formula expresses \( \int_{[M]_{\text{vir}}} P(\mathcal{E}) \) in terms of intersec.
numbers on \( S^{[n_1]} \times S^{[n_2]} \), and Seiberg-Witten invariants.
The Mochizuki formula states that

\[ e^{\text{vir}}(M), \chi^{\text{vir}}_y(M), E^{\text{vir}}(M) \text{ and } \{M\}^{\text{vir}} \] can all be expressed as

\[ \int_{[M]^{\text{vir}}} P(\mathcal{E}), \] for suitable polyn. \( P \), so can reduce computation to

Hilbert schemes.
$e^{\text{vir}}(M)$, $\chi^{\text{vir}}_-(M)$, $\text{Ell}^{\text{vir}}(M)$ and $\{M\}^{\text{vir}}$ can all be expressed as $\int_{[M]^{\text{vir}}} P(\mathcal{E})$, for suitable polyn. $P$, so can reduce computation to Hilbert schemes.

For $\chi^{\text{vir}}_-(M)$, $\text{Ell}^{\text{vir}}(M)$ use virtual Riemann-Roch formula

**Theorem (Fantechi-G.)**

For $V \in K^0(M)$ have

$$\chi^{\text{vir}}(M, V) = \int_{[M]^{\text{vir}}} \text{ch}(V) \text{td}(T^{\text{vir}}_M).$$
\[ S^{[n_1]} \times S^{[n_2]} = \{ \text{pairs } (Z_1, Z_2) \text{ of subsch. of deg. } (n_1, n_2) \text{ on } S \} \]

Work on \( S \times S^{[n_1]} \times S^{[n_2]} \), projection \( p \) to \( S^{[n_1]} \times S^{[n_2]} \)
Mochizuki formula

\[ S^{[n_1]} \times S^{[n_2]} = \{ \text{pairs } (Z_1, Z_2) \text{ of subsch. of deg. } (n_1, n_2) \text{ on } S \} \]

Work on \( S \times S^{[n_1]} \times S^{[n_2]} \), projection \( p \) to \( S^{[n_1]} \times S^{[n_2]} \)

Two universal sheaves: Let \( a_1, a_2 \in \text{Pic}(S) \)

1. \( \mathcal{I}_i(a) \) sheaf on \( S \times S^{[n_1]} \times S^{[n_2]} \) with \( \mathcal{I}_i(a_i)|_{S \times (Z_1, Z_2)} = l_{Z_i} \otimes a_9 \)

2. \( \mathcal{O}_i(a_i) \), vector bundle of rank \( n_i \) on \( S^{[n_1]} \times S^{[n_2]} \), with fibre \( \mathcal{O}_i(a_i)(Z_1, Z_2) = H^0(\mathcal{O}_{Z_i} \otimes a_i) \)
\[ \mathcal{I}_i(a) \text{ sheaf on } S \times S^{[n_1]} \times S^{[n_2]} \text{ with } \mathcal{I}_i(a_i)|_{S \times (Z_1, Z_2)} = l_{Z_i} \otimes a_9 \]

\[ \mathcal{O}_i(a_i), \text{ vector bundle of rank } n_i \text{ on } S^{[n_1]} \times S^{[n_2]}, \text{ with fibre} \]

\[ \mathcal{O}_i(a_i)(Z_1, Z_2) = H^0(\mathcal{O}_{Z_i} \otimes a_i) \]

Remember, we want to compute \( \int_{[M]_{\text{vir}}} \Psi(P) \) 

There is a (Laurent) polynomial \( \Psi_P(a_1, a_2, n_1, n_2, s) \) associated to \( P \) in a variable \( s \), the

\[ \tau_i(\alpha) := \rho_* (c_i(I_1(a_1) \oplus I_2(a_2)) \pi^*_S(\alpha)) \in H^{2i-4+k}(\mathcal{O}_Z, \mathcal{O}_S) \]

and the Chern classes of \( \mathcal{O}_1(a_1), \mathcal{O}_2(a_2) \), s.th following holds: Put

\[ A_P(a_1, a_2, c_2, s) = \sum_{n_1 + n_2 = c_2 - a_1 a_2} \int_{S^{[n_1]} \times S^{[n_2]}} \Psi_P(a_1, a_2, n_1, n_2, s) \in \mathbb{Q}[s, s^{-1}] \]
Mochizuki formula

\[ A_P(a_1, a_2, c_2, s) = \sum_{n_1 + n_2 = c_2 - a_1 a_2} \int_{S^{[n_1]} \times S^{[n_2]}} \psi_P(a_1, a_2, n_1, n_2, s) \]

**Theorem (Mochizuki)**

Assume \( \chi(E) > 0 \) for \( E \in M^S_{H}(c_1, c_2) \). Then

\[ \int_{[M^S_{H}(c_1, c_2)]^{\text{vir}}} P(\mathcal{E}) = \sum_{c_1 = a_1 + a_2} \text{SW}(a_1) \text{Coeff}_{s_0} A_P(a_1, a_2, c_2, s) \]

i.e. we replace a simple formula on a space where we cannot compute anything by a terrible formula on simpler space
Take now for $P(\mathcal{E}) = c_{vd}(T^\text{vir}_M)$ (works the same for the others)

Put

$$Z_S(a_1, a_2, s, q) = \sum_{n_1, n_2 \geq 0} \int_{S^{[n_1]} \times S^{[n_2]}} A(a_1, a_2, a_1 a_2 + n_1 + n_2, s) q^{n_1+n_2}$$
Take now for $P(\mathcal{E}) = c_{\text{vd}}(T^\text{vir}_M)$ (works the same for the others)

Put

$$Z_S(a_1, a_2, s, q) = \sum_{n_1, n_2 \geq 0} \int_{S^{[n_1]} \times S^{[n_2]}} A(a_1, a_2, a_1 a_2 + n_1 + n_2, s)q^{n_1+n_2}$$

**Proposition**

There exist univ. functions

$$A_1(s, q), \ldots, A_7(s, q) \in \mathbb{Q}[s, s^{-1}][[q]]$$

s.th. $\forall S, a_1, a_2$

$$Z_S(a_1, a_2, s, q) = F_0(a_1, a_2, s)A_1^{a_1^2} A_2^{a_1 a_2} A_3^{a_2^2} A_4^{a_1 K_S} A_5^{a_2 K_S} A_6^{K_S^2} A_7^\chi(\mathcal{O}_S),$$

(Where $F_0(a_1, a_2, s)$ is some explicit elementary function).
Proof: Modification of an argument of Elllingsrud-G-Lehn: "Intersection numbers of universal sheaves on $S^n$ are universal polynomials in intersection numbers on $S$".
Proof: Modification of an argument of Ellingsrud-G-Lehn: "Intersection numbers of universal sheaves on $S^{[n]}$ are universal polynomials in intersection numbers on $S$".

Reason: Intersection numbers on $S^{[n]}$ computed inductively: $Z_n(S) := \{(x, Z) \in S \times S^{[n]} | x \in Z\}$ universal subscheme
Blowup of $S \times S^{[n]}$ along $Z_n(S)$ is

$$S^{[n,n+1]} := \{(Z, W) \in S^{[n]} \times S^{[n+1]} | Z \in W\}$$

This allows to compute intersection numbers of $S^{[n+1]}$ in terms of inters. numbers on $S$ and $S^{[n]}$, and conclude by induction.
**Proof:** Modification of an argument of Elllingsrud-G-Lehn: "Intersection numbers of universal sheaves on $S^{[n]}$ are universal polynomials in intersection numbers on $S$".

**Reason:** Intersection numbers on $S^{[n]}$ computed inductively:

$Z_n(S) := \{(x, Z) \in S \times S^{[n]} | x \in Z\}$ universal subscheme

Blowup of $S \times S^{[n]}$ along $Z_n(S)$ is

$$S^{[n,n+1]} := \{(Z, W) \in S^{[n]} \times S^{[n+1]} | Z \in W\}$$

This allows to compute intersection numbers of $S^{[n+1]}$ in terms of inters. numbers on $S$ and $S^{[n]}$, and conclude by induction.

This gives:

$$\text{Coeff}_{q^{k_s}} Z_S(a_1, a_2, s, q) = P_{k,l}(a_1^2, a_1 a_2, a_2^2, a_1 K_S, a_1 K_S, K_S^2, \chi(O_S))$$

for some polynomial $P_{k,l}$ depending only on $k, l$.

For the multiplicativity use additional tricks.
Reduction to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

$A_1(s, q), \ldots A_7(s, q)$ are determined by value of $Z_S(a_1, a_2, s, q)$ for 7 triples $(S, a_1, a_2)$ ($S$ surface, $a_1, a_2 \in \text{Pic}(S)$) s.th. corresponding 7-tuples $(a_1^2, a_1 a_2, a_2^2, a_1 K_S, a_1 K_S, K_S^2, \chi(O_S))$ are linearly independent.
Reduction to \( \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \).

\( A_1(s, q), \ldots A_7(s, q) \) are determined by value of \( Z_S(a_1, a_2, s, q) \) for 7 triples \((S, a_1, a_2)\) \((S\text{ surface, } a_1, a_2 \in \text{Pic}(S))\) s.th.
corresponding 7-tuples \((a_1^2, a_1 a_2, a_2^2, a_1 K_S, a_1 K_S, K_S^2, \chi(O_S))\)
are linearly independent.

We take

\[
(\mathbb{P}^2, \mathcal{O}, \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}), (\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}), (\mathbb{P}^2, \mathcal{O}, \mathcal{O}(1)), \\
(\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}(1)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 0), \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}(1, 0))
\]
Reduction to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

\[ A_1(s, q), \ldots A_7(s, q) \] are determined by value of $Z_S(a_1, a_2, s, q)$
for 7 triples $(S, a_1, a_2)$ ($S$ surface, $a_1, a_2 \in \text{Pic}(S)$) s.th. corresponding 7-tuples $(a_1^2, a_1 a_2, a_2^2, a_1 K_S, a_1 K_S, K^2_S, \chi(O_S))$
are linearly independent.

We take

\[
(\mathbb{P}^2, O, O), (\mathbb{P}^1 \times \mathbb{P}^1, O, O), (\mathbb{P}^2, O(1), O), (\mathbb{P}^2, O, O(1)), \\
(\mathbb{P}^2, O(1), O(1)), (\mathbb{P}^1 \times \mathbb{P}^1, O(1, 0), O), (\mathbb{P}^1 \times \mathbb{P}^1, O, O(1, 0))
\]

In this case $S$ is a smooth toric, i.e. have an action of
$T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints,
Action of $T$ lifts to action on $S^{[n]}$ still with finitely many fixpoints described by partitions, compute by equivariant localization.
This computes $Z_S(a_1, a_2, s, q)$ in terms of combinatorics of partitions.
Reduction to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

**Computation:** Wrote a Pari/GP program
Reduction to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

**Computation:** Wrote a Pari/GP program

**Result:** Computed $A_1, \ldots A_7$

- mod $q^{31}$ for $e^{\text{vir}}(M)$
- mod $q^8$ for $\chi^{\text{vir}}_{-y}(M)$
- mod $q^7$ for $Eli^{\text{vir}}(M)$ and $\{M\}^{\text{vir}}$
Reduction to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

**Computation:** Wrote a Pari/GP program

**Result:** Computed $A_1, \ldots, A_7$

$$\mod q^{31} \text{ for } e^{\text{vir}}(M)$$

$$\mod q^8 \text{ for } \chi_{-y}^{\text{vir}}(M)$$

$$\mod q^7 \text{ for } Ell^{\text{vir}}(M) \text{ and } \{M\}^{\text{vir}}$$

This confirms conjectures for K3 surfaces, their blowups, elliptic surfaces, double covers of $\mathbb{P}^2$ and rational ruled surfaces, complete intersections, for $\text{vd}(M)$ smaller than roughly $\frac{3}{2}$ times the power of $q$. 
Let $X$ be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \ldots, p_e$

Let $E$ be equivariant vector bundle of rank $r$ on $X$. 
Let $X$ be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \ldots, p_e$.

Let $E$ be equivariant vector bundle of rank $r$ on $X$.

Fibre $E(p_i)$ of $X$ at fixp. $p_i$ has basis of eigenvect. for $T$-action $E(p_i) = \bigoplus_{k=1}^{r} \mathbb{C}v_i$, with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$.
Let $X$ be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \ldots, p_e$

Let $E$ be equivariant vector bundle of rank $r$ on $X$.

Fibre $E(p_i)$ of $X$ at fixp. $p_i$ has basis of eigenvect. for $T$-action $E(p_i) = \bigoplus_{k=1}^{r} \mathbb{C} v_i$, with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$

Equivariant Chern class of fibre at fixpoint:

$c^T(E(p_i)) = (1+c^T_1(E(p_i)) + \ldots + c^T_r(E(p_i)) = \prod_{i=1}^{r} (1+n_i \epsilon_1 + m_i \epsilon_2) \in \mathbb{Z}[\epsilon_1, \epsilon_2]$
Let $X$ be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \ldots, p_e$.

Let $E$ be equivariant vector bundle of rank $r$ on $X$.

Fibre $E(p_i)$ of $X$ at fixp. $p_i$ has basis of eigenvect. for $T$-action $E(p_i) = \bigoplus_{k=1}^r \mathbb{C} v_i$, with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$

Equivariant Chern class of fibre at fixpoint: 
$$c^T(E(p_i)) = (1+c_1^T(E(p_i))) + \ldots + c_r^T(E(p_i)) = \prod_{i=1}^r (1+n_i \epsilon_1 + m_i \epsilon_2) \in \mathbb{Z}[\epsilon_1, \epsilon_2]$$

Let $P(c(E)))$ polynomial in Chern classes of $E$, of degree $d = \dim(X)$

**Theorem (Bott residue formula)**

$$\int_{[X]} P(c(E)) = \sum_{k=1}^e \frac{P(c^T(E(p_k)))}{c^T_{\dim(X)}(TX(p_k))}$$

*(does not depend on $\epsilon_1, \epsilon_2$)*
For simplicity $S = \mathbb{P}^2$. $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{P}^2$ by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

Fixpoints are $p_0 = (1, 0, 0), p_1 = (0, 1, 0), p_2 = (0, 0, 1)$. 
For simplicity $S = \mathbb{P}^2$. $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{P}^2$ by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

Fixpoints are $p_0 = (1, 0, 0)$, $p_1 = (0, 1, 0)$, $p_2 = (0, 0, 1)$. Local (equivariant) coordinates near $p_0$ are $x = \frac{X_1}{X_0}$, $y = \frac{X_2}{X_0}$, $T$ action $(t_1, t_2)(x, y) = (t_1 x, t_2 y)$, similar for the $p_1, p_2$. 
For simplicity $S = \mathbb{P}^2$. $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{P}^2$ by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

Fixpoints are $p_0 = (1, 0, 0)$, $p_1 = (0, 1, 0)$, $p_2 = (0, 0, 1)$.

Local (equivariant) coordinates near $p_0$ are $x = \frac{X_1}{X_0}$, $y = \frac{X_2}{X_0}$, $T$ action $(t_1, t_2)(x, y) = (t_1 x, t_2 y)$, similar for the $p_1, p_2$.

$Z \in (\mathbb{P}^2)^n$ is $T$-invariant $\iff Z = Z_0 \sqcup Z_1 \sqcup Z_2$ $\quad$ $\text{supp}(Z_i) = p_i$. $\iff$ Reduce to case $\text{supp}(Z) = p_i$, e.g. $p_0$. 

Equivariant localization

For simplicity $S = \mathbb{P}^2$. $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{P}^2$ by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

Fixpoints are $p_0 = (1, 0, 0)$, $p_1 = (0, 1, 0)$, $p_2 = (0, 0, 1)$.

Local (equivariant) coordinates near $p_0$ are $x = \frac{X_1}{X_0}$, $y = \frac{X_2}{X_0}$,
$T$ action $(t_1, t_2)(x, y) = (t_1 x, t_2 y)$, similar for the $p_1, p_2$

$Z \in (\mathbb{P}^2)^[n]$ is $T$-invariant $\implies Z = Z_0 \sqcup Z_1 \sqcup Z_2$ \quad $\text{supp}(Z_i) = p_i$.
$\implies$ Reduce to case $\text{supp}(Z) = p_i$, e.g. $p_0$

Easy: $Z$ is $T$-invariant $\iff I_Z \in k[x, y]$ is gen. by monomials

Can write

$$I_Z = (y^{n_0}, xy^{n_1}, \ldots, x^r y^{n_r}, x^{r+1}) \quad (n_0, \ldots, n_r) \text{ partition of } n$$

Fixpoints on $(\mathbb{P}^2)^[n]$ are in bijections with triples $(P_0, P_1, P_2)$ of partitions of $3$ numbers adding up to $n$. 
Need to compute things like $c(\mathcal{O}^n)$

$\mathcal{O}^n$ vector bundle on $(\mathbb{P}^2)^n$ with fibre $\mathcal{O}^n(Z) = H^0(\mathcal{O}_Z)$
Need to compute things like $c(\mathcal{O}[n])$

$\mathcal{O}[n]$ vector bundle on $(\mathbb{P}^2)[n]$ with fibre $\mathcal{O}[n](Z) = H^0(\mathcal{O}_Z)$

If $Z = Z_0 \sqcup Z_1 \sqcup Z_2$, $\text{supp}(Z_i) = p_i$, then

$$\mathcal{O}[n](Z) = \mathcal{O}^{[n_0]}(Z_0) \oplus \mathcal{O}^{[n_1]}(Z_1) \oplus \mathcal{O}^{[n_2]}(Z_2)$$

$$c^T(\mathcal{O}[n](Z)) = c^T(\mathcal{O}^{[n_0]}(Z_0))c^T(\mathcal{O}^{[n_1]}(Z_1))c^T(\mathcal{O}^{[n_2]}(Z_2))$$
Need to compute things like $c(\mathcal{O}^n)$

$\mathcal{O}^n$ vector bundle on $(\mathbb{P}^2)^n$ with fibre $\mathcal{O}^n(Z) = H^0(\mathcal{O}_Z)$

If $Z = Z_0 \sqcup Z_1 \sqcup Z_2$, $\text{supp}(Z_i) = p_i$, then

\[
\mathcal{O}^n(Z) = \mathcal{O}^{[n_0]}(Z_0) \oplus \mathcal{O}^{[n_1]}(Z_1) \oplus \mathcal{O}^{[n_2]}(Z_2)
\]

\[
c^T(\mathcal{O}^n(Z)) = c^T(\mathcal{O}^{[n_0]}(Z_0))c^T(\mathcal{O}^{[n_1]}(Z_1))c^T(\mathcal{O}^{[n_2]}(Z_2))
\]

Let e.g. $Z = Z_0$, $I_Z = (y^4, xy^2, x^2y, x^3)$

Then the fibre

\[
\mathcal{O}^n(Z) = H^0(\mathcal{O}_Z) = \mathbb{C}[x, y]/(y^4, xy^2, x^2y, x^3)
\]

Thus basis of eigenvectors of fibre for $T$ action is

\[
\begin{bmatrix}
  1 & y & y^2 & y^3 \\
  x & xy & y^2 & y^3 \\
  x^2 & xy & y^2 & y^3
\end{bmatrix}
\]

with eigenvalues $t_1$, $t_1 t_2$, $t_2$, $t_2$.
Need to compute things like $c(\mathcal{O}^n)$

$\mathcal{O}^n$ vector bundle on $(\mathbb{P}^2)^n$ with fibre $\mathcal{O}^n(Z) = H^0(\mathcal{O}_Z)$

If $Z = Z_0 \sqcup Z_1 \sqcup Z_2$, supp($Z_i$) = $p_i$, then

$$\mathcal{O}^n(Z) = \mathcal{O}^{[n_0]}(Z_0) \oplus \mathcal{O}^{[n_1]}(Z_1) \oplus \mathcal{O}^{[n_2]}(Z_2)$$

$$c^T(\mathcal{O}^n(Z)) = c^T(\mathcal{O}^{[n_0]}(Z_0))c^T(\mathcal{O}^{[n_1]}(Z_1))c^T(\mathcal{O}^{[n_2]}(Z_2))$$

Let e.g. $Z = Z_0$, $I_Z = (y^4, xy^2, x^2y, x^3)$

Then the fibre $\mathcal{O}^n(Z) = H^0(\mathcal{O}_Z) = \mathbb{C}[x, y]/(y^4, xy^2, x^2y, x^3)$

Thus basis of eigenvectors of fibre for $T$ action is

$$\begin{pmatrix}
1 & y & y^2 & y^3 \\
x & xy & y^2 & y^3 \\
x^2 & x & xy & y^2 & y^3
\end{pmatrix}
\quad \text{with eigenvalues}
\begin{pmatrix}
1 & t_2 & t_2^2 & t_2^3 \\
t_1 & t_1t_2 & t_1^2 & t_1^2t_2 \\
t_1^2 & t_2^2 & t_1 & t_1^2
\end{pmatrix}$$

Thus

$$c^T(\mathcal{O}^n(Z)) = (1 + \epsilon_2)(1 + 2\epsilon_2)(1 + 3\epsilon_2)(1 + \epsilon_1)(1 + \epsilon_1 + \epsilon_2)(1 + 2\epsilon_1).$$
Now: state version of the Vafa-Witten formula for moduli space $M^H_S(3, c_1, c_2)$ of rank 3 sheaves. (There is a wrong physics prediction for all ranks $\geq 3$) Have formulas both for $\chi_{\text{vir}}(M)$ and $e_{\text{vir}}(M)$. For simplicity state only for $e_{\text{vir}}(M)$.

The formula again depends on the expected dimension

$$vd = vd(M^H_S(3, c_1, c_2) = 6c_2 - 2c_1^2 - 8\chi(\mathcal{O}_S).$$

Again assume $S$ algebraic surface with $b_1(S) = 0$ and $p_g(S) > 0$. For simplicity assume $S$ contains an irreducible canonical curve (zero set of a holomorphic 2 form).
The rank 3 case

$$\Theta_{A,0}(x) = \sum_{(n,m) \in \mathbb{Z}^2} x^{2(n^2 - nm + m^2)}, \quad \Theta_{A,1}(x) = \sum_{(n,m) \in \mathbb{Z}^2} \epsilon^{n+m} x^{2(n^2 - nm + m^2)}$$

Theta functions for $A_2$-lattice, here $\epsilon = e^{2\pi i/3}$. 
The rank 3 case

\[ \Theta_{A,0}(x) = \sum_{(n,m) \in \mathbb{Z}^2} x^{2(n^2 - nm + m^2)}, \quad \Theta_{A,1}(x) = \sum_{(n,m) \in \mathbb{Z}^2} \epsilon^{n+m} x^{2(n^2 - nm + m^2)} \]

Theta functions for \( A_2 \)-lattice, here \( \epsilon = e^{2\pi i/3} \). Define modular function

\[ z(x) := \frac{\Theta_{A,0}(x)}{\Theta_{A,1}(x)} = 1 + 9x^2 + 27x^4 + 81x^6 + O(x^8), \]

Define \( z_1(x), z_2(x) = z_1(-x) \) as the solutions of the equation

\[ w^2 - 4z(x)^2w + 4z(x) = 0. \]
Introduction
Virtual Euler number and its refinements
Examples
Check of conjectures
Further results

The rank 3 case

\[ \Theta_{A,0}(x) = \sum_{(n,m) \in \mathbb{Z}^2} x^{2(n^2-nm+m^2)}, \quad \Theta_{A,1}(x) = \sum_{(n,m) \in \mathbb{Z}^2} \epsilon^{n+m} x^{2(n^2-nm+m^2)} \]

Theta functions for \( A_2 \)-lattice, here \( \epsilon = e^{2\pi i/3} \). Define modular function

\[ z(x) := \frac{\Theta_{A,0}(x)}{\Theta_{A,1}(x)} = 1 + 9x^2 + 27x^4 + 81x^6 + O(x^8), \]

Define \( z_1(x), z_2(x) = z_1(-x) \) as the solutions of the equation

\[ w^2 - 4z(x)^2 w + 4z(x) = 0. \]

Recall \( \eta(x) = \prod_{n>0} (1 - x^n) \), and define

\[ \Psi_{S,c_1}(x) = 9 \left( \frac{1}{3\eta(x^2)^{12}} \right)^{\chi(O_S)} \left( \frac{3\eta(x^6)^3}{\Theta_{A,1}(x)} \right)^{K_S^2} \]

\[ \cdot \left( z_1(x)^{K_S^2} + z_2(x)^{K_S^2} + (-1)^{\chi(O_S)} (\epsilon^{c_1 K_S} + \epsilon^{-c_1 K_S}) \right). \]
The rank 3 case

\[ \Theta_{A,0}(x) = \sum_{(n,m) \in \mathbb{Z}^2} x^{2(n^2-nm+m^2)}, \quad \Theta_{A,1}(x) = \sum_{(n,m) \in \mathbb{Z}^2} \epsilon^{n+m} x^{2(n^2-nm+m^2)} \]

Theta functions for \( A_2 \)-lattice, here \( \epsilon = e^{2\pi i/3} \). Define modular function

\[ z(x) := \frac{\Theta_{A,0}(x)}{\Theta_{A,1}(x)} = 1 + 9x^2 + 27x^4 + 81x^6 + O(x^8), \]

Define \( z_1(x), z_2(x) = z_1(-x) \) as the solutions of the equation

\[ w^2 - 4z(x)^2 w + 4z(x) = 0. \]

Recall \( \bar{\eta}(x) = \prod_{n>0} (1 - x^n) \), and define

\[ \psi_{S,c_1}(x) = 9 \left( \frac{1}{3\bar{\eta}(x^2)^{12}} \right)^{\chi(O_S)} \left( \frac{3\bar{\eta}(x^6)^3}{\Theta_{A,1}(x)} \right)^{K_S^2} \times \left( z_1(x)^{K_S^2} + z_2(x)^{K_S^2} + (-1)^{\chi(O_S)} (\epsilon^{c_1 K_S} + \epsilon^{-c_1 K_S}) \right). \]

**Conjecture**

\[ e^{\text{vir}}(M_S^H (3, c_1, c_2)) = \text{Coeff}_{x^{vd}} [\psi_{S,c_1}(x)]. \]
$S$ algebraic surface with $b_1 = 0$, $p_g > 0$

$M^H_S(c_1, c_2) = H$-semi-stable rank 2 sheaves on $S$

$E$ universal sheaf on $S \times M$. For $\alpha \in H_2(S)$, put

$$\mu(\beta) = p_M^*(c_2(E) - c_1^2(E)/4)/\alpha \in H^2(M)$$
S algebraic surface with $b_1 = 0$, $p_g > 0$

$M^H_S(c_1, c_2) = H$-semi-stable rank 2 sheaves on $S$.

$E$ universal sheaf on $S \times M$. For $\alpha \in H_2(S)$, put

$$\mu(\beta) = p_{M^*}(c_2(E) - c_1^2(E)/4)/\alpha \in H^2(M)$$

**Donaldson invariant:**

$$D_{S,c_1}(\frac{\alpha^{vd}}{vd!}) = \int_{[M^H_S(c_1, c_2)]^{vir}} \frac{\mu(\alpha)^{vd}}{vd!}$$
$S$ algebraic surface with $b_1 = 0$, $p_g > 0$

$M^H_S(c_1, c_2) = H$-semi-stable rank 2 sheaves on $S$

$E$ universal sheaf on $S \times M$. For $\alpha \in H_2(S)$, put

$$\mu(\beta) = p_{M^*}(c_2(E) - c_1^2(E)/4)/\alpha \in H^2(M)$$

**Donaldson invariant:**

$$D_{S,c_1}(\frac{\alpha^{vd}}{vd!}) = \int_{[M^H_S(c_1,c_2)]^{vir}} \frac{\mu(\alpha)^{vd}}{vd!}$$

**Theorem (Witten conj., G.-Nakajima, Yoshioka)**

$$D_{S,c_1}(\frac{\alpha^{vd}}{vd!}) = 2^{2+K_S^2-\chi(O_S)} \text{Coeff}_{z^{vd}} \left[ \exp \left( \frac{Q(\alpha)}{2} z^2 \right) \right]$$

$$\sum_{a_i \text{ SWcl.}} \text{SW}(a_i)(-1)^{\langle c_1, a_i \rangle} \exp \left( \langle K_S - 2a_i, \alpha \rangle z \right)$$
Interpolate between VW and Don. invariants

Eisenstein series:

\[ G_2(x) := \sum_{n>0} \left( \sum_{d|n} d \right) x^n, \quad DG_2(x) := \sum_{n>0} \left( \sum_{d|n} nd \right) x^n \]

Conjecture

\[
\int \left[ M_S^H(c_1, c_2) \right]_{\text{vir}} \ c_{\text{vd}} - n \left( T_{\text{vir}}^{M_S^H(c_1, n)} \right) \frac{\mu(\alpha)^n}{n!} = \text{Coeff}_{x^{\text{vd}} z^n} \left[ 8 \left( \frac{1}{2\eta(x^2)12} \right) \chi(\mathcal{O}_S) \right.
\]

\[
\left( \frac{2\eta(x^4)^2}{\theta_3(x)} \right)^{K_S^2} \exp \left( \frac{1}{2} DG_2(x^2) Q(\alpha) z^2 - 2G_2(x^2) \langle K_S, \alpha \rangle z \right) \]

\[
\cdot \sum_{a_i \text{ SW cl.}} \text{SW}(a_i) (-1)^{\langle c_1, a_i \rangle} \left( \frac{\theta_3(x)}{\theta_3(-x)} \right)^{\langle K_S, a_i \rangle} e^{\left( \frac{1}{2} (G_2(x) - G_2(-x)) \langle K_S - 2a_i, \alpha \rangle z \right)}
\]

\[ z \to 0: \text{Vafa-Witten invariants}, \quad x \to 0, \quad xz \to 1: \text{Donaldson invariants} \]
Elliptic genus: (Introduced by Witten, motivated by physics). The elliptic genus is a refinement of the $\chi_{-y}$-genus. It associates to a smooth projective variety a Jacobi form (something like a modular form in two variables e.g. $\theta_3(x, y)$)
**Elliptic genus:** (Introduced by Witten, motivated by physics). The elliptic genus is a refinement of the $\chi_y$-genus. It associates to a smooth projective variety a Jacobi form (something like a modular form in two variables e.g. $\theta_3(x, y)$).

For vector bundle $E$ put

$$Ell_{q,y}(E) = y^{-\text{rk}(E)/2} \bigotimes_{n \geq 1} (\Lambda_{-yq^n} E^\vee \otimes \Lambda_{-yq^n} E \otimes S_{q^n} E^\vee \otimes S_{q^n} E),$$

$$\Lambda_t(E) = \bigoplus_{n \geq 0} t^n \Lambda^n E, \quad S_t(E) = \bigoplus_{n \geq 0} t^n S^n E.$$
Elliptic genus: (Introduced by Witten, motivated by physics). The elliptic genus is a refinement of the $\chi_{-y}$-genus. It associates to a smooth projective variety a Jacobi form (something like a modular form in two variables e.g. $\theta_3(x, y)$)

For vector bundle $E$ put

$$E\ell_{q,y}(E) = y^{-\text{rk}(E)/2} \bigotimes_{n \geq 1} (\Lambda_{-yq^n-1} E^\vee \otimes \Lambda_{-yq^n} E \otimes S_{q^n} E^\vee \otimes S_{q^n} E),$$

$$\Lambda_t(E) = \bigoplus_{n \geq 0} t^n \Lambda^n E, \quad S_t(E) = \bigoplus_{n \geq 0} t^n S^n E.$$

$E\ell(X) := \chi(X, E\ell_{q,y}(T_X))$ elliptic genus.

$E\ell^\text{vir}(M) := \chi^\text{vir}(M, E\ell_{q,y}(T_M^\text{vir}))$ virtual elliptic genus.

for $q = 0$ $E\ell^\text{vir}(M)$ specializes to $\chi_{-y}^\text{vir}(M)$. 
**DMVV formula** (conj. Dijkgraaf-Moore-Verlinde-Verlinde '97), (proof: Borisov-Libgober '00)

Put

$$L\left(\sum_{m,l} c_{m,l} y^l q^m\right) := \prod_{n>0} \prod_{m,l} (1 - x^n y^l q^m)^{c_{nm,l}}$$

Borcherds type lift, Jacobi form $\mapsto$ Siegel modular form
**DMVV formula** (conj. Dijkgraaf-Moore-Verlinde-Verlinde ’97),
(proof: Borisov-Libgober ’00)
Put
\[
L\left(\sum_{m,l} c_{m,l} y^l q^m\right) := \prod_{n>0} \prod_{m,l} (1 - x^n y^l q^m)^{c_{nm,l}}
\]
Borcherds type lift, Jacobi form \(\mapsto\) Siegel modular form

Then
\[
\sum_{n\geq 0} \text{Ell}(S[n]) x^n = \frac{1}{L(\text{Ell}(S))} = \left(\frac{1}{L(24\phi_2)} \text{ for } S = K3\right).
\]
Elliptic genus

\[ G_{1,0}(q, y) = -\frac{1}{2}\frac{y+1}{y-1} + \sum_{n>0} \sum_{d|n} (y^d - y^{-d})q^n, \quad G_2(q) = -\frac{1}{24} + \sum_{n>0} \sum_{d|n} dq^n \]

\[ G_{2,0}(q, y) = y \frac{\partial G_{1,0}(q, y)}{\partial y} - 2G_2(q) = \wp(q, y), \quad G_{3,0}(q, y) = y \frac{\partial \wp(q, y)}{\partial y} \]

\[ \phi_i(q, y) := G_{i,0}(q, y) \left( (y^{1/2} - y^{-1/2}) \prod_{n>0} \frac{(1-q^n y)(1-q^n/y)}{(1-q^n)^2} \right)^i \]

\[ L\left( \sum_{m,l} c_{m,l}y^l q^m \right) := \prod_{n>0} \prod_{m,l} (1 - x^n y^l q^m)^{c_{nm,l}}, \quad L_n(\phi) = L(\phi)|_{x=x^n} \]
Elliptic genus

\[ G_{1,0}(q, y) = -\frac{1}{2} \frac{y + 1}{y - 1} + \sum_{n>0} \sum_{d|n} (y^d - y^{-d}) q^n, \quad G_2(q) = -\frac{1}{24} + \sum_{n>0} \sum_{d|n} dq^n \]

\[ G_{2,0}(q, y) = y \frac{\partial G_{1,0}(q, y)}{\partial y} - 2G_2(q) = \wp(q, y), \quad G_{3,0}(q, y) = y \frac{\partial \wp(q, y)}{\partial y} \]

\[ \phi_i(q, y) := G_{i,0}(q, y) \left( (y^{1/2} - y^{-1/2}) \prod_{n>0} \frac{(1 - q^ny)(1 - q^n/y)}{(1 - q^n)^2} \right)^i \]

\[ L\left( \sum_{m,l} c_{m,l} y^l q^m \right) := \prod_{n>0} \prod_{m,l} (1 - x^n y^l q^m)^{c_{nm,l}}, \quad L_n(\phi) = L(\phi)|_{x=x^n} \]

Conjecture

\[ Ell_{\text{vir}}^v(M_S^H(c_1, c_2)) = \text{Coeff}_{x^{vd}} \left[ 8 \left( \frac{1}{2} \frac{1}{L_2(12\phi_2)} \right)^{\chi(O_S)} \right. \]

\[ . \left( \frac{2L_4(\phi_1\phi_3)L(-2\phi_1)}{L_2(-2\phi_{1,\text{ev}}(q^{1/2}, y) - \phi_1(q^2, y^2) + 2\phi_1^2)} \right) ^{K_S^2} \]