Tropical refined curve counting

Lothar Göttsche

Joint works with Vivek Shende, Florian Block, Franziska Schroeter

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Topics:

1. Introduction: Severi degrees, refined invariants
2. Tropical curve counting and tropical refined Severi degrees
3. Fock spaces and refined curve counting
4. Logarithmic Gromov-Witten invariants with λ-classes and refined Severi degrees
5. Refined descendent invariants

Severi degree:
S smooth proj. alg. surface, L line bundle on S

$$|L| = \{ C = Z(s) \mid s \text{ section of } L \} = \mathbb{P}^{h^0(L)-1}$$

$\mathbb{P}^\delta \subset |L|$ gen. δ-dim. lin. subsp. (curves through $\dim|L| - \delta$ pts)
Severi degree: $n_{(S,L),\delta} := \#\{\delta\text{-nodal curves in } \mathbb{P}^\delta \subset |L|\}$

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Kool-Shende-Thomas compute $n^{(S,L),\delta}$ using relative Hilbert schemes of points on the universal curve $\mathcal{C}/\mathbb{P}^\delta$
Note: The Pandharipande-Thomas moduli space of stable pairs is isomorphic to the relative Hilbert scheme of points.

Refinement: $\mathbb{P}^\delta \subset |L|$ general δ-dimensional linear subspace
$\mathcal{C}^{[n]}$ relative Hilbert scheme of universal curve $\mathcal{C}$ over $\mathbb{P}^\delta$
parametrizes sets of n-points on the curves of $\mathbb{P}^\delta$

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χ_{-y}-genus: $\chi_{-y}(X) = \sum_{p,q}(-1)^{p+q}h^{p,q}(X)y^q$

Write

$$\sum_{n\geq 0} \chi_{-y}(\mathcal{C}^{[n]})t^{n-g(L)+1} = \sum_{l\geq 0}^{\infty} N_l^{\mathcal{C}}(y)\left(\frac{t}{(1-t)(1-yt)}\right)^{l-g(L)+1}$$

$g(L)=$genus of smooth curve in $|L|$

Refined invariants

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Refined invariants: $N^{(S,L),\delta}(y) := N_\delta^{\mathcal{C}}(y)/y^\delta$

Note: $N^{(S,L),\delta}(1) = n_{(S,L),\delta}$ (if L suff. ample), so refinement of Severi degrees

More surprisingly observe $N^{(S,L),\delta}(-1)$ = Welschinger number.

Interpretation as refined K-theoretic PT invariants (Afgani)

X total space of K_S

X is noncompact CY with $\mathbb{C}^*$ action by rescaling the fibres

$i : S \hookrightarrow X$, zero section, image $S = X^{\mathbb{C}^*}$.

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Let $P_\chi(X, i_*\beta)$ moduli space of stable pairs

($\mathcal{F}$ dim 1 sheaf on X, $s : \mathcal{O}_X \to \mathcal{F}$ section with 0 dim coker)

$\chi(\mathcal{F}) = \chi$, $[supp(\mathcal{F})] = \beta$

$P_\chi = P_\chi(X, i_*\beta)$ carries self-dual $\mathbb{C}^*$ equiv obstr theory

For $s_1, \ldots, s_m \in S$ compute via virtual localization

$$M_\chi := \chi\left(\mathcal{O}^{vir}_{P_\chi} \otimes (K^{\mathrm{vir}}_{P_\chi})^{1/2} \otimes \prod_{i=1}^{m} \gamma(\mathcal{O}_{s_i})\right) \in Z[t^{1/2}, t^{-1/2}]$$

$\gamma(\alpha) = p_*(\mathcal{O}_{\mathrm{div}\,\mathbb{F}} \cdot q^*\alpha)$

$$\begin{tikzcd} P_\chi & P_\chi \times S \arrow[l, "p"'] \arrow[r, "q"] & S \end{tikzcd}$$

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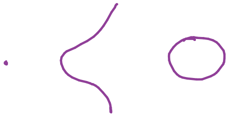
t is the equivariant parameter of the action. Under suitable assumptions $N^{(S,L),\delta}(y)$ expressed in terms gen. fct of $M_\chi\big|_{t=y}$, by BPS like formula

Let S real algebraic surface; P configuration of dim $|L| - \delta$ *real* points of S

Welschinger numbers: $W_{(S,L),\delta}(P) = \sum_C (-1)^{s(C)}$

sum is over all real δ-nodal curves C in $|L|$ though P

$s(C) = \#\{\text{isolated real nodes of } C\}$



Curves can be counted by counting piecewise linear objects: the tropical curves

A **lattice polygon** Δ in $\mathbb{R}^2$ is a polygon with vertices with integer coordinates
To a convex lattice polygon Δ one can associate a pair $(S(\Delta), L(\Delta))$ of a toric surface and a toric line bundle on S

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$h^0(S, L) = \#(\Delta \cap \mathbb{Z}^2)$, arithmetic genus $\#int(\Delta \cap \mathbb{Z}^2)$

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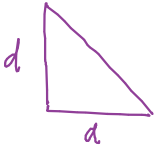
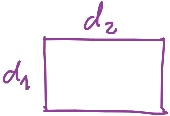
Examples:

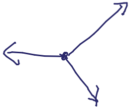
1. $(\mathbb{P}^2, \mathcal{O}(d))$

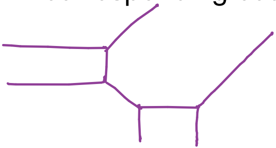


2. $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d_1, d_2))$



plane tropical curve of degree Δ:
piecewise linear graph Γ immersed in $\mathbb{R}^2$ s.t.

1. the edges e of Γ have rational slope
2. they have weight $w(e) \in \mathbb{Z}_{>0}$
3. **balancing condition:**
   let $p(e)$ primitive integer vector in direction of e;
   for all vertices v of Γ:
   $$\sum_{e \text{ at } v} p(e)w(e) = 0.$$
   
4. For every edge of Δ (of lattice length n) Γ has n unbounded edges in corresponding outer normal direction



conic in $\mathbb{P}^2$

Simple tropical curves: analogues of nodal curves, trivalent

genus of $\Gamma \to \mathbb{R}^2$ is $g(\Gamma) = h^1(\Gamma) - h^0(\Gamma) + 1$
Number of nodes $\#int(\Delta) - g(\Gamma)$
through $\#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$ general points in $\mathbb{R}^2$, there are finitely many δ-nodal degree Δ tropical curves, all simple

Count these curves with certain multiplicities

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Different counts; same principle:

1. for every vertex v of Γ define **vertex multiplicity** $u(v)$
2. multiplicity of Γ is $u(\Gamma) = \prod_{v \text{ vertex}} u(v)$
3. corresponding curve count is

$$u(\Delta, \delta) := \sum_{\Gamma} u(\Gamma)$$

(sum over all δ-nodal, degree Δ tropical curves through $\#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$ general points in $\mathbb{R}^2$)

1. v vertex of Γ, e_1, e_2, e_3 edges at v. vertex multiplicity

$$m(v) := w(e_1)w(e_2)|\det(p(e_1), p(e_2))|, \quad m(\Gamma) = \prod_{v \text{ vertex}} m(v)$$

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2. vertex multiplicity $\omega(v) := \begin{cases} (-1)^{(m(v)-1)/2} & m(v) \text{ odd} \\ 0 & m(v) \text{ even} \end{cases}$,

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Tropical Welschinger inv. $W^{\mathrm{trop}}_{\Delta,\delta} := \sum_{\Gamma} \omega(\Gamma)$

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Mikhalkin: The Severi degree is equal to the tropical Severi degree and the Welschinger numbers are equal to the tropical Welschinger invariants.

$$n_{(S(\Delta),L(\Delta)),\delta} = n^{\mathrm{trop}}_{\Delta,\delta}, \qquad W_{(S(\Delta),L(\Delta)),\delta}(P) = W^{\mathrm{trop}}_{\Delta,\delta} \text{ for suitable } P$$

quantum number: $[n]_y := \frac{y^{n/2}-y^{-n/2}}{y^{1/2}-y^{-1/2}}$

By definition $[n]_1 = n$, $[n]_{-1} = \begin{cases} (-1)^{(n-1)/2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

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Refined Severi degree: $N^{\text{trop}}_{\Delta,\delta}(y) := \sum_\Gamma M(\Gamma)$ sum as above

$$N^{\text{trop}}_{\Delta,\delta}(1) = n^{\text{trop}}_{\Delta,\delta}, \quad N^{\text{trop}}_{\Delta,\delta}(-1) = W^{\text{trop}}_{\Delta,\delta}$$

Itenberg-Mikhalkin: $N^{\text{trop}}_{\Delta,\delta}(y)$ is a tropical invariant, i.e. independent of the position of the points

Conjecture (Tropical refined $= \chi_y$(Hilbert scheme) refined)

If $L(\Delta)$ is sufficiently ample (δ-very ample), then

$S(\Delta)$ nonsingular

$$N^{\text{trop}}_{\Delta,\delta}(y) = N^{(S(\Delta),L(\Delta)),\delta}(y)$$

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Heisenberg algebra gen. by $a_n, b_n, \quad n \in \mathbb{Z}$
commutation relations

$$[a_n, a_m] = 0 = [b_n, b_m], \quad [a_n, b_m] = [n]_y \delta_{n,-m}, \quad [n]_y = \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}}$$

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Fock space: F generated by **creation operators** a_{-n}, b_{-n}
acting on vacuum vector $v_\emptyset$
elements of F are $f\, v_\emptyset$, where f is a polynomial
(with coefficients in $y^{\pm 1/2}$ in the a_{-n}, b_{-n})
H-module by $a_n v_\emptyset := 0$, $b_n v_\emptyset := 0$ for $n \geq 0$
(concatenate and apply commutation relations) e.g.
$a_2(a_{-1}b_{-2}v_\emptyset) = a_{-1}(b_{-2}a_2 + [2]_y a_{-1})v_\emptyset = (y^{1/2} + y^{-1/2})a_{-1}v_\emptyset$.

Basis paramtr. by pairs of partitions

$\mu = (1^{\mu_1}, 2^{\mu_2}, \dots)$, $\nu = (1^{\nu_1}, 2^{\nu_2}, \dots)$

$a_\mu := \prod_i \frac{a_i^{\mu_i}}{\mu_i!}$, $a_{-\mu} := \prod_i \frac{a_{-i}^{\mu_i}}{\mu_i!}$, similarly for b_ν, $b_{-\nu}$

$v_{\mu,\nu} := a_{-\mu} b_{-\nu} v_\emptyset$ basis for F

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inner product $\langle v_\emptyset | v_\emptyset \rangle = 1$; a_n, b_n adjoint to a_{-n}, b_{-n}

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Example: cases $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$:

$$H_m(t) := \sum_{k>0} b_{-k} b_k + t \sum_{\|\mu\| = \|\nu\| - m} a_{-\mu} a_\nu, \qquad \|\mu\| := \textstyle\sum_i i\mu_i$$

Theorem

$$\sum_{d \geq 0} \sum_{\delta \geq 0} \frac{t^d q^{d(d+3)/2-\delta}}{(d(d+3)/2-\delta)!} N^{\mathrm{trop}}_{d,\delta}(y) = \langle v_\emptyset | \exp(qH_1(t)) \exp(a_{-1}) v_\emptyset \rangle$$

$$\sum_{d_1, d_2 \geq 0} \sum_{\delta \geq 0} \frac{s^{d_1} t^{d_2} q^{d_1 d_2 + d_1 + d_2 - \delta}}{(d_1 d_2 + d_1 + d_2 - \delta)!} N^{\mathrm{trop}}_{d_1, d_2, \delta}(y)$$

$$= \langle v_\emptyset | \exp(a_1 s) \exp(qH_0(t)) \exp(a_{-1}) v_\emptyset \rangle$$

Idea of proof: The tropical Severi degrees can be computed using floor diagrams, encoding the combinatorics of "horizontally stretched" tropical curves.

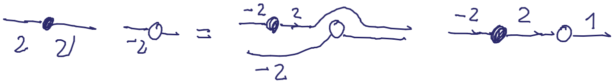
The commutation of the operators in expressions in Heisenberg operators can be encoded in Feynman diagrams.

$b_{-k} b_k$ $\quad a_{-(2\,1)} a_{(1^2)}$

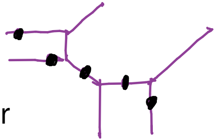
Feynman diagrams: connect vertices respecting order and weight so that black and white alternate

Now show that Feynman diagrams and floor diagrams are the same.

$b_{-2} b_2 a_{-2} a_1 = b_{-2} a_{-2} b_2 a_1 + (y^{1/2}+y^{-1/2}) b_{-2} a_1$



To Γ tropical curve through horizontally stretched conf. of points associate marked floor diagram.



escalators: horizontal segments of Γ
floors: conn. comp. of complem. of escalators. One marked point on every floor and escalator

Floor diagram: black vertex for escalator
white vertex for floor
connect if escalator connects to floor, keep weight

Put $m(\Lambda) := \prod_{e \text{ edges}} [w(e)]_y$ (same as multiplicity of Feynman diagr.)

Proposition

$$N^{\text{trop}}_{d,\delta}(y) = \sum_{\Lambda \text{ floor diagrams}} m(\Lambda)$$

Bousseau relates refined Severi deg. to log-Gromov-Witten inv.

Let Δ be a lattice polygon and $S = S(\Delta), L = L(\Delta)$ corresp. toric surface with toric line bundle. S is naturally a log scheme

$$\text{Let} \quad M = M^{\log}_{g,n}(\Delta) = \{(C, p_1, \dots, p_n, f)\}$$

moduli space log-stable maps of genus g to S of class L
Deligne-Mumford stack of expected dimension $g - 1 + n + K_S L$

$M^{\log}_{g,n}(\Delta)$ comes with the following data:

1. **evaluation maps:** $ev_i : M \to S(\Delta);\ (C, p_1, \dots, p_n) \mapsto f(p_i)$
2. **virtual fundamental class** $[M]^{\mathrm{vir}} \in A_{g-1+n+K_S L}(M)$
3. λ**-classes**: Let $\pi : \mathcal{C} \to M$ be the universal curve
   $\Omega_{\mathcal{C}/M}$ the relative dualizing sheaf $\mathbb{E} = \pi_*(\Omega_{\mathcal{C}/M})$
   The λ-classes are $\lambda_i = c_i(\mathbb{E})$.

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If $n = g - 1 + K_S L$, the **log Gromov-Witten invariants** are

$$n^{\log}_{\Delta,g} = \langle \tau_0(pt)^n \rangle_{g,\Delta} = \int_{[M]^{\mathrm{vir}}} \prod_{i=1}^{n} ev_i^*(pt).$$

Severi degrees of toric varieties are log GW-invariants
write $n_{\Delta,g} := n^{\text{trop}}_{\Delta,\delta}$, $N_{\Delta,g}(y) = N^{\text{trop}}_{\Delta,\delta}(y)$
with $\delta + g = g(L)$ arithmetic genus

Theorem (log-trop Corresp. theorem, Ranganathan, Mandel-Ruddat)

$$n^{\log}_{\Delta,g} = n^{\text{trop}}_{\Delta,g}$$

Bousseau extends this to refined invariants, using λ-classes:

The refined Severi degree $N^{\text{trop}}_{\Delta,g}(y)$ counts curves of all genera $g' \geq g$ through $n = g - 1 + K_S L$ points

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Expected dimension of space of these curves is $g' - g > 0$
Use the lambda class $\lambda_{g'-g}$ to cut down to a finite number

$$n^{\log}_{\Delta,g',\lambda_{g'-g}} = \int_{[M^{\log}_{g',n}(\Delta)]^{\mathrm{vir}}} (-1)^{g'-g} \lambda_{g'-g} \prod_{i=1}^{n} ev_i^*(pt).$$

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Theorem (Refined correspondence theorem, Boisseau)

$$\sum_{g' \geq g} n^{\log}_{\Delta,g',\lambda_{g'-g}} u^{2g'-2-LK_S} = N^{\mathrm{trop}}_{\Delta,g}(y) \left((-i)(y^{\frac{1}{2}} - y^{-\frac{1}{2}})\right)^{2g-2-LK_S}$$

with the identification $y = e^{iu}$.

Refined correspondence theorem

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Remark

1. *This gives a nontropical meaning to the $N^{\mathrm{trop}}_{\Delta,g}(y)$, and makes them manifestly invariant.*
2. *The change of variables $y = e^{iu}$ means knowing $N^{\mathrm{trop}}_{\Delta,g}(y)$ is eq. to knowing the infinitely many log-GW invariants $n_{\Delta,g',\lambda_{g'-g}}$*

Welschinger invariants: For simplicity work on $\mathbb{P}^2$
Until now only considered **totally real** Welschinger invariants, i.e. the curves are required to pass through real points
Now allow pairs of complex conjugate points

Count irreducible curves of genus 0, i.e. $\delta = (d-1)(d-2)/2$
P configuration of r real points in $\mathbb{P}^2$ and s pairs of complex conjugate points with $(r+2s) = 3d-1$

Welschinger invariants: $W^0_{d,r,s} = \sum_C (-1)^{s(C)}$

sum over all real nodal rational degree d curves C though P

$s(C) = \#\{\text{isolated real nodes of } C\}$

The $W^0_{d,r,s}$ can be computed via tropical geometry
Make refinement, replacing the number by polynomial $N^{0,trop}_{d,r,s}(y)$
Just count degree d tropical curves in $\mathbb{R}^2$ as before
But point conditions change:

The $W^0_{d,r,s}$ can be computed via tropical geometry

Make refinement, replacing the number by polynomial $N^{0,trop}_{d,r,s}(y)$

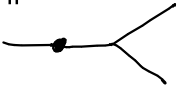
Just count degree d tropical curves in $\mathbb{R}^2$ as before

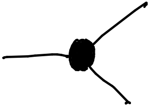
But point conditions change:

Let P configuration of r *thin* and s *fat* points in $\mathbb{R}^2$

We say a tropical curve Γ passes through P, if

1. the thin points lie on Γ,
2. the fat points are vertices of Γ.





For P general, there are finitely many δ-nodal degree d tropical curves through P. Count them with multiplicities

Count them again with a vertex multiplicity
Two kinds of vertices:

Standard vertex: $M(v) = [m(v)]_y$, $\quad [n]_y = \dfrac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}}$

Fat vertex: $M(v) = \{m(v)\}_y$, $\quad \{n\}_y = \dfrac{y^{n/2} + y^{-n/2}}{y^{1/2} + y^{-1/2}}$

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$$M(\Gamma) = \prod_{v \text{ vertex}} M(v),$$

$$N^{0,trop}_{d,r,s}(y) = \sum_{\Gamma} M(\Gamma)$$

sum over all genus 0 degree d simple tropical curves through P

Theorem

1. $N^{0,trop}_{d,r,s}(y) \in \mathbb{Z}[y, y^{-1}]$
2. $N^{0,trop}_{d,r,s}(y)$ *is a tropical invariant (independent of point position)*
3. $N^{0,trop}_{d,r,s}(-1) = W^0_{d,r,s}$
4. $N^{0,trop}_{d,r,s}(1) = \langle \tau_0(pt)^r \tau_1(pt)^s \rangle_{0,\mathbb{P}^2,d}$ *(descendent Gromov-Witten invariants).*

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$$\langle \tau_{a_1}(pt) \cdots \tau_{a_n}(pt) \rangle_{0,\mathbb{P}^2,d} = \int_{M_{0,n}(\mathbb{P}^2,d)} \psi_1^{a_1} ev_1^*(pt) \cdots \psi_n^{a_n} ev_n^*(pt)$$

$M_{0,n}(\mathbb{P}^2, d) = \{f : (C, x_1, \ldots, x_n) \to \mathbb{P}^2\}$, $ev_i(f, C, \underline{x}) = f(x_i)$,
$\psi_i = c_1(L_i)$, L_i line bundle on M with fibre T^*_{C,x_i}, at $(f, C, \underline{x})$

Boisseau: hints that $N^{0,trop}_{d,r,s}(e^{iu})$ is in same way gener. function of the

$$\langle \tau_0(pt)^r \tau_1(pt)^s \lambda_{g'-g} \rangle_{\Delta, g', r, s} = \int_{[M_{g',r+s}(\Delta)^{\log}]^{\mathrm{vir}}} \psi_1 \cdots \psi_s ev_1^*(pt) \cdots ev_{r+s}^*(pt)(-1)^g \lambda_{g'-g}$$