Hilbert schemes of points on surfaces

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Hilbert scheme of points on surface is
- simple example of moduli space
- nice resolution of singularities

Important example of many conjectures and results

Connections to other fields:

theoretical physics

Moduli of sheaves

Hilbert schemes of points

infinite-dim Lie algebras

enumerative geom of curves

1. Hilbert schemes of points

S smooth projective surface/$\mathbf{C}$
A finite subscheme of S is:

$$Z = \Big\{(x_1, Q_1), \ldots, (x_k, Q_k)\Big\}$$

where: $\{x_1, \ldots x_k\} \subset S$,
$Q_i \neq 0$ finite dim quot ring of holom fcts at x_i

$$\text{length of } Z = \sum_i dim(Q_i)$$

$$(x, \text{constants } \mathbf{C}) \simeq x$$

$dim(\mathcal{O}_{Z,x}) = 2 \Longrightarrow (x, \mathcal{O}_{Z,x}) \simeq \bullet\!\searrow$ tangent dir at x

Hilbert scheme of points

$$S^{[n]} := \Big\{ Z \subset S \;\Big|\; \text{finite subscheme of length } n \Big\}$$

Examples: $S^{[0]} = \{\emptyset\} = \text{pt}$

$$S^{[1]} = \{\ \bullet\ \} = S$$

$$S^{[2]} = \{\ \bullet \quad \bullet\ \} \cup \{\ \bullet\!\searrow\ \} = Bl_{\Delta}(S \times S)/G_2$$

G_n symmetric group

Symmetric power:

$$S^{(n)} := S^n/G_n = \left\{(x_1, n_1), \dots, (x_k, n_k)\right\}$$

$n_i \in \mathbf{Z}_{>0}$

Hilbert Chow morphism

$$\rho : S^{[n]} \to S^{(n)}; \;\; Z \mapsto supp(Z)$$

- $S^{(n)}$ is singular along big diagonal $S^{[n]}$ is nonsingular and ρ a resolution of singularities
- ρ is crepant: $\rho^*(K_{S^{(n)}}) = K_{S^{[n]}}$
- If $K_S = 0$ (abelian or K3) $\Longrightarrow$ $S^{[n]}$ compact holom symplectic (hyperkähler) (there is holom 2-form whose n-fold wedge product is nowhere 0.)
  Very few examples of such manifolds exist.

2. Betti numbers

Should look at all $S^{[n]}$ at the same time. Extra structures emerge.
Betti numbers and Euler numbers: generating function in terms of modular forms.

Modular form: $f : \mathbf{H} \to \mathbf{C}$ holom

$$\mathbf{H} := \left\{ \tau \in \mathbf{C} \,\middle|\, \Im(\tau) > 0 \right\}$$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbf{Z})$$

$k =$ weight
Fourier development

$$f(\tau) := \sum_{n \geq 0} a_n q^n, \quad q = e^{2\pi i \tau}$$

Example:

$$\Delta(\tau) = q \prod_{n>0} (1 - q^n)^{24}$$

$\eta = \Delta^{1/24}$

Shift Betti numbers symmetric around degr 0

$$b_i(X) := h^{i+dim_{\mathbf{C}}(X)}(X, \mathbf{Q})$$
$$p(X, y) := \sum (-1)^i b_i(X) y^i$$

Betti numbers

$$\sum_{n \geq 0} p(S^{[n]}, y) q^n = \prod_{k \geq 1} \prod_i (1 - y^i q^k)^{(-1)^{i+1} b_i(X)}$$

Euler numbers

$$\sum_{n \geq 0} e(S^{[n]}) q^{n - e(S)/24} = \eta(\tau)^{-e(S)}$$

Refined to Hodge numbers
motives, Chow groups (de Cataldo-Migliorini)
Elliptic genus (Dijkgraaf, Moore, Verlinde, Verlinde), (Borisov-Libgober)

3. Infinite-dim Lie algebras

All cohomology with $\mathbf{Q}$-coefficients

$$\mathbf{H} := \bigoplus_n H^*(S^{[n]})$$

Heisenberg algebra: (Nakajima-Grojnowski)

$$p_n(\alpha) : \mathbf{H} \to \mathbf{H}, \quad n \in \mathbf{Z},\ \alpha \in H^*(S)$$

Assume cohomology classes repr by submfds.
$p_n([A])$: add fat point of length n in $A \subset S$
Fat point = subscheme concentrated in a point

$$p_n([A])[Y] = \left[\overline{\left\{ Z \sqcup P_n \;\middle|\; Z \in Y,\ P_n \text{ fat point in } A \right\}}\right]$$

for $[Y] \in H^*(S^{[m]})$
$p_{-n}([A])$: subtract fat point of length n in A

Heisenberg relation

$$[p_n(\alpha), p_m(\beta)] = n\delta_{n,-m}\left(\textstyle\int_S \alpha\beta\right) id_{\mathbf{H}}$$

Irreducible module: $\mathbf{H}$ is generated from $\mathbf{1} \in H^0(S^{[0]})$ via "creation operators" $p_n(\alpha)$, $n > 0$

Tautological bundles:
$\mathcal{O}_n$ vector bundle of rank n on $S^{[n]}$
fibre over $\{(x_1, Q_1), \ldots (x_k, Q_k)\}$

$$Q_1 \oplus \ldots \oplus Q_k$$

For V vector bundle on S analogously V_n vector bundle on $S^{[n]}$
fibre over $\{x_1, \ldots, x_n\}$

$$V(x_1) \oplus \ldots \oplus V(x_n)$$

These tautological bundles play important role

- enumerative geometry
- Donaldson invariants

For $\alpha \in H^*(S)$

$$\alpha_\bullet : \mathbf{H} \to \mathbf{H}$$

on $H^*(S^{[n]})$ multipl by α_n generalization of Chern character of tautological bundle (Li-Qin-Wang)

Virasoro algebra: (Lehn,Li-Qin-Wang)
$\delta : H^*(S) \to H^*(S \times S)$ diagonal pushforward

$$L_n(\alpha) := \sum_{\nu \in \mathbf{Z}} p_\nu p_{n-\nu} \delta(a) : \mathbf{H} \to \mathbf{H}$$

Virasoro relation

$$[L_n(\alpha), L_m(\beta)] = (n-m)L_{n+m}(\alpha\beta), \quad n \neq -m \tag{1}$$

Tie to ring structure
$\partial : \mathbf{H} \to \mathbf{H}$ on each $H^*(S^{[n]})$ multipl by $c_1(\mathcal{O}_n)$

$$[\partial, p_n(\alpha)] = nL_n(\alpha) + \binom{n}{2} p_n(K_S\alpha), \quad n \geq 0 \tag{2}$$

$$\begin{aligned}[\alpha_\bullet, p_1(\beta)] = & p_1(\alpha\beta) + [\partial, p_1(\alpha\beta)] \\ & + \frac{1}{2}[\partial, [\partial, p_1(\alpha\beta)]] + \dots\end{aligned} \tag{3}$$

(1),(2),(3) determine ring structure of $S^{[n]}$

Lehn-Sorger: simple descr of $H^*(S^{[n]})$ if
$K_S = 0$

4. Orbifold cohomology

X compact compl manif, G finite group
X/G quotient
(Dixon-Harvey-Vafa-Witten, Zaslow):

$$H^*_{orb}([X/G]) := \bigoplus_{[g]} H^*(X^g/C(g))$$

$[g]$ conjugacy class, $C(g)$ centralizer
grading:
shift of $H^*(X^g/C(g))$, Poincaré duality

Batyrev-Borisov: $Y \to X/G$ crepant resol $\Longrightarrow$
$H^*_{orb}([X/G]) = H^*(Y)$ graded vector spaces.

One of first examples: $\rho : S^{[n]} \to S^{(n)}$

Orbifold cohomology ring:
(Chen-Ruan, Abramovich-Graber-Vistoli)
ring structure on $H^*_{orb}([X/G])$:
part of quantum cohomology of orbifold

Conjecture: (Ruan)

1. $\pi : Y \to X/G$ hyperkähler resolution
$\Longrightarrow H^*_{orb}([X/G]) = H^*(Y)$ as rings

2. $\pi : Y \to X/G$ crepant resolution
$\Longrightarrow H^*_{orb}([X/G]) = H^*(Y)$ after explicit Gromov-Witten corrections w.r.t. curve classes in Y contracted by π

1. shown for $S^{[n]} \to S^{(n)}$ $\quad (K_S = 0)$

2. true for $S^{[2]} \to S^{(2)}$

5. Moduli of vector bundles

$$M_S^H(r, c_1, c_2) = \left\{H\text{-stable sheaves on } S\right\}$$
$$\supset \left\{H\text{-stable vector bundles }\right\}$$

Connections to $S^{[n]}$

- $S^{[n]} = M(1, 0, n)$
- Serre correspondence: Vector bundles as extensions of ideal sheaves
- Fourier-Mukai transform: Birational maps between moduli of sheaves on S $(K_S = 0)$
- Wallcrossing for H : "walls" in $H^2(S, \mathbf{R})$, $M_S^H(c_1, c_2)$ depends on "chamber" of H

"Flip" when H passes through wall: replace projective bundle over $S^{[k]} \times S^{[l]}$ by another one.

Used to compute Betti numbers, Donaldson invariants.

Physics: S-duality (Vafa-Witten):

$$\sum_n e(M_S^H(c_1, c_2)q^{c_2 - c_1^2/4} \quad \text{partition function}$$

should be modular form.

Idea:

$$(\text{strong} \leftrightarrow \text{weak coupling}) \iff (\tau \leftrightarrow -\frac{1}{\tau})$$

Physics (partition function) invariant.

Motivating examples:

- $S^{[n]}$
- $\mathbf{P}_2$ (Yoshioka)
- blowup formulas (Qin-Li)
- K3 surfaces: $M_S^H(r, c_1, c_2)$ compact $\Longrightarrow$ same Betti numbers as $S^{[n]}$ (Yoshioka)

6. Enumerative Geometry of curves

S K3 surface, L primitive line bundle
g genus of general curve in $|L|$
finite number n_g of rational curves in $|L|$

$$\sum_{g>0} n_g q^{g-1} = \frac{1}{\Delta}.$$

(i.e. $n_g = e(S^{[g]})$) (Yau-Zaslow, Beauville):

Conjectural generalization:
Universal generating function for δ-nodal curves in δ-dim subspace of sufficiently ample $|L|$

(Bryan-Leung): K3 and abelian surfaces
(Liu): general case.

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$\mathcal{J} \to |\mathcal{L}|$ rel compactified Jacobian; fibre $J(C)$
$e(J(C)) = 0$ unless $g(C) = 0 \Longrightarrow n_g = e(\mathcal{J})$
$\mathcal{J}$ birational to $S^{[g]} \Longrightarrow e(\mathcal{J}) = e(S^{[g]})$

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Krichever-Höhn elliptic genus

$$\phi(X) = \text{"}\, S^1\text{-equivariant } \chi_y\text{-genus of } \mathcal{L}X\text{"}$$

Conjecture (D-M-V-V) (proved B-L):

$$\sum_{N \geq 0} \phi(S^{[N]}) p^N = \prod_{n \geq 0, m, l} (1 - y p^m q^n)^{-c(nm, l)}$$

where $\phi(S) = \sum_{n \geq 0, l} c(n, l) y^l q^n$

r.h.s. orbifold elliptic genus, use as loop space

$$\left\{ c : [0,1] \to S^n \;\middle|\; \exists g \in G_n \; c(1) = g(c(0)) \right\} / G_n$$