Introduction O	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000

Lehn and Verlinde formulas for moduli spaces of sheaves on surfaces

Lothar Göttsche, joint work with Martijn Kool

Zürich, 24 June 2020

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Marian-Oprea-Pandharipande:

generating functions for Segre integrals $\int_{S^{[n]}} c_{2n}(V^{[n]})$

+ their conj. relation to Verlinde numbers $\chi(S^{[n]}, \mu(L) \otimes E^{\otimes r})$.

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 $Hilb^n(S)$ is a moduli space of rank 1 stable sheaves on S

Aim: Extend above results to moduli spaces of sheaves $M_S^H(\rho, c_1, c_2)$ of higher rank:

Use Mochizuki's formula computing virtual inters. numbers on $M_S^H(\rho, c_1, c_2)$ in terms of inters. numbers on $S^{[n]}$.

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Hilbert scheme of po	ints			

 $S^{[n]} = Hilb^n(S) = \{$ zero dim. subschemes of degree n on $S\}$

 $S^{[n]}$ is smooth projective, of dimension 2nClosely related to symmetric power $S^{(n)} = S^n/(\text{perm. of factors})$ $\pi: S^{[n]} \to S^{(n)}, Z \mapsto supp(Z)$ is a crepant resolution.

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$$Z_n(S) = \{(x, [Z]) \mid x \in Z\} \subset S imes S^{[n]},$$

 $p: Z_n(S) o S^{[n]}, \quad q: Z_n(S) o S$ projections
Fibre $p^{-1}([Z]) = Z.$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000
Hilbert scheme of	points			
$Z_n(S)$ $p: Z_n(S)$	$egin{aligned} &= ig\{(x, [Z]) \mid x \in Z, \ (S) o S^{[n]}, q:Z, \end{aligned}$	$\{\mathcal{S}\}\subset \mathcal{S} imes \mathcal{S}^{[n]}, \ \mathcal{S}) o \mathcal{S}$ projection	S	
Tauto	logical sheaves:	V vector bundle of r	ank <i>r</i> on <i>S</i>	

 $V^{[n]} := p_*q^*(V)$ vector bundle of rank *rn* on $S^{[n]}$ $V^{[n]}([Z]) = H^0(V|_Z)$, in particular $\mathcal{O}_S^{[n]}([Z]) = H^0(\mathcal{O}_Z)$ Introduction Hilbert scheme of points Moduli spaces of sheaves Algebraicity Checks 00000 Hilbert scheme of points $Z_n(S) = \{(x, [Z]) \mid x \in Z\} \subset S \times S^{[n]}$ $p: Z_n(S) \to S^{[n]}, \quad q: Z_n(S) \to S$ projections **Tautological sheaves:** V vector bundle of rank r on S $V^{[n]} := p_* q^*(V)$ vector bundle of rank *rn* on $S^{[n]}$

 $V^{[n]}([Z]) = H^0(V|_Z)$, in particular $\mathcal{O}_{S}^{[n]}([Z]) = H^0(\mathcal{O}_Z)$

Line bundles on $S^{[n]}$: Pic $(S^{[n]}) = \mu(\text{Pic}(S)) \oplus \mathbb{Z}E$ with $E = \det(\mathcal{O}_{S}^{[n]} \text{ and } \mu(L) \text{ pullback from } S^{(n)} \text{ of equiva.}$ pushforward of $L \boxtimes \ldots, \boxtimes L$ from S^n to $S^{(n)}$. We have

$$\det(\mathit{V}^{[\mathit{n}]}) = \mu(\det(\mathit{V})) \otimes \mathit{E}^{\otimes \operatorname{\mathsf{rk}}(\mathit{V})}, \quad \mathit{V} \in \mathit{K}(\mathcal{S})$$

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$$\det(V^{[n]}) = \mu(\det(V)) \otimes E^{\otimes \operatorname{rk}(V)}, \quad V \in K(S)$$

Want formulas for

$$\chi(\mathcal{S}^{[n]},\mu(L)\otimes E^{\otimes r})$$
 Verlinde formula $\int_{\mathcal{S}^{[n]}} c_{2n}(V^{[n]})$ Lehn formula

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Cobordism invariance				

Tool:

Theorem (Ellingsrud-G-Lehn)

Let $P(x_1, \ldots, x_{2n}, y_1, \ldots, y_n)$ polynomial. Put

$$P[S^{[n]}, L] := \int_{S^{[n]}} P(c_1(S^{[n]}), ..., c_{2n}(S^{[n]}), c_1(L^{[n]}), ..., c_n(L^{[n]}))$$

There is a polynomial $\tilde{P}(x, y, z, w)$, such that for all surfaces *S*, all line bundles *L* on *S* we have

$$\mathbf{P}[\mathbf{S}^{[n]}, L] = \widetilde{\mathbf{P}}(\mathbf{K}_{\mathcal{S}}^2, \chi(\mathcal{O}_{\mathcal{S}}), L\mathbf{K}_{\mathcal{S}}, \mathbf{K}_{\mathcal{S}}^2).$$

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Usually have sequence of polynomials $P_n(x_1, ..., x_{2n}, y_1, ..., y_n), n \ge 0$, "nicely organized", then

$$\sum_{n\geq 0} P_n[S^{[n]}, L] x^n = A_1(x)^{L^2} A_2(x)^{LK_S} A_3(x)^{K_S^2} A_4(x)^{\chi(O_S)}$$

for universal power series $A_1, \ldots, A_4 \in \mathbb{Q}[[x]]$

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Lehn's conjecture

For L a line bundle on S consider the top Segre class

$$\int_{S^{[n]}} s_{2n}(L^{[n]}) = \int_{S^{[n]}} c_{2n}((-L)^{[n]})$$

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Conjecture (Lehn 1999)

$$\sum_{n=0}^{\infty} \int_{S^{[n]}} s_{2n}(L^{[n]}) z^n = \frac{(1-w)^a (1-2w)^b}{(1-6w+6w^2)^c},$$

with the change of variable

$$z = rac{w(1-w)(1-2w)^4}{(1-6w+6w^2)^3},$$

with
$$a = LK_S - 2K_S^2$$
, $b = (L - K_S)^2 + 3\chi(\mathcal{O}_S)$,
 $c = \chi(S, L) = \frac{1}{2}L(L - K_S) + \chi(\mathcal{O}_S)$

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Theorem (Marian-Oprea-Pandharipande, Voisin)

Lehn's conjecture is true.

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ehn's conjecture				
Marian- a formu	Oprea-Pandharipar la for $\sum_{n\geq 0} \int_{\mathcal{S}^{[n]}} c_{2n}$	nde consider a genera $(\alpha^{[n]})z^n, \alpha \in K(S)$	lized Lehn fo	rmula:

Theorem (Marian-Oprea-Pandharipande)

For any $s \in \mathbb{Z}$, there exist V_s , W_s , X_s , Y_s , $Z_s \in \mathbb{Q}[[z]]$ s.th. for any $\alpha \in K(S)$ of rank s on S, we have

$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} c(\alpha^{[n]}) = V_s^{c_2(\alpha)} W_s^{c_1(\alpha)^2} X_s^{\chi(\mathcal{O}_S)} Y_s^{c_1(\alpha)K_S} Z_s^{K_S^2}$$

ntroduction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000	
_ehn's conjecture)				
Marian-Oprea-Pandharipande consider a generalized Lehn formula: a formula for $\sum_{n\geq 0} \int_{S^{[n]}} c_{2n}(\alpha^{[n]}) z^n$, $\alpha \in K(S)$					
Theor	em (manan-Oprea-	Panunanpanue)			
For an $\alpha \in K$	<i>ny</i> s ∈ \mathbb{Z} , there exist (S) of rank s on S. w	$V_s, W_s, X_s, Y_s, Z_s \in \mathbb{Q}$ ve have	[[z]] s.th. for a	any	

$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} c(\alpha^{[n]}) = V_s^{c_2(\alpha)} W_s^{c_1(\alpha)^2} X_s^{\chi(\mathcal{O}_S)} Y_s^{c_1(\alpha)K_S} Z_s^{K_S^2}.$$

With the change of variables $z = t(1 + (1 - s)t)^{1-s}$, one has

$$V_{s}(z) = (1 + (1 - s)t)^{1-s}(1 + (2 - s)t)^{s},$$

$$W_{s}(z) = (1 + (1 - s)t)^{\frac{1}{2}s-1}(1 + (2 - s)t)^{\frac{1}{2}(1-s)},$$

$$X_{s}(z) = (1 + (1 - s)t)^{\frac{1}{2}s^{2}-s}(1 + (2 - s)t)^{-\frac{1}{2}s^{2}+\frac{1}{2}}(1 + (2 - s)(1 - s)t)^{-\frac{1}{2}}.$$

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$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} c(\alpha^{[n]}) = V_s^{c_2(\alpha)} W_s^{c_1(\alpha)^2} X_s^{\chi(\mathcal{O}_S)} Y_s^{c_1(\alpha)K_S} Z_s^{K_S^2}.$$

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They showed explicit expressions for Y_s , Z_s for $s \in \{-2, -1, 0, 1, 2\}$, and conjecture that Y_s , Z_s are algebraic functions for all $s \in \mathbb{Z}$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000
Verlinde formula for Hilbert schemes				

Consider the generating series $\sum_{n=0}^{\infty} w^n \chi(S^{[n]}, \mu(L) \otimes E^{\otimes r})$.

Theorem (Ellingsrud-G-Lehn)

For any $r \in \mathbb{Z}$, there exist $g_r, f_r, A_r, B_r \in \mathbb{Q}[[w]]$ such that for any $L \in Pic(S)$, we have

$$\sum_{n=0}^{\infty} w^n \chi(S^{[n]}, \mu(L) \otimes E^{\otimes r}) = g_r^{\chi(L)} f_r^{\frac{1}{2}\chi(\mathcal{O}_S)} A_r^{LK_S} B_r^{K_S^2}$$

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With the change of variables $w = v(1 + v)^{r^2-1}$, we have

$$g_r(w) = 1 + v, \quad f_r(w) = \frac{(1 + v)^{r^2}}{1 + r^2 v}.$$

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$$g_r(w) = 1 + v, \quad f_r(w) = \frac{(1 + v)^{r^2}}{1 + r^2 v}.$$

Serre duality implies $A_r = B_{-r}/B_r$ for all *r*. Furthermore, $A_r = B_r = 1$ for $r = 0, \pm 1$. In general the A_r , B_r are unknown.

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000		
Segre-Verlinde correspondence						
We be						

We have seen

$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} c(\alpha^{[n]}) = V_s^{c_2(\alpha)} W_s^{c_1(\alpha)^2} X_s^{\chi(\mathcal{O}_S)} Y_s^{c_1(\alpha)K_S} Z_s^{K_S^2}, \quad s = \mathsf{rk}(\alpha)$$
$$\sum_{n=0}^{\infty} w^n \chi(S^{[n]}, \mu(L) \otimes E^{\otimes r}) = g_r^{\chi(L)} f_r^{\frac{1}{2}\chi(\mathcal{O}_S)} A_r^{LK_S} B_r^{K_S^2},$$

with V_s , W_s , $X_s \in \mathbb{Q}[[z]]$, f_r , $g_r \in \mathbb{Q}[[w]]$ known algebraic functions, and Y_s , $Z_s \in \mathbb{Q}[[z]]$, A_r , $B_r \in \mathbb{Q}[[w]]$ unknown

Introduction o	Hilbert scheme of points ○○○○○●	Moduli spaces of sheaves	Algebraicity	Checks 00000000000	
Segre-Verlinde correspondence					

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$$\sum_{n=0}^{\infty} z^n \int_{S^{[n]}} c(\alpha^{[n]}) = V_s^{c_2(\alpha)} W_s^{c_1(\alpha)^2} X_s^{\chi(\mathcal{O}_S)} Y_s^{c_1(\alpha)K_S} Z_s^{K_S^2}, \quad s = \mathsf{rk}(\alpha)$$
$$\sum_{n=0}^{\infty} w^n \chi(S^{[n]}, \mu(L) \otimes E^{\otimes r}) = g_r^{\chi(L)} f_r^{\frac{1}{2}\chi(\mathcal{O}_S)} A_r^{LK_S} B_r^{K_S^2},$$

with V_s , W_s , $X_s \in \mathbb{Q}[[z]]$, f_r , $g_r \in \mathbb{Q}[[w]]$ known algebraic functions, and Y_s , $Z_s \in \mathbb{Q}[[z]]$, A_r , $B_r \in \mathbb{Q}[[w]]$ unknown Based on strange duality there is a conjectural relation between these two generating functions

Conjecture (Johnson, Marian-Oprea-Pandharipande)

For any $r \in \mathbb{Z}$, we have

$$A_r(w) = W_s(z) Y_s(z), \quad B_r(w) = Z_s(z),$$

where s = 1 - r and $w = v(1 + v)^{r^2 - 1}$, $z = t(1 + (1 - s)t)^{1 - s}$, and $v = t(1 + rt)^{-1}$.

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Moduli spaces of sheaves					

Aim: Find analogues of all these results for higher rank moduli spaces

Let (S, H) polarized surface. A torsion free coherent sheaf \mathcal{E} on S is called H-semistable, if for all subsheaves $\mathcal{F} \subset \mathcal{E}$, we have

$$\frac{\chi(\boldsymbol{\mathcal{S}}, \mathcal{F} \otimes \boldsymbol{H}^{\otimes n})}{\mathsf{rk}(\mathcal{F})} \leq \frac{\chi(\boldsymbol{\mathcal{S}}, \mathcal{E} \otimes \boldsymbol{H}^{\otimes n})}{\mathsf{rk}(\mathcal{E})}, \quad n \gg 0$$

For $\rho \in \mathbb{Z}_{>0}$, $c_1 \in H^2(S, \mathbb{Z})$, and $c_2 \in H^4(S, \mathbb{Z})$, let $M := M_S^H(\rho, c_1, c_2)$ moduli space of rank ρ *H*-semistable sheaves on *S* with Chern classes c_1, c_2

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For $\rho \in \mathbb{Z}_{>0}$, $c_1 \in H^2(S, \mathbb{Z})$, and $c_2 \in H^4(S, \mathbb{Z})$, let $M := M_S^H(\rho, c_1, c_2)$ moduli space of rank ρ *H*-semistable sheaves on *S* with Chern classes c_1, c_2 **Note:** via $Z \mapsto I_Z$, we have $S^{[n]} = M_S^H(1, 0, n)$. Assume *M* contains no strictly semistable sheaves For simplicity also assume there exists a universal sheaf \mathcal{E} on $S \times M$, (i.e. $\mathcal{E}|_{S \times \{[E]\}} = E$)

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Moduli spaces of she	aves			

For simplicity we assume in the following that $p_g(S) > 0$, $b_1(S) = 0$ and *S* has a smooth connected canonical divisor

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Moduli spaces of she	eaves			

For simplicity we assume in the following that $p_g(S) > 0$, $b_1(S) = 0$ and *S* has a smooth connected canonical divisor $M = M_S^H(\rho, c_1, c_2)$ has a perfect obstruction theory of expected dimension

$$vd(M) := 2\rho c_2 - (\rho - 1)c_1^2 - (\rho^2 - 1)\chi(\mathcal{O}_S)$$

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$$vd(M) := 2\rho c_2 - (\rho - 1)c_1^2 - (\rho^2 - 1)\chi(\mathcal{O}_S)$$

In particular

- it carries a virtual class $[M]^{\text{vir}} \in H_{2\text{vd}(M)}(M)$
- has a virtual Tangent bundle $T_M^{\text{vir}} \in K^0(M)$
- has a virtual structure sheaf O^{vir}_M ∈ K₀(S)
 For any V ∈ K⁰(M) the virtual holomorphic Euler characteristic of V is χ^{vir}(M, V) = χ(M, V ⊗ O^{vir}_M)

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Virtual Segre numbe	ers of moduli spaces			

For any class $\alpha \in K^0(S)$, we define

$$\mathrm{ch}(\alpha_{M}) := -\mathrm{ch}(\pi_{M!}(\pi_{S}^{*}\alpha \cdot \mathcal{E} \cdot \mathrm{det}(\mathcal{E})^{-\frac{1}{\rho}}))$$

On $M := M_S^H(1, 0, n) \cong S^{[n]}$, we have $\alpha_M = \alpha^{[n]}$

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ho}}))$$

On $M := M_S^H(1, 0, n) \cong S^{[n]}$, we have $\alpha_M = \alpha^{[n]}$ For any $\sigma \in H^k(S, \mathbb{Q})$ the μ -class of Donaldson theory is

$$\mu(\sigma) := \left(c_2(\mathcal{E}) - \frac{\rho - 1}{2\rho}c_1(\mathcal{E})^2\right) / \mathrm{PD}(\sigma) \in H^k(M, \mathbb{Q}),$$

For $\alpha \in K^0(S)$, $L \in H^2(S, \mathbb{Z})$, and $pt \in H^4(S, \mathbb{Z})$ the Poincaré dual of a point, the *virtual Segre number* of *M* is

$$\int_{[M]^{\mathrm{vir}}} \boldsymbol{c}(\alpha_M) \exp \left(\mu(\boldsymbol{L}) + \boldsymbol{u} \mu(\boldsymbol{\rho} t) \right) \in \mathbb{Q}[\boldsymbol{u}]$$

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Virtual Segre numbers of moduli spaces

For simplicity we assume in the following that $p_g(S) > 0$, $b_1(S) = 0$ and *S* has a smooth connected canonical divisor Write $\varepsilon_{\rho} := \exp(2\pi i/\rho)$ and $[n] := \{1, \ldots, n\}$. For any $J \subset [n]$, write |J|for its cardinality and $||J|| := \sum_{i \in J} j$

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Conjecture

Let $\rho \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$. There exist V_s , W_s , X_s , Q_s , R_s , $T_s \in \mathbb{C}[[z]]$, $Y_{J,s}$, $Z_{J,s}$, $S_{J,s} \in \mathbb{C}[[z^{\frac{1}{2}}]]$, s.th. for all $J \subset [\rho - 1]$ for all S as above, any $\alpha \in K^0(S)$ with $\mathrm{rk}(\alpha) = s$ and $L \in \mathrm{Pic}(S)$ we have that

$$\int_{[M_S^H(\rho,c_1,c_2)]^{\mathrm{vir}}} \boldsymbol{c}(\alpha_M) \exp(\mu(L) + u\,\mu(pt))$$

is the coefficient of $z^{\frac{1}{2}vd(M)}$ of

$$\rho^{2-\chi(\mathcal{O}_{S})+K_{S}^{c}} V_{s}^{c_{2}(\alpha)} W_{s}^{c_{1}(\alpha)^{2}} X_{s}^{\chi(\mathcal{O}_{S})} e^{L^{2}Q_{s}+(c_{1}(\alpha)L)R_{s}+uT_{s}} \\ \sum_{J \subset [\rho-1]} (-1)^{|J|\chi(\mathcal{O}_{S})} \varepsilon_{\rho}^{\|J\|K_{S}c_{1}} Y_{J,s}^{c_{1}(\alpha)K_{S}} Z_{J,s}^{K_{S}^{2}} e^{(K_{S}L)S_{J,s}}$$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 0000000000
Virtual Segre nur	nbers of moduli spaces			
Conje	ecture			
$\int_{[M]^{\mathrm{vir}}}$	$c(\alpha_M) \exp(\mu(L) + u \mu)$	(pt)) is the coefficient	of $z^{\frac{1}{2}vd(M)}$ of	
	$ ho^{2-\chi(\mathcal{O}_S)+\mathcal{K}_S^2} V_s^{c_2(lpha)}$	$W_s^{c_1(\alpha)^2} X_s^{\chi(\mathcal{O}_S)} e^{L^2 Q_s + (Q_s)}$	$c_1(\alpha)L)R_s+uT_s$	

 $\sum (-1)^{|J|\chi(\mathcal{O}_S)} \varepsilon_{\rho}^{\|J\|\mathcal{K}_S c_1} Y_{J,s}^{c_1(\alpha)\mathcal{K}_S} Z_{J,s}^{\mathcal{K}_S^2} e^{(\mathcal{K}_S L)S_{J,s}}.$

 $J \subset [\rho - 1]$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity 0000	Checks 0000000000
Virtual Segre nu	mbers of moduli spaces			
Conj	ecture			
$\int_{[M]^{\mathrm{vir}}}$	$c(\alpha_M) \exp(\mu(L) + u \mu)$	(pt)) is the coefficient	of $z^{\frac{1}{2}vd(M)}$ of	
	$\rho^{2-\chi(\mathcal{O}_S)+K_S^2} V_s^{c_2(\alpha)}$	$W_s^{c_1(\alpha)^2} X_s^{\chi(\mathcal{O}_S)} e^{L^2 Q_s + (Q_s)}$	$c_1(\alpha)L)R_s+uT_s$	
	$\sum_{J \in [n-1]} (-1)^{ J }$	$\chi(\mathcal{O}_{\mathcal{S}}) \in_{\rho}^{\ J\ K_{\mathcal{S}}c_{1}} Y_{J,s}^{c_{1}(\alpha)K_{\mathcal{S}}}$	$Z_{J,s}^{K_S^2} e^{(K_S L)S_{J,s}}.$	
With .	$z = t(1 + (1 - \frac{s}{\rho})t)^{1-1}$	$\frac{s}{\rho}$, we have		
$V_s(z)$	$=(1+(1-rac{s}{ ho})t)^{ ho-s}(1)$	$+\left(2-\frac{s}{ ho}\right)t ight)^{s},$		
$W_s(z)$	$t = (1 + (1 - \frac{s}{\rho})t)^{\frac{1}{2}(s-1-1)}$	$(1+(2-\frac{s}{\rho})t)^{\frac{1}{2}(1-s)},$		
$X_s(z)$	$t = (1 + (1 - \frac{s}{\rho})t)^{\frac{1}{2}(s^2 - t)}$	$(1 + (2 - \frac{s}{\rho})t)^{-\frac{1}{2}s^2}$	$+\frac{1}{2}(1+(1-\frac{s}{\rho}))$	$(2-\frac{s}{\rho})t)^{-\frac{1}{2}},$
$Q_s(z)$	$= \frac{1}{2}t(1+(1-\frac{s}{\rho})t),$	$R_s(z) = t, T_s(z) = \rho t(z)$	$1 + \frac{1}{2}(1 - \frac{s}{\rho})(2$	$(-\frac{s}{\rho})t).$
Furth	hermore, Y _{J,s} , Z _{J,s} , S	_{J,s} are all algebraic fui	nctions	

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Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 0000000000
Virtual Segre nui	mbers of moduli spaces			
Conje	ecture			
$\int_{[M]^{\mathrm{vir}}}$	$c(\alpha_M) \exp(\mu(L) + u \mu)$	(pt)) is the coefficient	of $z^{\frac{1}{2}vd(M)}$ of	
	$\rho^{2-\chi(\mathcal{O}_S)+\mathcal{K}_S^2} V_s^{c_2(\alpha)}$	$W_s^{c_1(\alpha)^2} X_s^{\chi(\mathcal{O}_S)} e^{L^2 Q_s + (Q_s)}$	$c_1(\alpha)L)R_s+uT_s$	
	$\sum_{J \subset [n-1]} (-1)^{ J }$	$\chi(\mathcal{O}_{\mathcal{S}}) \varepsilon_{\rho}^{\ J\ K_{\mathcal{S}}c_{1}} Y_{J,s}^{c_{1}(\alpha)K_{\mathcal{S}}}$	$Z_{J,s}^{K_S^2} e^{(K_S L)S_{J,s}}.$	
With 2	$z = t(1 + (1 - \frac{s}{\rho})t)^{1-1}$	$\frac{s}{\rho}$, we have		
$V_s(z)$	$= (1 + (1 - \frac{s}{\rho})t)^{\rho-s}(1$	$+\left(2-\frac{s}{ ho}\right)t)^{s},$		
$W_s(z)$	$=(1+(1-rac{s}{ ho})t)^{rac{1}{2}(s-1-r)}$	$(1+(2-\frac{s}{\rho})t)^{\frac{1}{2}(1-s)},$		
$X_s(z)$	$=(1+(1-\frac{s}{\rho})t)^{\frac{1}{2}(s^2-t)}$	$(1 + (2 - \frac{s}{\rho})t)^{-\frac{1}{2}s^2}$	$+\frac{1}{2}(1+(1-\frac{s}{\rho}))$	$(2-\frac{s}{\rho})t)^{-\frac{1}{2}},$
$Q_{s}(z)$	$= \frac{1}{2}t(1+(1-\frac{s}{\rho})t),$	$R_s(z) = t, T_s(z) = \rho t(z)$	$1 + \frac{1}{2}(1 - \frac{s}{\rho})(2$	$(-\frac{s}{\rho})t).$
Furth	nermore, Y _{J.s} , Z _{J.s} , S	_{J,s} are all algebraic fui	nctions	

The fact that $R_s(z) = t$ explains the variable change: *z* counts the virtual dimension; *t* counts $c_1(\alpha)L$ For $S^{[n]}$ we have $Y_s = Y_{\emptyset,s}$, $Z_s = Z_{\emptyset,s}$

troduction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks
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Virtual Verlinde formula

Determinant bundles: Let $c \in K(S)$ be the class of $E \in M = M_S^H(\rho, c_1, c_2)$ and $K_c := \{v \in K(S) : \chi(S, c \otimes v) = 0\}$ For $\alpha \in K_c$ put with $\pi_S : S \times M \to S, \pi_M : S \times M \to M$ projections

$$\lambda(\alpha) := \det \left(\pi_{\mathcal{M}!} \left(\pi_{\mathcal{S}}^* \alpha \cdot [\mathcal{E}] \right) \right)^{-1} \in \operatorname{Pic}(\mathcal{M})$$

roduction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks
		000000000		

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$$\lambda(\alpha) := \det \left(\pi_{M!} \left(\pi_{\mathcal{S}}^* \alpha \cdot [\mathcal{E}] \right) \right)^{-1} \in \operatorname{Pic}(M)$$

Fix $r \in \mathbb{Z}$, $L \in \operatorname{Pic}(S) \otimes \mathbb{Q}$ with $\mathcal{L} := L \otimes \det(c)^{-\frac{r}{\rho}} \in \operatorname{Pic}(S)$ take $v \in K_c$ such that $\operatorname{rk}(v) = r$ and $c_1(v) = \mathcal{L}$, put

$$\mu(L)\otimes E^{\otimes r}:=\lambda(v)\in \operatorname{Pic}(M).$$

On $M_S^H(1,0,n) \cong S^{[n]}$ this is previous definition of $\mu(L) \otimes E^{\otimes r}$ Relation to Donaldson μ class in cohom.: $\mu(c_1(L)) = c_1(\mu(L))$
roduction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks
		000000000		

Virtual Verlinde formula

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$$\lambda(\alpha) := \det \left(\pi_{M!} \left(\pi_{\mathcal{S}}^* \alpha \cdot [\mathcal{E}] \right) \right)^{-1} \in \operatorname{Pic}(M)$$

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$$\mu(L)\otimes E^{\otimes r}:=\lambda(\nu)\in \operatorname{Pic}(M).$$

On $M_S^H(1,0,n) \cong S^{[n]}$ this is previous definition of $\mu(L) \otimes E^{\otimes r}$ Relation to Donaldson μ class in cohom.: $\mu(c_1(L)) = c_1(\mu(L))$ Denote by $\mathcal{O}_M^{\text{vir}}$ the virtual structure sheaf of MThe *virtual Verlinde numbers* of S are the virtual holomorphic Euler characteristics

$$\chi^{\mathrm{vir}}(M,\mu(L)\otimes E^{\otimes r}):=\chi(M,\mu(L)\otimes E^{\otimes r}\otimes \mathcal{O}_M^{\mathrm{vir}})$$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000
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Virtual Verlinde formula

For simplicity we assume in the following that $p_g(S) > 0$, $b_1(S) = 0$ and *S* has a smooth connected canonical divisor Write $\varepsilon_{\rho} := \exp(2\pi i/\rho)$ and $[n] := \{1, \ldots, n\}$. For any $J \subset [n]$, write |J|for its cardinality and $||J|| := \sum_{j \in J} j$

ntroduction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000
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Conjecture

Let $\rho \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}$. There exist $A_{J,r}$, $B_{J,r} \in \mathbb{C}[[w^{\frac{1}{2}}]]$ for all $J \subset [\rho - 1]$ such that $\chi^{\text{vir}}(M, \mu(L) \otimes E^{\otimes r})$ equals the coefficient of $w^{\frac{1}{2}\text{vd}(M)}$ of

$$\rho^{2-\chi(\mathcal{O}_{\mathcal{S}})+K_{\mathcal{S}}^{2}} G_{r}^{\chi(L)} F_{r}^{\frac{1}{2}\chi(\mathcal{O}_{\mathcal{S}})} \sum_{J \subset [\rho-1]} (-1)^{|J|\chi(\mathcal{O}_{\mathcal{S}})} \varepsilon_{\rho}^{\|J\|K_{\mathcal{S}}c_{1}} A_{J,r}^{K_{\mathcal{S}}L} B_{J,r}^{K_{\mathcal{S}}^{2}}$$

ntroduction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000
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Virtual Verlinde formula

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Here $G_r(w) = 1 + v$, $F_r(w) = \frac{(1+v)^{\frac{r^2}{\rho^2}}}{1+\frac{r^2}{\rho^2}v}$ with $w = v(1+v)^{\frac{r^2}{\rho^2}-1}$ Furthermore, $A_{J,r}$, $B_{J,r}$ are all algebraic functions.

ntroduction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000
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Conjecture

Let $\rho \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}$. There exist $A_{J,r}$, $B_{J,r} \in \mathbb{C}[[w^{\frac{1}{2}}]]$ for all $J \subset [\rho - 1]$ such that $\chi^{\text{vir}}(M, \mu(L) \otimes E^{\otimes r})$ equals the coefficient of $w^{\frac{1}{2}\text{vd}(M)}$ of

$$\rho^{2-\chi(\mathcal{O}_{\mathcal{S}})+K_{\mathcal{S}}^{2}} G_{r}^{\chi(L)} F_{r}^{\frac{1}{2}\chi(\mathcal{O}_{\mathcal{S}})} \sum_{J \subset [\rho-1]} (-1)^{|J|\chi(\mathcal{O}_{\mathcal{S}})} \varepsilon_{\rho}^{\|J\|K_{\mathcal{S}}c_{1}} A_{J,r}^{K_{\mathcal{S}}L} B_{J,r}^{K_{\mathcal{S}}^{2}}.$$

Here $G_r(w) = 1 + v$, $F_r(w) = \frac{(1+v)^{\frac{r^2}{\rho^2}}}{1+\frac{r^2}{\rho^2}v}$ with $w = v(1+v)^{\frac{r^2}{\rho^2}-1}$ Furthermore, $A_{J,r}$, $B_{J,r}$ are all algebraic functions.

This conjecture is true for K3 surfaces

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000
Virtual Serre duality				

Virtual Serre duality $\chi^{\text{vir}}(M, L) = (-1)^{\text{vd}(M)}\chi^{\text{vir}}(M, K_M^{\text{vir}} \otimes L^{-1})$ gives

Conjecture

For any $\rho > 0$, we have

$$\frac{B_{J,-r}(w^{\frac{1}{2}})}{B_{J,r}(-w^{\frac{1}{2}})} = G_r(w)^{\binom{\rho}{2}} A_{J,r}(-w^{\frac{1}{2}})^{\rho}$$
$$A_{J,-r}(w^{\frac{1}{2}}) = \frac{1}{A_{J,r}(-w^{\frac{1}{2}})G_r(w)^{\rho-1}}$$

for all $J \subset [\rho - 1]$ and $r \in \mathbb{Z}$.

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000		
Virtual Segre-Verlind	Virtual Segre-Verlinde correspondence					

We get the following analogue of the Segre-Verlinde correspondence for Hilbert schemes

Conjecture

For any $\rho \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}$, for all $J \subset [\rho - 1]$, we have

$$A_{J,r}(w^{\frac{1}{2}}) = W_{
ho-r}(z) Y_{J,
ho-r}(z^{\frac{1}{2}}), \quad B_{J,r}(w^{\frac{1}{2}}) = Z_{J,
ho-r}(z^{\frac{1}{2}}),$$

with

$$w = v(1+v)^{\frac{r^2}{\rho^2}-1}, \quad z = t(1+(1-\frac{s}{\rho})t)^{1-\frac{s}{\rho}}, \quad v = t(1+\frac{r}{\rho}t)^{-1}.$$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity ●○○○	Checks 00000000000
Algebraicity				

We conjecturally determined the $Y_{J,s}$, $Z_{J,s}$, $S_{J,s}$ as algebraic functions for $\rho = 2$, s = -1, ..., 5, $\rho = 3$, s = 0, ..., 6, and $\rho = 4$, s = 0, 4, and the $A_{J,s}$, $B_{J,s}$ corresponding to them under the Segre-Verlinde correspondence

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity ●○○○	Checks 00000000000
Algebraicity				

We conjecturally determined the $Y_{J,s}$, $Z_{J,s}$, $S_{J,s}$ as algebraic functions for $\rho = 2$, $s = -1, \ldots, 5$, $\rho = 3$, $s = 0, \ldots, 6$, and $\rho = 4$, s = 0, 4, and the $A_{J,s}$, $B_{J,s}$ corresponding to them under the Segre-Verlinde correspondence Below we list these functions for $M_S^H(\rho, c_1, c_2)$, and $rk(\alpha) = s$ with

 $\rho = 2, s = 1, -1$ $\rho = 3, s = 1, \rho = 4, s = 0.$

o O	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000
Rank $\rho =$	2			
for W	T = Y, Z, S we have	$W_{\emptyset,s}(-z^{\frac{1}{2}}) = W_{\{1\}}$	$s(Z^{\frac{1}{2}})$	

We write $W_s := W_{\emptyset,s}$

They are solutions of

$$t^{4}x^{4} - 2t^{2}(1+2t)x^{3} + (1+\frac{3}{2}t)^{2}x^{2} - 2(1+2t)x + 1 = 0$$

$$y^{4} - 2(1+\frac{15}{4}t)y^{3} + (1+\frac{5}{2}t)(1+\frac{15}{4}t)y^{2} - t(1-\frac{5}{2}t)(1+\frac{15}{4}t)^{2} = 0$$

Introduction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000
Rank $\rho = 3$				

s=1: For $z = t(1 + \frac{2}{3}t)^{\frac{2}{3}}$, the power series $S_{\emptyset,1}$, $S_{\{1,2\},1}$, $S_{\{1\},1}$, $S_{\{2\},1}$ are the four solutions of

$$x^{4} + 2tx^{3} - (3t + t^{2})x^{2} - (3t^{2} + 2t^{3})x - (t^{3} + \frac{2}{3}t^{4}) = 0.$$

 $Y_{\varnothing,1},\;Y_{\{1,2\},1},\;Y_{\{1\},1},\;Y_{\{2\},1}$ are the four solutions of

$$\begin{aligned} x^4 - (4 + \frac{17}{3}t)(1 + \frac{2}{3}t)^{\frac{1}{2}}x^3 + (6 + 18t + 16t^2 + \frac{31}{9}t^3)x^2 \\ - (4 + \frac{17}{3}t)(1 + \frac{2}{3}t)^{\frac{7}{2}}x + (1 + \frac{2}{3}t)^6 = 0. \end{aligned}$$

 $Z_{\varnothing,1},$ $Z_{\{1,2\},1},$ $Z_{\{1\},1},$ $Z_{\{2\},1}$ are the four solutions of

$$\begin{aligned} x^4 - 6 \frac{1 + \frac{10}{9}t}{1 + \frac{2}{3}t} x^3 + \frac{(13 + \frac{58}{3}t + \frac{55}{9}t^2)(1 + \frac{10}{9}t)}{(1 + \frac{2}{3}t)^3} x^2 \\ + \frac{(4 + \frac{5}{3}t)(1 + \frac{10}{9}t)^2}{(1 + \frac{2}{3}t)^3} (-3x + 1) = 0. \end{aligned}$$

Introduction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks
			0000	
Rank $\rho = 4$				

s=0: For z = t(1 + t), we conjecturally have

$$\begin{split} S_{\varnothing,0} &= (1+2^{\frac{1}{2}})t^{\frac{1}{2}}(1+t)^{\frac{1}{2}}, \quad S_{\{1\},0} = t^{\frac{1}{2}}(1+t)^{\frac{1}{2}}, \\ Z_{\varnothing,0} &= 2(2+2^{\frac{1}{2}}), \quad Z_{\{1\},0} = 2, \\ Y_{\varnothing,0} &= \frac{(1+t)^4((1+t)^{\frac{1}{2}}+t^{\frac{1}{2}})}{(1+2t)^{\frac{1}{2}}((1+2t)-2^{\frac{1}{2}}t^{\frac{1}{2}}(1+t)^{\frac{1}{2}})}, \\ Y_{\{1\},0} &= \frac{(1+t)^4((1+t)^{\frac{1}{2}}+t^{\frac{1}{2}})}{(1+2t)^{\frac{1}{2}}(1+(1-i)t)}. \end{split}$$

The other power series are obtained by $z^{\frac{1}{2}} \mapsto -z^{\frac{1}{2}}, 2^{\frac{1}{2}} \mapsto -2^{\frac{1}{2}}$ and $i \mapsto -i$

Introduction O	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ●0000000000
Mochizuki formula				

Main tool: Mochizuki's formula:

Compute intersection numbers on $M = M_S^H(\rho, c_1, c_2)$ in terms of intersection numbers on Hilbert scheme of points.

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ●oooooooooo
Mochizuki formula				

Main tool: Mochizuki's formula:

Compute intersection numbers on $M = M_S^H(\rho, c_1, c_2)$ in terms of intersection numbers on Hilbert scheme of points.

On $S \times M$ have \mathcal{E} universal sheaf i.e. if $[E] \in M$ corresponds to a sheaf E on S then $\mathcal{E}|_{S \times [E]} = E$. For $\alpha \in H^k(S)$, put

$$au_i(lpha) := \pi_{M_*}(c_i(\mathcal{E})\pi^*_{\mathcal{S}}(lpha)) \in H^{2i-4+k}(M)$$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ●oooooooooo
Mochizuki formula				

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On $S \times M$ have \mathcal{E} universal sheaf i.e. if $[E] \in M$ corresponds to a sheaf E on S then $\mathcal{E}|_{S \times [E]} = E$. For $\alpha \in H^k(S)$, put

$$\tau_i(\alpha) := \pi_{M_*}(c_i(\mathcal{E})\pi^*_{\mathcal{S}}(\alpha)) \in H^{2i-4+k}(M)$$

Let $P(\mathcal{E})$ be any polynomial in the $\tau_i(\alpha)$ Mochizuki's formula expresses $\int_{[M]^{\text{vir}}} P(\mathcal{E})$ in terms of intersec. numbers on $S^{[n_1]} \times S^{[n_2]} \times \ldots \times S^{[n_{\rho}]}$, and Seiberg-Witten invariants.

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○●○○○○○○○○
Mochizuki formula				

$$\int_{[M]^{\mathrm{vir}}} c(\alpha_M) \exp(\mu(L) + u\mu(\rho t)), \quad \chi_{-y}^{\mathrm{vir}}(M, \mu(L) + E^{\otimes r})$$

can both be expressed as $\int_{[M]^{vir}} P(\mathcal{E})$, for suitable polyn. *P*, so can reduce computation to Hilbert schemes.

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○●○○○○○○○○
Mochizuki formula				

$$\int_{[M]^{\mathrm{vir}}} c(\alpha_M) \exp(\mu(L) + u\mu(\rho t)), \quad \chi^{\mathrm{vir}}_{-y}(M, \mu(L) + E^{\otimes r})$$

can both be expressed as $\int_{[M]^{vir}} P(\mathcal{E})$, for suitable polyn. *P*, so can reduce computation to Hilbert schemes.

For $\chi_{-\nu}^{\text{vir}}(M, \mu(L) + E^{\otimes r})$ use virtual Riemann-Roch formula

Theorem (Fantechi-G., Kapronov Ciocan-Fontanine)

For $V \in K^0(M)$ have

$$\chi^{\mathrm{vir}}(\boldsymbol{M},\boldsymbol{V}) = \int_{[\boldsymbol{M}]^{\mathrm{vir}}} \mathrm{ch}(\boldsymbol{V}) \mathrm{td}(\boldsymbol{T}_{\boldsymbol{M}}^{\mathrm{vir}}).$$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks oo●ooooooo
Mochizuki formula				

Seiberg-Witten invariants:

invariants of differentiable 4-manifolds *S* projective algebraic surface $H^2(S, \mathbb{Z}) \ni a \mapsto SW(a) \in \mathbb{Z}$, *a* is called SW class if $SW(a) \neq 0$.

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks oo●ooooooo
Mochizuki formula				

Seiberg-Witten invariants:

invariants of differentiable 4-manifolds *S* projective algebraic surface $H^2(S, \mathbb{Z}) \ni a \mapsto SW(a) \in \mathbb{Z}$, *a* is called SW class if $SW(a) \neq 0$.

In general for alg. surfaces they are easy to compute, e.g. if $b_1(S) = 0$, $p_g(S) > 0$ and $|K_S|$ contains smooth connected curve, then SW cl. of *S* are 0, K_S with

$$SW(0) = 1$$
, $SW(K_S) = (-1)^{\chi(\mathcal{O}_S)}$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks
Mochizuki formula				

Seiberg-Witten invariants:

invariants of differentiable 4-manifolds *S* projective algebraic surface $H^2(S, \mathbb{Z}) \ni a \mapsto SW(a) \in \mathbb{Z}$, *a* is called SW class if $SW(a) \neq 0$.

In general for alg. surfaces they are easy to compute, e.g. if $b_1(S) = 0$, $p_g(S) > 0$ and $|K_S|$ contains smooth connected curve, then SW cl. of *S* are 0, K_S with

$$SW(0) = 1$$
, $SW(K_S) = (-1)^{\chi(\mathcal{O}_S)}$

This is the reason for assumption $|K_S|$ contains smooth connected curve, otherwise our results look more complicated: They are expressed in terms of the Seiberg-Witten inv. of *S*

ntroduction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 0000000000
Mochizuki formula				
For sim comput	plicity look at case te $\int_{[M]^{vir}} c(L_M)$.	$\rho = 2, s = 1, \alpha =$	$L \in \operatorname{Pic}(S)$ a	Ind

 $S^{[n_1]} \times S^{[n_2]} = \{ \text{pairs } (Z_1, Z_2) \text{ of subsch. of deg. } (n_1, n_2) \text{ on } S \}$

Work on $S \times S^{[n_1]} \times S^{[n_2]}$, projection *p* to $S^{[n_1]} \times S^{[n_2]}$

ntroduction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○●○○○○○○
Nochizuki formula				

For simplicity look at case $\rho = 2$, s = 1, $\alpha = L \in \text{Pic}(S)$ and compute $\int_{[M]^{\text{vir}}} c(L_M)$.

 $S^{[n_1]} \times S^{[n_2]} = \{ \text{pairs } (Z_1, Z_2) \text{ of subsch. of deg. } (n_1, n_2) \text{ on } S \}$

Work on $S \times S^{[n_1]} \times S^{[n_2]}$, projection *p* to $S^{[n_1]} \times S^{[n_2]}$ Two universal sheaves: Let $a \in Pic(S)$

- $\mathcal{I}_i(a)$ sheaf on $S \times S^{[n_1]} \times S^{[n_2]}$ with $\mathcal{I}_i(a)|_{S \times (Z_1, Z_2)} = I_{Z_i} \otimes a$
- ② $\mathcal{O}_i(a)$, vector bundle of rank n_i on $S^{[n_1]} \times S^{[n_2]}$, with fibre $\mathcal{O}_i(a)(Z_1, Z_2) = H^0(\mathcal{O}_{Z_i} \otimes a)$

For simplicity look at case $\rho = 2$, s = 1, $\alpha = L \in \text{Pic}(S)$ and compute $\int_{[M]^{\text{vir}}} c(L_M)$.

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For a vector bundle E of rank r and variable s put

$$c_i(E\otimes s) = \sum_{k=0}^i {r-i \choose k} s^{i-k} c_k(E), \quad Eu(E) = c_r(E)$$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks 00000000000
Mochizuki formula				

For sheaves $\mathcal{E}_1, \mathcal{E}_2$ on $\mathcal{S} \times \mathcal{S}^{[n_1]} \times \mathcal{S}^{[n_2]}$ put

$$Q(\mathcal{E}_1, \mathcal{E}_2) = Eu(-RHom_p(\mathcal{E}_1, \mathcal{E}_2) - RHom_p(\mathcal{E}_2, \mathcal{E}_1))$$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○●○○○○○○
Mochizuki formula	a			
For sh	neaves $\mathcal{E}_1, \mathcal{E}_2$ on $\boldsymbol{S} imes$.	$\mathcal{S}^{[n_1]} imes \mathcal{S}^{[n_2]}$ put		
	$Q(\mathcal{E}_1,\mathcal{E}_2) = Eu(-$	$RHom_p(\mathcal{E}_1,\mathcal{E}_2)-RHom_p(\mathcal{E}_1,\mathcal{E}_2)$	$om_p(\mathcal{E}_2,\mathcal{E}_1))$	
For a ₁	$, a_2 \in \operatorname{Pic}(S)$ put			
Ψ(<i>a</i> ,	$a_2 (n_1, n_2, s) = \frac{P(\mathcal{I})}{P(\mathcal{I})}$	$\mathcal{I}_1(a_1) \otimes s^{-1} \oplus \mathcal{I}_2(a_2) \otimes s^{-1}$	$E Eu(\mathcal{O}_1(a_1))E$	$\exists u(\mathcal{O}_2(a_2)\otimes s^2)$
• (u),	a ₂ , <u>,</u> , ,, ,, ,, ,, ,, ,, ,, ,, ,, ,, ,, ,,	$Q(\mathcal{I}_1(a_1)\otimes s^{-1},\mathcal{I}_2(a_1))$	$_2)\otimes s)\cdot (2s)^n$	$_1+n_2-\chi(\mathcal{O}_S)$

 $A(a_1, a_2, L, c_2, s) = \sum_{n_1 + n_2 = c_2 - a_1 a_2} \int_{S^{[n_1]} \times S^{[n_2]}} \Psi(a_1, a_2, n_1, n_2, s) \in \mathbb{Q}[s, s^{-1}]$

Introduction O	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○●○○○○○○
Mochizuki formula	a			
For sh	neaves $\mathcal{E}_1, \mathcal{E}_2$ on $oldsymbol{S} imes$	$\mathcal{S}^{[n_1]} imes \mathcal{S}^{[n_2]}$ put		
	$Q(\mathcal{E}_1,\mathcal{E}_2)=Eu(-$	$RHom_p(\mathcal{E}_1,\mathcal{E}_2)-RHo$	$om_p(\mathcal{E}_2,\mathcal{E}_1))$	
For a ₁	$,a_2\in\operatorname{Pic}(\mathcal{S})$ put			
Ψ(<i>a</i> ,	$a_{2} (n_{1} n_{2} s) = \frac{P(\mathcal{I})}{P(\mathcal{I})}$	$a_1(a_1) \otimes s^{-1} \oplus \mathcal{I}_2(a_2) \otimes s^{-1}$) <i>Eu</i> (<i>O</i> ₁ (<i>a</i> ₁)) <i>E</i>	$\exists u(\mathcal{O}_2(a_2)\otimes s^2)$
f(u),	a ₂ , <u>-</u> ,, ₂ , 0) –	$Q(\mathcal{I}_1(a_1)\otimes s^{-1},\mathcal{I}_2(a_1)\otimes s^{-1})$	$_2)\otimes s)\cdot (2s)^n$	$h_1+n_2-\chi(\mathcal{O}_S)$

$$A(a_1, a_2, L, c_2, s) = \sum_{n_1+n_2=c_2-a_1a_2} \int_{S^{[n_1]} \times S^{[n_2]}} \Psi(a_1, a_2, n_1, n_2, s) \in \mathbb{Q}[s, s^{-1}]$$

In our case $\int_{[M]^{\text{vir}}} c(L_M)$, we have essentially $P(\mathcal{I}_1(a_1) \otimes s^{-1} \oplus \mathcal{I}_2(a_2) \otimes s) = c(\mathcal{O}_1((a_1 - a_2)/2 + L) \otimes s^{-1})c(\mathcal{O}_2((a_2 - a_1)/2 + L) \otimes s)$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○●○○○○○○
Mochizuki formula	a			
For sh	neaves $\mathcal{E}_1, \mathcal{E}_2$ on $oldsymbol{S} imes$	$\mathcal{S}^{[n_1]} imes \mathcal{S}^{[n_2]}$ put		
	$Q(\mathcal{E}_1,\mathcal{E}_2)=Eu(-$	$-RHom_p(\mathcal{E}_1,\mathcal{E}_2)-RH$	$om_p(\mathcal{E}_2,\mathcal{E}_1))$	
For a ₁	$, a_2 \in \operatorname{Pic}(S)$ put			
	- (-			

 $\Psi(a_{1}, a_{2}, L, n_{1}, n_{2}, s) = \frac{P(\mathcal{I}_{1}(a_{1}) \otimes s^{-1} \oplus \mathcal{I}_{2}(a_{2}) \otimes s) Eu(\mathcal{O}_{1}(a_{1})) Eu(\mathcal{O}_{2}(a_{2}) \otimes s^{2})}{Q(\mathcal{I}_{1}(a_{1}) \otimes s^{-1}, \mathcal{I}_{2}(a_{2}) \otimes s) \cdot (2s)^{n_{1}+n_{2}-\chi(\mathcal{O}_{S})}}$ $A(a_{1}, a_{2}, L, c_{2}, s) = \sum_{n_{1}+n_{2}=c_{2}-a_{1}a_{2}} \int_{S^{[n_{1}]} \times S^{[n_{2}]}} \Psi(a_{1}, a_{2}, n_{1}, n_{2}, s) \in \mathbb{Q}[s, s^{-1}]$

In our case $\int_{[M]^{\text{vir}}} c(L_M)$, we have essentially $P(\mathcal{I}_1(a_1) \otimes s^{-1} \oplus \mathcal{I}_2(a_2) \otimes s) = c(\mathcal{O}_1((a_1 - a_2)/2 + L) \otimes s^{-1})c(\mathcal{O}_2((a_2 - a_1)/2 + L) \otimes s)$

Theorem (Mochizuki)

Assume $\chi(E) > 0$ for $E \in M^S_H(c_1, c_2)$. Then

$$\int_{[M_{S}^{H}(c_{1},c_{2})]^{\text{vir}}} P(\mathcal{E}) = \sum_{\substack{c_{1}=a_{1}+a_{2}\\a_{1}H < a_{2}H}} SW(a_{1}) \text{Coeff}_{s^{0}} A(a_{1},a_{2},L,c_{2},s)$$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○○●○○○○○
Universality				

Universality: Put

$$Z_{S}(a_{1}, a_{2}, L, s, q) = \sum_{n_{1}, n_{2} \geq 0} \int_{S^{[n_{1}]} \times S^{[n_{2}]}} A(a_{1}, a_{2}, L, a_{1}a_{2} + n_{1} + n_{2}, s)q^{n_{1} + n_{2}}$$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○●○○○○○
Universality				

Universality: Put

$$Z_{\mathcal{S}}(a_1, a_2, L, s, q) = \sum_{n_1, n_2 \ge 0} \int_{\mathcal{S}^{[n_1]} \times \mathcal{S}^{[n_2]}} A(a_1, a_2, L, a_1a_2 + n_1 + n_2, s) q^{n_1 + n_2}$$

Proposition

There exist univ. functions $A_1(s,q), \ldots, A_{11}(s,q) \in \mathbb{Q}[s,s^{-1}][[q]]$ s.th. $\forall_{S,a_1,a_2,L}$

$$Z_{S}(a_{1}, a_{2}, L, s, q) = F_{0}(a_{1}, a_{2}, L, s) A_{1}^{a_{1}^{2}} A_{2}^{a_{1}a_{2}} A_{3}^{a_{2}^{2}} A_{4}^{a_{1}K_{S}} A_{5}^{a_{2}K_{S}} A_{6}^{K_{S}^{2}} A_{7}^{\chi(\mathcal{O}_{S})} \\ \cdot A_{8}^{L^{2}} A_{9}^{LK_{S}} A_{10}^{La_{1}} A_{11}^{La_{2}},$$

(where $F_0(a_1, a_2, L, s)$ is some explicit elementary function).

Proof: Modification of the cobordism argument for Hilbert schemes of points

ntroduction Hilbert scheme of points		Moduli spaces of sheaves	Algebraicity	Checks ○○○○○●○○○○
Reduction to \mathbb{P}^2 and	$d \mathbb{P}^1 \times \mathbb{P}^1.$			
A ₁ (s, c) for 11 corres	$(y), \dots A_{11}(s, q)$ are triples (S, a_1, a_2, L) ponding 11-tuples	e determ. by value of) (<i>S</i> surface, <i>a</i> ₁ , <i>a</i> ₂ ,	f $Z_{\mathcal{S}}(a_1, a_2, L$ $L \in \operatorname{Pic}(\mathcal{S}))$ s	., <i>s</i> , <i>q</i>) s.th.

 $(a_1^2, a_1a_2, a_2^2, a_1K_S, a_1K_S, K_S^2, \chi(O_S)), L^2, LK_S, La_1, La_1)$

are linearly independent.

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○○● ○ ○○○
Reduction to \mathbb{P}^2 a	and $\mathbb{P}^1 \times \mathbb{P}^1$.			
A.(s	a) $A_{11}(s, a)$ ar	e determ by value of	fZ _o (a ₁ a ₀ l	s a)

for 11 triples (S, a_1, a_2, L) (S surface, $a_1, a_2, L \in Pic(S))$ s.th. corresponding 11-tuples

 $\begin{aligned} &(a_1^2, a_1a_2, a_2^2, a_1K_S, a_1K_S, K_S^2, \chi(\mathcal{O}_S)), L^2, LK_S, La_1, La_1) \\ \text{are linearly independent. We take} \\ &(\mathbb{P}^2, \mathcal{O}, \mathcal{O}, \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}, \mathcal{O}), (\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}, \mathcal{O}), (\mathbb{P}^2, \mathcal{O}, \mathcal{O}(1), \mathcal{O}), \\ &(\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}(1), \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 0), \mathcal{O}, \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}) \\ &(\mathbb{P}^2, \mathcal{O}, \mathcal{O}, \mathcal{O}(1)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}, \mathcal{O}(1, 0)), (\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}, \mathcal{O}(1)), \\ &(\mathbb{P}^2, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(1)), \end{aligned}$

Introduction o	Hilbert schei 000000	me of points	Mo	oduli spaces of	sheaves	A	lgebraicity	0	Checks ○○○○○●○○○○
Reduction to \mathbb{P}^2 and	$\mathbb{P}^1 \times \mathbb{P}^1.$								
	· -								

 $A_1(s,q), \ldots A_{11}(s,q)$ are determ. by value of $Z_S(a_1, a_2, L, s, q)$ for 11 triples (S, a_1, a_2, L) (S surface, $a_1, a_2, L \in Pic(S)$) s.th. corresponding 11-tuples

 $(a_1^2, a_1a_2, a_2^2, a_1K_S, a_1K_S, K_S^2, \chi(O_S)), L^2, LK_S, La_1, La_1)$

are linearly independent. We take

$$\begin{split} (\mathbb{P}^2,\mathcal{O},\mathcal{O},\mathcal{O}), (\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O},\mathcal{O},\mathcal{O}), (\mathbb{P}^2,\mathcal{O}(1),\mathcal{O},\mathcal{O}), (\mathbb{P}^2,\mathcal{O},\mathcal{O}(1),\mathcal{O}), \\ (\mathbb{P}^2,\mathcal{O}(1),\mathcal{O}(1),\mathcal{O}), (\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(1,0),\mathcal{O},\mathcal{O}), (\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O},\mathcal{O}(1,0),\mathcal{O}) \\ (\mathbb{P}^2,\mathcal{O},\mathcal{O},\mathcal{O}(1)), (\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O},\mathcal{O},\mathcal{O}(1,0)), (\mathbb{P}^2,\mathcal{O}(1),\mathcal{O},\mathcal{O}(1)), \\ (\mathbb{P}^2,\mathcal{O},\mathcal{O}(1),\mathcal{O}(1)), \end{split}$$

In this case S is a smooth toric, i.e. have an action of

 $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints,

Action of *T* lifts to action on $S^{[n]}$ still with finitely many fixpoints described by partitions, compute by equivariant localization. This computes $Z_S(a_1, a_2, L, s, q)$ in terms of combinatorics of partitions.

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○○○●○○○
Reduction to \mathbb{P}^2 and	$\mathbb{P}^1 \times \mathbb{P}^1.$			

We determined $Z_S(a_1, a_2, \ldots, a_\rho, \alpha, L, s, q)$

- for $\rho = 2 \mod q^{11}$
- for $\rho = 3 \mod q^9$
- for $\rho = 4 \mod q^8$

This shows the conjectures e.g. for the blowup of a K3 surface in a point for

- for $\rho = 2$ up to virtual dimension 16
- for $\rho = 3$ up to virtual dimension 14
- for $\rho = 4$ up to virtual dimension 6

Introduction O	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks		
Equivariant localization						

Let *X* be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, p_1, \ldots, p_e Let *E* be equivariant vector bundle of rank *r* on *X*.
Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○○○○●○○
Equivariant localizati	on			

Let *X* be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, p_1, \ldots, p_e

Let *E* be equivariant vector bundle of rank r on *X*.

Fibre $E(p_i)$ of X at fixp. p_i has basis of eigenvect. for T-action $E(p_i) = \bigoplus_{k=1}^r \mathbb{C}v_i$, with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○○○●○○	
Equivariant localization					

Let *X* be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, p_1, \ldots, p_e Let *E* be equivariant vector bundle of rank *r* on *X*.

Fibre $E(p_i)$ of X at fixp. p_i has basis of eigenvect. for T-action $E(p_i) = \bigoplus_{k=1}^r \mathbb{C}v_i$, with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$

Equivariant chern class of fibre at fixpoint:

$$c^{\mathsf{T}}(E(p_i)) = (1 + c_1^{\mathsf{T}}(E(p_i)) + \ldots + c_r^{\mathsf{T}}(E(p_i))) = \prod_{i=1}^{\mathsf{T}} (1 + n_i \epsilon_1 + m_i \epsilon_2) \in \mathbb{Z}[\epsilon_1, \epsilon_2]$$

r

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○○○○●○○
Equivariant localization	on			

Let *X* be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, p_1, \ldots, p_e

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Fibre $E(p_i)$ of X at fixp. p_i has basis of eigenvect. for T-action $E(p_i) = \bigoplus_{k=1}^r \mathbb{C}v_i$, with action $(t_1, t_2) \cdot v_i = t_1^{n_i} t_2^{m_i} v_i$, $n_i, m_i \in \mathbb{Z}$

Equivariant chern class of fibre at fixpoint:

$$c^{\mathsf{T}}(\mathsf{E}(p_i)) = (1 + c_1^{\mathsf{T}}(\mathsf{E}(p_i)) + \ldots + c_r^{\mathsf{T}}(\mathsf{E}(p_i))) = \prod_{i=1}^{\mathsf{T}} (1 + n_i \epsilon_1 + m_i \epsilon_2) \in \mathbb{Z}[\epsilon_1, \epsilon_2]$$

Let P(c(E)))polynomial in Chern classes of E, of degree $d = \dim(X)$

Theorem (Bott residue formula)

$$\int_{[X]} P(c(E)) = \sum_{k=1}^{e} \frac{P(c^{T}(E(p_k)))}{c_d^{T}(T_X(p_k))}$$

(does not depend on ϵ_1, ϵ_2)

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○○○○○●○			
Equivariant localization							
For s	implicity $S = \mathbb{P}^2$. T	$=\mathbb{C}^* imes\mathbb{C}^*$ acts on I	₽ ² by				

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

Fixpoints are $p_0 = (1, 0, 0)$, $p_1 = (0, 1, 0)$, $p_2 = (0, 0, 1)$.

Introduction O	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○○○○○●○
Equivariant localizati	on			

For simplicity $S = \mathbb{P}^2$. $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on \mathbb{P}^2 by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

Fixpoints are $p_0 = (1, 0, 0)$, $p_1 = (0, 1, 0)$, $p_2 = (0, 0, 1)$. Local (equivariant) coordinates near p_0 are $x = \frac{X_1}{X_0}$, $y = \frac{X_2}{X_0}$, *T* action $(t_1, t_2)(x, y) = (t_1 x, t_2 y)$, similar for the p_1, p_2

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○○○○○●○
Equivariant localizati	on			

For simplicity $S = \mathbb{P}^2$. $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on \mathbb{P}^2 by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

Fixpoints are $p_0 = (1, 0, 0)$, $p_1 = (0, 1, 0)$, $p_2 = (0, 0, 1)$. Local (equivariant) coordinates near p_0 are $x = \frac{X_1}{X_0}$, $y = \frac{X_2}{X_0}$, T action $(t_1, t_2)(x, y) = (t_1 x, t_2 y)$, similar for the p_1, p_2 $Z \in (\mathbb{P}^2)^{[n]}$ is T-invariant $\Longrightarrow Z = Z_0 \sqcup Z_1 \sqcup Z_2 \quad supp(Z_i) = p_i$. \Longrightarrow Reduce to case $supp(Z) = p_i$, e.g. p_0

Introduction O	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○○○○○●○
Equivariant localizati	on			

For simplicity $S = \mathbb{P}^2$. $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on \mathbb{P}^2 by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2)$$

Fixpoints are $p_0 = (1, 0, 0)$, $p_1 = (0, 1, 0)$, $p_2 = (0, 0, 1)$. Local (equivariant) coordinates near p_0 are $x = \frac{X_1}{X_0}$, $y = \frac{X_2}{X_0}$, T action $(t_1, t_2)(x, y) = (t_1x, t_2y)$, similar for the p_1, p_2 $Z \in (\mathbb{P}^2)^{[n]}$ is T-invariant $\Longrightarrow Z = Z_0 \sqcup Z_1 \sqcup Z_2 \quad supp(Z_i) = p_i$. \Longrightarrow Reduce to case $supp(Z) = p_i$, e.g. p_0 Easy: Z is T-invariant $\iff I_Z \in k[x, y]$ is gen. by monomials Can write

$$I_Z = (y^{n_0}, xy^{n_1}, ..., x^r y^{n_r}, x^{r+1})$$
 $(n_0, ..., n_r)$ partition of n

Fixpoints on $(\mathbb{P}^2)^{[n]}$ are in bijections with triples (P_0, P_1, P_2) of partitions of 3 numbers adding up to *n*.

Introduction o	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity 0000	Checks ○○○○○○○○○●		
Equivariant localization						
Need to compute things like $c(\mathcal{O}^{[n]})$ $\mathcal{O}^{[n]}$ vector bundle on $(\mathbb{P}^2)^{[n]}$ with fibre $\mathcal{O}^{[n]}(Z) = H^0(\mathcal{O}_Z)$						

Introdu o	uction	Hilbert scheme of points	Moduli spaces of sheaves	Algebraicity	Checks ○○○○○○○○○●
Equiva	ariant localizati	on			
	Need to $\mathcal{O}^{[n]}$ vec	compute things like α tor bundle on $(\mathbb{P}^2)^{[n]}$ of $U = Z_1 \sqcup Z_2$, $supp(Z_i)$	$\mathcal{C}(\mathcal{O}^{[n]})$ with fibre $\mathcal{O}^{[n]}(Z)=H^0$ $= oldsymbol{p}_i,$ then	$\mathcal{O}(\mathcal{O}_Z)$	
		$\mathcal{O}^{[n]}(Z) = \mathcal{O}^{[n_0]}(Z)$ $c^T(\mathcal{O}^{[n]}(Z)) = c^T(\mathcal{O}^{[n]}(Z))$	$(Z_0) \oplus \mathcal{O}^{[n_1]}(Z_1) \oplus \mathcal{O}^{[n_2]}(Z_0)) c^T (\mathcal{O}^{[n_1]}(Z_1)) c^T$	$(\mathcal{O}^{[n_2]}(Z_2))$	

Introduction	Hilbert scheme of points	Moduli spaces of sheave	s Algebraicity	Checks ○○○○○○○○●
Equivariant loc	alization			
Need $\mathcal{O}^{[n]}$ If Z :	d to compute things like vector bundle on (\mathbb{P}^2) = $Z_0 \sqcup Z_1 \sqcup Z_2$, supp(2)	$ke \; m{c}(\mathcal{O}^{[n]}) \ {}^{[n]} \; ext{with fibre } \mathcal{O}^{[n]}(Z_i) = m{ ho}_i, ext{ then }$	$H^{0}(\mathcal{O}_{Z})$	
	$\mathcal{O}^{[n]}(Z) = \mathcal{O}^{[n]}(Z)$ $c^{T}(\mathcal{O}^{[n]}(Z)) = c^{T}(C)$	$\mathcal{O}^{[n_0]}(Z_0)\oplus \mathcal{O}^{[n_1]}(Z_1)\oplus \mathcal{O}^{[n_0]}(Z_0))c^{\mathcal{T}}(\mathcal{O}^{[n_1]}(Z_0))$	$\mathcal{O}^{[n_2]}(Z_2)$ $Z_1))c^T(\mathcal{O}^{[n_2]}(Z_2))$	
Let e Ther Thus	e.g. $Z = Z_0$, $I_Z = (y^4,)$ In the fibre $\mathcal{O}^{[n]}(Z) = F$ is basis of eigenvectors	$xy^2, x^2y, x^3)$ $f^0(\mathcal{O}_Z) = \mathbb{C}[x, y]/(x)$ s of fibre for T actions	y^4, xy^2, x^2y, x^3) on is	
	1 y y ² y ³ x xy x ²	with eigenvalues	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	

Introduction 0		ilbert sche 00000	eme of p	points	Moduli space	es of sheave	S	Algebraic	city	Checks ○○○○○○○○○●
Equivariant	localization									
N€ ⊘⊡ If ∠	eed to c $n^{[n]}$ vecto $Z = Z_0$	omput r bunc ⊔ <i>Z</i> 1 ∟	te thi dle or <i>Z</i> 2, s	ngs like n (ℙ²) ^[n] supp(Z,	$c(\mathcal{O}^{[n]})$ with fibre $c(\mathcal{O}^{[n]}) = p_i$, the	e $\mathcal{O}^{[n]}(Z)$ en) = H	$^{0}(\mathcal{O}_{Z})$		
	c	$\mathcal{O}^{[n]}$	[]] (Z) = (Z)) =	$= \mathcal{O}^{[n_0]}($ $= \boldsymbol{c}^T(\mathcal{O})$	$\mathcal{C}(Z_0)\oplus\mathcal{O}^{[n]}$	$^{(1)}(Z_1)\oplus ^T(\mathcal{O}^{[n_1]}(Z_1))$	$\mathcal{O}^{[n_2]}(Z_1))c^{\overline{1}}$	(Z_2) $\int (\mathcal{O}^{[n_2]})$	(<i>Z</i> ₂))	
Le Th Th	t e.g. Z ien the ius basi	$f = Z_0$ fibre <i>C</i> s of e	, <i>I_Z =</i> 2 ^[n] (<i>Z</i> igenv	(y^4, x) $Z) = H^0$ vectors	$(\mathcal{O}_Z) = \mathbb{C}$ of fibre for) [<i>x</i> , <i>y</i>]/() · <i>T</i> actio	y⁴, xy² on is	² , x ² y,	x ³)	
	1 <i>x</i> <i>x</i> ²	y xy	<i>y</i> ²	y ³ w	ith eigenv	alues	$\begin{array}{ccc} 1 & t \\ t_1 & t_1 \\ t_1^2 \\ t_1^2 \end{array}$	t ₂ t ₂ ² t ₂	t ₂ ³	

Thus

 $c^{T}(\mathcal{O}^{[n]}(Z)) = (1+\epsilon_2)(1+2\epsilon_2)(1+3\epsilon_2)(1+\epsilon_1)(1+\epsilon_1+\epsilon_2)(1+2\epsilon_1).$