Holomorphic Euler Characteristics of line bundles on moduli spaces of sheaves on surfaces

Lothar Göttsche
joint work with (1) H.Nakajima, K.Yoshioka,
(2) D.Zagier

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Invariants of moduli spaces: A moduli space is a variety M parametrizing objects, we are interested in.
Invariants are intersection numbers $\int_M \alpha$ on moduli spaces.
Often interesting in theoretical physics.
Examples of Moduli spaces

1. Moduli space of stable maps $M_{g,n}(X, \beta)$ parametrizes morphism from a curve of genus g to a smooth variety X.
   The corresponding invariants are Gromov-Witten invariants
2. Moduli space of stable vector bundles/sheaves on curve/surface/threefold
   corresponding invariants: conformal blocks/Donaldson inv./Donaldson-Thomas inv.

For a long time motivation for studying moduli spaces of sheaves on surfaces were Donaldson invariants. Refinement:

Let Y compact complex manifold of dimension d, $L \in Pic(X)$.
Can define two related invariants of (Y, L):

1. $\int_Y c_1(L)^d$ (degree)
2. $\chi(Y, L) = \sum_{i=0}^d (-1)^i h^i(Y, L)$ (holom. Euler char.)

Riemann-Roch: $\chi(Y, L^{\otimes n}) = \frac{n^d}{d!} \int_Y c_1(L)^d + O(n^{d-1})$.
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For C curve $\chi(C, L) = deg(L) + 1 - g$, i.e. determines genus
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1. Donaldson inv.=deg. of line bundle L on moduli of sheaves
2. K-theory Don. inv.= holomorphic Euler characteristic of L

Let C nonsingular projective curve of genus g
$M_C(r, L) =$
Moduli space of rk r bundles E on C with $det(E) = L$

One shows $Pic(M_C(r, L)) = \mathbb{Z} \cdot \theta$

Theorem

(Verlinde formula)

$$h^0(M_C(r, \mathcal{O}), \theta^k) = \chi(M_C(r, \mathcal{O}), \theta^k)$$
$$= \frac{r^g}{(r+k)^g} \sum_{\substack{S \sqcup T = \{1,\ldots,r+k\} \\ |S| = k}} \prod_{s \in S} \prod_{t \in T} \left| 2 \sin\left(\pi \frac{s-t}{r+k}\right) \right|^{g-1}$$

In some sense we are trying to find the analogue of the Verlinde formula for surfaces.

X simply conn. proj. algebraic surface, H ample on X
Fix $c_1 \in H^2(X,\mathbb{Z})$, $c_2 \in H^4(X,\mathbb{Z})$.

$$M := M_X^H(c_1, c_2)$$
$$= \text{moduli space of } H\text{-stable rk 2 torsion-free sheaves on } X$$

torsion free sheaf="vector bundle with singularities"
H-stable: "all subsheaves of $\mathcal{E}$ are small":
$\frac{h^0(\mathcal{F}\otimes H^{\otimes n})}{\text{rk}(\mathcal{F})} < \frac{h^0(\mathcal{E}\otimes H^{\otimes n})}{\text{rk}(\mathcal{E})}$ for all $0 \neq \mathcal{F} \subset \mathcal{E}$, $n \gg 0$

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Moduli space means:

1. As set M is the set of isomorphism classes $[E]$ of H-stable torsion free sheaves on X
2. If $\mathcal{E}/X \times S$ flat family of sheaves, then $S \to M; s \mapsto [\mathcal{E}|_{X\times\{s\}}]$ is a morphism.

Simplifying assumptions:

- M is compact
- There is a universal sheaf $\mathcal{E}$ on $X \times M$
  i.e. $\mathcal{E}|_{X\times[E]} = E$ for all $[E] \in M$
- M is nonsingular of the expected dimension
  $d = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_X)$

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All these assumptions can be removed.

$K^0(X) :=$ Grothendieck group of vector bundles $=$
free abelian group gen. by vector bundles on $X/\equiv$
Here, if $0 \to E \to F \to G \to 0$ is an exact sequence, then
$F \equiv E + G$.
This may look complicated, but is just a way to keep track of rank and Chern classes

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Let $c \in K^0(X)$ class of $E \in M$.

$$X \xleftarrow{q} X \times M \xrightarrow{p} M$$

Definition

Let $v \in K^0(X)$ with $\chi(X, c \otimes v) = 0$ (write $v \in c^\perp$)
The determinant bundle for v is

$$\lambda(v) := \det(Rp_*(\mathcal{E} \otimes q^*(v))^{-1} \in Pic(M)$$

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Let $L \in Pic(X)$. Assume Lc_1 even. Put

$$v(L) := L^{-1} - \mathcal{O}_X + k\mathcal{O}_{pt} \in c^\perp$$

The Donaldson line bundle for L is $\widetilde{L} := \lambda(v(L))$.

Definition

The K-theoretic Donaldson invariant for L is $\chi(M, \widetilde{L})$.
Generating function:

$$\chi^X_{c_1,H}(L) := \sum_{c_2} \chi\left(M^H_X(c_1, c_2), \widetilde{L}\right) t^{c_2} \in \mathbb{Z}[[t]].$$

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Theorem

(Jun Li) For $n \gg 0$ the line bundle $\widetilde{nH}$ on $M^H_X(c_1, c_2)$ is spanned, and defines a birational morphism onto the Uhlenbeck compactification.

$M_X^H(c_1, c_2)$ depends on H:
There are walls (hyperplanes) in $H^2(X, \mathbb{R})$.
$M_X^H(c_1, c_2)$ and invariants change only when H crosses a wall.

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Aims:

- Prove wallcrossing formula for $\chi\big(M_X^H(c_1, c_2), \widetilde{L}\big)$
- For X for $\mathbb{P}^2$ and rational ruled surfaces compute generating function as rational function
- Relate result to Le Potiers strange duality conjecture

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Definition

Let $C_X \subset H^2(X, \mathbb{R})$ be the ample cone.
$\xi \in H^2(X, \mathbb{Z})$ defines wall of type (c_1, c_2) if

1. $\xi \equiv c_1 \mod 2H^2(X, \mathbb{Z})$
2. $4c_2 - c_1^2 + \xi^2 \geq 0$

The wall is

$$W^\xi := \{H \in C_X \mid H \cdot \xi = 0\}$$

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Chambers=connected components of $C_X \setminus$ walls
$M_X^H(c_1, c_2)$ and invariants constant on chambers, change when H crosses wall (i.e. $H_- \to H_+$ with $H_-\xi < 0 < H_+\xi$)

Definition

Let ξ define a wall of type (c_1, c_2). Put $d := 4c_2 - c_1^2 - 3$
The wallcrossing term is
$\Delta^X_{\xi,d}(L) := \chi(M_X^{H_+}(c_1, c_2)), \widetilde{L}) - \chi(M_X^{H_-}(c_1, c_2)), \widetilde{L}).$

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First aim: give a generating function for the wallcrossing terms in terms of elliptic functions.

Theta functions:

$$\theta_{ab}(h) = \sum_{n\equiv a\ (2)} q^{n^2}(i^b y)^n,\ a, b \in \{0, 1\},\ y = e^{h/2}$$

$$\theta_{ab} := \theta_{ab}(0),\ u := -\frac{\theta_{00}^2}{\theta_{10}^2} - \frac{\theta_{10}^2}{\theta_{00}^2}, \quad \Lambda := \frac{\theta_{11}(h)}{\theta_{01}(h)}$$

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Theorem

Write

$$q^{-\xi^2} y^{\xi(L-K_X)} \left(\frac{\theta_{01}(h)}{\theta_{01}}\right)^{(L-K_X)^2} \theta_{01}^{\sigma(X)} q \frac{du}{dq}\frac{dh}{d\Lambda} = \sum_{d\in\mathbb{Z}_{\geq 0}} f_d(q)\Lambda^d.$$

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Then

$\Delta^X_{\xi,d}(L) := \chi(M_X^{H_+}(c_1, c_2), \widetilde{L}) - \chi(M_X^{H_-}(c_1, c_2), \widetilde{L}) = \pm \mathrm{Coeff}_{q^0} f_d(q).$

Generating function

$\Delta^X_\xi(L) := \sum_d \Delta^X_{\xi,d}(L)\Lambda^d = \sum_d \mathrm{Coeff}_{q^0} f_d(q)\Lambda^d$

Before, to compute $\chi(M_X^H(c_1, c_2), \widetilde{L})$, need to understand $M_X^H(c_1, c_2)$ very well, only possible for small c_2.
Want to determine generating function $\sum_{c_2} \chi(M_X^H(c_1, c_2), \widetilde{L}) t^{c_2}$
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Remark

1. *The walls W^ξ of type (c_1,c_2) are locally finite in C_X.*
2. *(bad news) If one wants to consider generating function $\sum_n \chi(M_X^H(c_1,n),\widetilde{L})t^n$, one has to consider all walls W^ξ for all $\xi\in c_1+2H^2(X,\mathbb{Z})$. These are everywhere dense in C_X.*

Corollary

(good news) Let ξ class of type (c_1, c_2)

1. $\Delta^X_{\xi,d}(L) = 0$ *for* d *large*
   (a wall contributes only in finitely may degrees)
2. *If* $|\xi(L - K_X)| + 1 \leq -\xi^2$ *then* $\Delta^X_{\xi,d}(L) = 0$ *for all* d
   "Most walls do not contribute at all".

This is very different from the usual Donaldson invariants.

Let X algebraic surface, let $c, v \in K^0(X)$ with $\chi(c \otimes v) = 0$.
Assume $H^2(E \otimes F) = 0$ for all $[E] \in M(c)$, $[F] \in M(v)$.

$$\Theta := \{(E, F) \in M(c) \times M(v) \mid h^0(E \otimes F) \neq 0\}$$

Assume Θ is zero set of $\sigma \in H^0(M(c) \times M(v), \lambda(v) \boxtimes \lambda(c))$
$\Longrightarrow$ Duality morphism $D : H^0(M(c), \lambda(v))^\vee \to H^0(M(v), \lambda(c))$

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Conjecture/Question

1. *When is $D : H^0(M(c), \lambda(v))^\vee \to H^0(M(v), \lambda(c))$ an isomorphism? (strong strange duality)*
2. *When is $\chi(M(c), \lambda(v)) = \chi(M(v), \lambda(c))$? (weak strange duality)*

Let $L \in Pic(X)$, $c(n)$ class of $E \in M_X^H(c_1, n)$
$v(L) = \mathcal{O}_X - L^{-1} + k\mathcal{O}_{pt}$ independent of n.
$c(n) = \mathcal{O} + \mathcal{O}(c_1) - n\mathcal{O}_{pt}$, thus
$\lambda(c(n)) = \lambda(\mathcal{O} + \mathcal{O}(c_1)) \otimes \lambda(\mathcal{O}_{pt})^{\otimes -n}$.
It follows $\chi(M(v(L), \lambda(c(n))t^n \in \mathbb{Q}(t)$ is a rational function in t.
Strange duality implies $\chi(M_X^H(c_1, n), \widetilde{L}) = \chi(M(v(L), \lambda(c(n))$

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Remark

*There is natural morphism $\pi : M(v(L)) \to |L|, \mathcal{F} \mapsto supp(\mathcal{F})$
General fibre over $[C]$ is $Pic^d(C)$.*

Let X be a rational surface. Possibly after blowing up X there is an H_0 with $\chi(M_X^{H_0}(c_1, c_2), \widetilde{L}) = 0$. $\implies$ Everything is determined by wallcrossing:

$$\chi(M_X^H(c_1, c_2), \widetilde{L}) = \sum_{\xi H_0 < 0 < \xi H} \Delta_{\xi,d}^X(L)$$

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Problem: If we want to look at generating functions, i.e. consider all c_2, the sum becomes infinite. Need arguments about elliptic functions/modular forms to carry out the sum.

Let X rational ruled surface. F fibre, G section. Let $L \in Pic(X)$. Put $s = L \cdot F$, $n = L \cdot G - G^2$. E.g. if $X = \mathbb{P}^1 \times \mathbb{P}^1$, then $L = nF + sG$

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Theorem

1. *Let $c_1 = 0, F$ assume $\frac{H\cdot G - G^2}{H\cdot F} > \frac{n}{4}$, then*

$$1 + \sum_{c_2>0} \chi\left(M_X^H(c_1,c_2),\widetilde{L}\right)t^{c_2} = \begin{cases} \frac{1}{(1-t)^{n+1}} & s=0 \\ \frac{1}{(1-t)^{2n+2}} & s=1 \\ \frac{1}{2}\frac{(1+t)^n + (-1)^{c_1\cdot L/2}(1-t)^n}{(1-t)^{3n+3}} & s=2 \end{cases}$$

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2. *Let $c_1 \cdot F$ odd, assume $\frac{n}{4} < \frac{H \cdot G - G^2}{H \cdot F} < n$, then*

$$\sum_{c_2 > 0} \chi(M_X^H(c_1, c_2), \widetilde{L}) t^{c_2 - c_1^2/4} = \begin{cases} 0 & s = 0 \\ \frac{t^{n/2}}{(1-t)^{2n+2}} & s = 1 \\ \frac{t^{n/4}}{2} \frac{(1+t^{1/2})^n + (-1)^{c_1 L/2}(1-t^{1/2})^n}{(1-t)^{3n+3}} & s = 2 \end{cases}$$

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Let X be a rational surface, $\widehat{X}$ the blowup of X in a point. E the exceptional divisor. Let H ample on X and $L \in Pic(X)$

Theorem

$$\sum_n \chi(M^H_{\widehat{X}}(c_1, n), \widetilde{L})t^n = \sum_n \chi(M^H_X(c_1, n), \widetilde{L})t^n$$

$$\sum_n \chi(M^H_{\widehat{X}}(c_1, n), \widetilde{L - E})t^n = (1-t)\sum_n \chi(M^H_X(c_1, n), \widetilde{L})t^n$$

There are also "higher blowup formulas" for $\widetilde{L - nE}$, which are more difficult to write

Theorem

$$1+\sum_{c_2>0}\chi\left(M_{\mathbb{P}^2}(0,c_2),\widetilde{nH}\right)t^{c_2}=\begin{cases}\frac{1}{(1-t)^3} & n=1\\ \frac{1}{(1-t)^6} & n=2\\ \frac{1+t^2}{(1-t)^{10}} & n=3\\ \frac{1+6t^2+t^3}{(1-t)^{15}} & n=4\\ \frac{1+21t^2+20t^3+21t^4+t^6}{(1-t)^{21}} & n=5\end{cases}$$

$$\sum_{c_2>0}\chi\left(M_{\mathbb{P}^2}(H,c_2),\widetilde{nH}\right)t^{c_2-1}=\begin{cases}\frac{1}{(1-t)^6} & n=2\\ \frac{1+6t+t^3}{(1-t)^{15}} & n=4\end{cases}$$

For any $\widetilde{nH}$ there is an algorithm to determine the generating function as explicit rational function in t (computed for $n\leq 10$)

Danila determined other side of strange duality for H, $2H$, $3H$ and checked strange duality in this case for small c_2.

Corollary

(Strong) strange duality holds for $c_1 = 0, H$*,* $L = H, 2H, 3H$*.*

Yuan Yau is using the result to prove strange duality for larger n and for rational ruled surfaces.
Strange duality for K3 and abelian surfaces has been studied by Marian-Oprea. Belkale and Marian-Oprea, proved strange duality for curves (also called rank-level duality).

Main ingredient in proof of wallcrossing formula:
Nekrasov partition function.

Instanton moduli space:
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Torus action: $\mathbb{C}^* \times \mathbb{C}^*$ acts on $(\mathbb{P}^2, l_\infty)$:
$(t_1, t_2)(z_0 : z_1 : z_2) = (z_0 : t_1 z_1 : t_2 z_2)$.
Extra $\mathbb{C}^*$ acts by $s(E,\phi) = (E, diag(s^{-1}, s) \circ \phi)$.

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Character: Let V vector space with $(\mathbb{C}^*)^3$ action. $\Longrightarrow V = \sum_i V_{M_i}$
V_{M_i} eigenspace with eigenvalue M_i Laurent monomial in t_1, t_2, s.
The Character of V is $ch(V) := \sum_i dim(V_{M_i}) M_i$

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$$Z^{inst}(\epsilon_1, \epsilon_2, a, \Lambda) := \sum_{n\geq 0} \left(\frac{\Lambda^4}{t_1 t_2}\right)^n ch(H^0(M(n), \mathcal{O}))|_{t_1=e^{\epsilon_1}, t_2=e^{\epsilon_2}, s=e^a}$$

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Nekrasov Conjecture (Nekrasov-Okounkov, Nakajima-Yoshioka, Braverman-Etingof):

1. $Z=\exp(\frac{F(\epsilon_1,\epsilon_2,a,\Lambda)}{\epsilon_1\epsilon_2})$, F regular at ϵ_1,ϵ_2
2. $F|_{\epsilon_1=\epsilon_2=0}$ can be expressed in terms of elliptic functions.

Proof of wallcrossing formula

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Reduce to case X is toric. In this case use localization to compute intersection number on $Hilb^{n_1}(X) \times Hilb^{n_2}(X)$ in terms of weights at the fixpoints. The expression obtained this way is equal to what one gets by computing product of Nekrasov partition functions over the fixpoints of the action on X with $\epsilon_1, \epsilon_2, a$ replaced by weights of the action.