Refined curve counting

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If C singular curve: (geometric) genus $g(C)$ genus of normalization

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$$n_{d,g} := \#\{\text{curves of degree } d, \text{ genus } g \text{ in } \mathbb{P}^2 \text{ through } 3d-1+g \text{ general points}\}$$

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Caporaso-Harris recursion: A recursion formula that determines all $n_{d,g}$ recursively.

Why care?
These are classical questions, but for me the most important reason is the following:
These and similar numbers come up in physics, and there they are closely related to each other and to other subjects of mathematics
These relations will be reflected and are in some sense most visible in the generating functions of curve counting and other invariants.

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(0) For Severi degrees: curves are elements of $|\mathcal{O}(d)|$
Count curves with given genus through correct number of points as points in proj. space $|\mathcal{O}(d)|$

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e.g. Let X quintic 3-fold in $\mathbb{P}^4$
$N_d :=$"$\#\{$ rational curves of degree d in $X\}$"
$:= deg([M_0(X, dl)]^{vir})$
$N_1 = 5$, $N_2 = 2875$, $N_3 = 609250, \dots$

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First computed by physicists using Mirror symmetry
(There are pairs X, X' of Calabi-Yau 3-folds (i.e. $K_X = \mathcal{O}_X$) which are closely related. By physics counting curves on X is equivalent to variations of Hodge structures on X')

(2) Gopakumar-Vafa (BPS)-invariants:
Gromov-Witten invariants count maps
$\Longrightarrow$ complicated formulas for multiple covers
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More conjecturally: there should be **refined BPS invariants**: GV invariants should be something like the Euler number of some physics moduli space and the refined invariants should be something like Betti numbers

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P-T moduli space:
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Conjectural PT-GW correspondence:
PT and GW invariants conjectured equivalent (generating
functions related by explicit change of variables)
PT have conjecturally defined GV-inv. in terms of PT-inv.
So all 3 sets of invariants are conjectured to be equivalent.

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$L \in Pic(S)$ very ample line bundle
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Severi degree:

$$\begin{aligned} n_{L,g(L)-\delta} &:= \#\{\delta\text{-nodal curves in } V_\delta\} \\ &= \#\{\text{ curves of genus } g(L)-\delta \text{ in } V_\delta\} \end{aligned}$$

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Formulas for this as polynomial in $L^2, LK_S, K_S^2, c_2(S)$ computed by Avritzer-Vainsencher for $\delta \leq 6$ and by Kleimann-Piene for larger δ.

Yau-Zaslow formula from physics: Let S K3-surface, L line bundle on S, then the number n_{L^2} of rational curves in $|L|$ depends only on L^2 and

$$\sum_d n_d q^d = \frac{1}{\Delta(q)}$$

$(\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24})$.

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Conjecture (G 1997)

(1) Exists universal polynomial $n^L_{g(L)-\delta}$ in $L^2, LK_S, K_S^2, c_2(S)$, such that $n_{L,g(L)-\delta} = n^L_{g(L)-\delta}$ for L is sufficiently ample wrt δ.
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Part (1) of the conjecture (existence of univ. polyn. $n^L_{g(L)-\delta}$) was proven by Tzeng, Kool-Shende-Thomas (2010)

Aim: Give a refined version of conj, inspired by KST proof.
Replace $n^L_{g(L)-\delta}$ by polynomials $N_{g(L)-\delta}(y) \in \mathbb{Z}[y]$
"Refined curve counting invariants"
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If S is a smooth projective surface, then $S^{[n]}$ is smooth projective variety of dimension $2n$.
For any line bundle $L \in Pic(S)$ have a tautological vector bundle $L^{[n]}$ of rank n on $S^{[n]}$ with fibre $L^{[n]}([Z]) = H^0(L|_Z)$.

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$\mathcal{C} := \{(p,[C]) \in S \times V_\delta \mid p \in C\}$ universal curve
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KST show:

1. $\exists_{n_l \in \mathbb{Z}}$, $l = g(L) - \delta, \ldots, g(L)$ s.th.
$$\sum_{n \geq 0} e(\mathcal{C}^{[n]}) q^n = \sum_{l=g(L)-\delta}^{g(L)} n_l q^{g(L)-l} (1-q)^{2l-2}.$$
$$e(X) = \sum_{i=0}^{2dim(X)} (-1)^i rk(H^i(X, \mathbb{Z})) \text{ topological Euler number}$$
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Note: This computes $n^L_{g-\delta}$ as a BPS invariant:
Pandharipande-Thomas defined BPS-invariants by formula (1).

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This allows to compute $e(\mathcal{C}^{[n]})$ as intersection number on $S^{[n]}$:

$$e(\mathcal{C}^{[n]}) = \int_{S^{[n]} \times V_\delta} \frac{c(T_{S^{[n]}})c_n(L^{[n]} \otimes H)}{c(L^{[n]} \otimes H)}$$

($c(E) = 1 + c_1(E) + \ldots + c_{rk(E)}(E)$ Chern class).
Ellingsrud-G-Lehn: such "tautological" integrals are always given by universal polynomials in $L^2, LK_S, K_S^2, c_2(S)$.

Refinement: $L \in Pic(S)$ suff ample, $V_\delta \subset |L|$ general δ-dim linear subspace.

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Idea: Replace everywhere Euler number by χ_{-y}-genus

$$\chi_{-y}(X) = \sum_{p,q} (-1)^{p+q} y^q h^{pq}(X) = \sum_q (-y)^q \chi(X, \Omega^q(X))$$

Note: $\chi_{-1}(X) = e(X)$

Refined curve counting conjecture

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Conjecture

$\exists_{N_l(y)\in\mathbb{Z}[y]}$, $l = g(L)-\delta, \ldots, g(L)$ *s.th.*
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Obvious if one allows $\sum_{l=-\infty}^{g(L)}$

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Definition

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Example

Nodal cubics in $\mathbb{P}^2$: $\mathcal{C}/\mathbb{P}^1$ pencil of cubics
$n_0^{3H} = \#\{$rational curves in pencil$\}$=12,
$N_0^{3H}(y) = \chi_{-y}(\mathcal{C}) = 1+10y+y^2$

Check of Conjecture: $\chi_{-y}(\mathcal{C}^{[n]})$ computed by very similar integral on $S^{[n]}$ as $e(\mathcal{C}^{[n]})$.
EGL: coeff. of $\chi_{-y}(\mathcal{C}^{[n]})$ are univ. polyn. in $L^2, LK_S, K_S^2, c_2(S)$.

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$\Longrightarrow$ determined by values for
$(S, L) = (\mathbb{P}^2, \mathcal{O}), (\mathbb{P}^2, \mathcal{O}(1)), (\mathbb{P}^2, \mathcal{O}(-1)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O})$.
These are toric surface: action of $T = \mathbb{C}^* \times \mathbb{C}^*$ on S with finitely many fixpoints. Action lifts to $S^{[n]}$ with finitely many fixpoints $p_1, \ldots, p_e$.
Bott Residue formula: Integral for $\chi_{-y}(\mathcal{C}^{[n]})$ on $S^{[n]}$ can be computed in terms of the weights of action of T on the fibres $T_{S^{[n]}}(p_i)$, $L^{[n]}(p_i)$. Programmed on computer:

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Result: Conjecture is true modulo q^{11}.

$$\sum_{n\geq 0} \chi_{-y}(\mathcal{C}^{[n]})q^n \equiv \sum_{l=g(L)-\delta}^{g(L)} N_l(y) q^{g(L)-l}\big((1-q)(1-qy)\big)^{l-1}$$

The computation gives conjectural generating function for refined invariants $N^L_{g(L)-\delta}(y)$. Let $\overline{N}^L_{g(L)-\delta}(y) := N^L_{g(L)-\delta}(y)/y^\delta$

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$D := q\frac{q}{dq}$
$\Delta(y,q) = q\prod_{n\geq 1}(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2,$
$\widetilde{DG}_2(y,q) = \sum_{n\geq 1} q^n\Big(\sum_{d|n}\frac{n}{d}\frac{y^d-2+y^{-d}}{y-2+y}\Big)$
$B_1(y,q) = 1-q-(y+3+y^{-1})q^2+\ldots,$
$B_2(y,q) = 1+(y+3+y^{-1})q+(y^2+y^{-2})q^2+\ldots$

The computation gives conjectural generating function for refined invariants $N^L_{g(L)-\delta}(y)$. Let $\overline{N}^L_{g(L)-\delta}(y) := N^L_{g(L)-\delta}(y)/y^\delta$

$D := q\frac{q}{dq}$
$\Delta(y,q) = q\prod_{n\geq 1}(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2,$
$\widetilde{DG}_2(y,q) = \sum_{n\geq 1} q^n\Big(\sum_{d|n}\frac{n}{d}\frac{y^d-2+y^{-d}}{y-2+y}\Big)$
$B_1(y,q) = 1 - q - (y+3+y^{-1})q^2 + \ldots,$
$B_2(y,q) = 1 + (y+3+y^{-1})q + (y^2+y^{-2})q^2 + \ldots$

Conjecture

$$\sum_\delta \overline{N}^L_{g(L)-\delta}(y)(\widetilde{DG}_2(y,q))^\delta = \frac{(\widetilde{DG}_2(y,q)/q)^{\chi(L)}B_1(y,q)^{K_X^2}B_2(y,q)^{LK_X}}{\big(\Delta(y,q)\,D\widetilde{DG}_2(y,q)/q^2\big)^{\chi(\mathcal{O}_X)/2}}$$

Putting $y=1$ recovers the previous conjecture

Interpretation of the $N^L_{g(L)-\delta}(y)$: what do they count?

$n^L_{g(L)-\delta}$ are the BPS (Gopakumar-Vafa)-invariants from Physics

Hope: $N^L_{g(L)-\delta}(y)$ are refined BPS-invariants

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The Gromov-Witten and Pandharipande-Thomas invariants were computed in this case by Maulik-Pandharipande-Thomas
By the GW-PT correspondence (proven in this case) both are equivalent, so we only state PT invariants.

Pandharipande-Thomas invariants:
$\mathcal{C}$ universal curve over $|L|$.
P-T moduli space:
$P_n(S,L) \simeq \mathcal{C}^{[n+g(L)-1]}/|L|$.
H pullback of hyperplane class on $|L|$.
PT invariants with point insertions:
$C^k_{n,g} := \int_{P_n(S,L)} c(\Omega_P)H^k$ for $g = g(L) = \frac{L^2}{2} + 1$.

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Theorem (MPT)

$$\sum_{n\geq 0}\sum_{g\geq 0} C^k_{n,g}(-y)^n q^{g-1} = (-1)^{k-1}(y-2+y^{-1})^{k-1}\frac{\widetilde{DG_2}(y,q)^k}{\Delta(y,q)}$$

Modulo the conjecture this says
$\overline{N}^g_k(y) = (-1)^{k-1}(y-2+y^{-1})^{k-1}\sum_{n\geq 0} C^k_{n,g}(-y)^n$
Thus $N^g_k(y)$ counts PT invariants with k point insertions.

Let S be a real toric surface (S defined over $\mathbb{R}$, $\mathbb{C}^* \times \mathbb{C}^*$-action defined over $\mathbb{R}$, fixed points real (for simplicity assume $S = \mathbb{P}^2$, but what follows works more generally).
We conjecture that the refined invariants $N_g^L(y)$ specialized at $y = -1$ compute the Welschinger invariants, counting real algebraic curves

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Severi degree:

$$n_{d,g(d)-\delta} := \#\{\text{curves of degree } d, \text{ genus } g(d)-\delta \text{ through } \tbinom{d+2}{2} - 1 - \delta \text{ points}\}$$

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Caporaso-Harris recursion: $n_{d,g}$ computed by recursion.
The recursion involves relative Severi degrees $n_{d,g}(\alpha, \beta)$
($\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots)$, α_i, $\beta_j \in \mathbb{Z}_{\geq 0}$)
(tangency conditions along fixed line)
$n_{d,g}((d), (0)) = n_{d,g}$.

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Complex conjugation maps X to itself and $X^{\mathbb{R}}$ is fixpoint locus
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Theorem (Itenberg-Kharlamov-Shustin)

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Conjecture

$$w_{d,g(d)-\delta} = (-1)^{\delta} N^{dH}_{g(d)-\delta}(-1) (= \overline{N}^{dH}_{g(d)-\delta}(-1)), \text{ if } \delta \le 3d-3$$

(using the recursion this is checked for $d < 15$, $\delta < 11$).

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Found recursion defining polynomials $N_{d,g}(\alpha,\beta) \in Z_{\geq 0}[y]$, s.th. $N_{d,g}(\alpha,\beta)(1) = n_{d,g}(\alpha,\beta)$, $N_{d,g}(\alpha,\beta)(-1) = \pm w_{d,g}(\alpha,\beta)$.

Definition

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$N_{d,g(d)-\delta}(y) = N^{dH}_{g(d)-\delta}(y)$ *for* $\delta \leq 2d-2$*.*

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Example

$N_{3,0}(y) = N^{3H}_0(y) = 1 + 10y + y^2,\ n_{3,0} = 12,\ w_{3,0} = 8.$

Other specialization: $y=0$: $\chi_0(X)=\chi(X,\mathcal{O}_X)$.

Proposition

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Question

1. *What is the enumerative meaning of $N_{d,g}(y)$?*
2. *Are the $N_{d,g}(y)$ related to tropical geometry?*
3. *For general surfaces, is there an interpretation of $N^L_{g(L)-\delta}(-1)$?*
4. *Can the $N^L_{g(L)-\delta}(y)$ be related to open Gromov-Witten invariants?*

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Open Gromov-Witten invariants count maps from Riemann surfaces with boundary (C,B) to X, s.th. boundary B is mapped to X^τ.
Welschinger invariants are a special case of this: Given $f:(C,B)\to(X,X^\tau)$ by the Schwarz reflection principle get $\overline{f}:\overline{C}\to X$, where $\overline{C}$ is the curve obtained by gluing C to C along B. If τ is complex conjugation, real algebraic curve.

Recursion formula

$$N_{d,g}(\alpha,\beta)(y) = \sum_{k|\beta_k>0} \frac{1-y^k}{1-y} N_{d,g}(\alpha+e_k,\beta-e_k)(y)$$

$$+ \sum_{\beta',\alpha',g'} y^{I_\beta + I_{\alpha'}} \prod_i \left(\frac{1-y^i}{1-y}\right)^{\beta'_i-\beta_i} \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} N_{d-1,g'}(\alpha',\beta')(y)$$

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$\alpha = (\alpha_1,\alpha_2,\ldots)$, $\beta = (\beta_1,\beta_2,\ldots)$, $I_\alpha = \sum_i i\alpha_i$, $\binom{\alpha}{\beta} = \prod_i \binom{\alpha_i}{\beta_i}$,
$e_k = (0,\ldots,0,1,0,\ldots)$: 1 in position k.

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$\alpha=(\alpha_1,\alpha_2,\dots)$, $\beta=(\beta_1,\beta_2,\dots)$, $I_\alpha=\sum_i i\alpha_i$, $\binom{\alpha}{\beta}=\prod_i\binom{\alpha_i}{\beta_i}$,
$e_k=(0,\dots,0,1,0,\dots)$: 1 in position k.
Second sum is over all $\alpha'\le\alpha$, $\beta'\ge\beta$ and g' such that

$$I_{\alpha'}+I_{\beta'}=d-1, \quad g-g'=\sum_i(\beta'_i-\beta_i).$$

$y=1$ gives Caporaso-Harris recursion,
$y=-1$ gives recursion for Welschinger invariants.

Let X variety with $\mathbb{C}^*$-action by with fixpointset $\{p_1, \ldots, p_e\}$ finite
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Write
$\prod_{i=1}^r (1 + w(e_i)) =: (1 + c_1(E(p_i)) + c_2(E(p_i)) + \ldots + c_n(E(p_i)))$.
Let $P(c_1(E), \ldots, c_r(E))$ polynomial of weight $d = dim(X)$ in Chern classes of E.
Then $\int_X P(c_1(E), \ldots, c_r(E)) = \sum_{i=1}^e \frac{P(c_1(E(p_i)),\ldots,c_r(E(p_i)))}{c_d(T_X(p_i))}$