# Virtual Riemann-Roch and Applications

Lothar Göttsche (ICTP)
joint work with Barbara Fantechi (SISSA)
arXiv:0706.0988

Principal bundles, Stacks and Gerbes,
Bad Honnef, June 2007

Invariants of moduli spaces $X$ (Donaldson inv, Donaldson-Thomas inv.)

- $\int_X \alpha \in \mathbb{Q}, \quad \alpha \in A^*(X)$,
- $\chi(X, V) \in \mathbb{Z}, \quad V$ vector bundle on $X$.

But only works well if $X$ is nonsingular of expected dimension

Invariants of moduli spaces $X$ (Donaldson inv, Donaldson-Thomas inv.)

- $\int_X \alpha \in \mathbb{Q}, \quad \alpha \in A^*(X)$,
- $\chi(X, V) \in \mathbb{Z}, \quad V$ vector bundle on $X$.

But only works well if $X$ is nonsingular of expected dimension

Good situation: $X$ has perfect obstruction theory $[E^{-1} \to E^0]$
$d = \mathrm{rk}(E^0) - rk(E^{-1})$ expected dimension

- virtual fundamental class $[X]^{\mathrm{vir}} \in A_d(X)$
  compute $\int_{[X]^{\mathrm{vir}}} \alpha$.
- virtual structure sheaf $\mathcal{O}_X^{\mathrm{vir}} \in K_0(X)$
  compute $\chi^{\mathrm{vir}}(X, V) := \chi(X, V \otimes \mathcal{O}_X^{\mathrm{vir}})$

**Philosophy:** Pair $(X, E^\bullet)$ of proper scheme and perfect obstruction theory is viewed as "virtually smooth scheme". Should behave like a nonsingular variety.

**Philosophy:** Pair $(X, E^\bullet)$ of proper scheme and perfect obstruction theory is viewed as "virtually smooth scheme". Should behave like a nonsingular variety.
We show Riemann Roch:

$$\chi^{\mathrm{vir}}(X, V) = \int_{[X]^{\mathrm{vir}}} \mathrm{ch}(V)\mathrm{td}(T_X^{\mathrm{vir}}).$$

**Philosophy:** Pair $(X, E^\bullet)$ of proper scheme and perfect obstruction theory is viewed as "virtually smooth scheme". Should behave like a nonsingular variety.
We show Riemann Roch:

$$\chi^{\mathrm{vir}}(X, V) = \int_{[X]^{\mathrm{vir}}} \mathrm{ch}(V)\mathrm{td}(T_X^{\mathrm{vir}}).$$

**Applications:** Study virtual analogues of $\chi_y$-genus, Euler characteristic, Elliptic genus. They behave as if $X$ was nonsingular.
Applications to moduli of sheaves on surfaces and $K$-theory Donaldson invariants.

**Philosophy:** Pair $(X, E^\bullet)$ of proper scheme and perfect obstruction theory is viewed as "virtually smooth scheme". Should behave like a nonsingular variety.
We show Riemann Roch:

$$\chi^{\mathrm{vir}}(X, V) = \int_{[X]^{\mathrm{vir}}} \mathrm{ch}(V)\mathrm{td}(T_X^{\mathrm{vir}}).$$

**Applications:** Study virtual analogues of $\chi_y$-genus, Euler characteristic, Elliptic genus. They behave as if $X$ was nonsingular.
Applications to moduli of sheaves on surfaces and $K$-theory Donaldson invariants.

**Other work** Similar results are obtained by Ciocan-Fontanine and Kapranov for DG-schemes.

**Definition**

Let $X$ proper scheme. A perfect obstruction theory on $X$ is a complex $[E^{-1} \to E^0]$ of vector bundles on $X$ with morphism $\phi : E^\bullet \to L_X$ in derived category such that

- $h^0(\phi)$ is isomorphism
- $h^{-1}(\phi)$ is surjective

Here $L_X$ =cotangent complex. Only need

$$\tau_{\geq -1} L_X = [I/I^2 \to \Omega_M|X]$$

for $X \subset_{closed} M$ nonsingular , $I = I_{X/M}$.

**Definition**

A pair $(X, E^\bullet)$ of proper scheme and perfect obstruction theory is called virtually smooth of dimension $d := \mathrm{rk}(E^0) - \mathrm{rk}(E^{-1})$. Let $[E_0 \to E_1]$ dual complex to $[E^{-1} \to E^0]$.

$$T_X^{\mathrm{vir}} := E_0 - E_1 \in K^0(X), \quad \text{virtual tangent bundle}$$

$$\Omega_X^{\mathrm{vir}} := E^0 - E^{-1} \in K^0(X), \quad \text{virtual cotangent bundle}$$

$$K_X^{\mathrm{vir}} := \det(E^0) \otimes \det(E^{-1})^{-1} \quad \text{virtual canonical bundle}$$

From now on, let $(X, E^\bullet)$ virtually smooth of dimension $d$.

Assume $X$ can be embedded into smooth scheme $M$.
Let $C_{X/M}$ normal cone. Then $C_{X/M} \subset_{closed} \mathcal{N}_{X/M}$.
Intrinsic normal cone: $\mathfrak{C}_X := [C_{X/M}/i^*T_M]$
$E^\bullet$ obstruction theory implies $[\mathcal{N}_{X/M}/i^*T_M] \subset_{closed} [E_1/E_0]$

$$\mathfrak{C}_X := [C_{X/M}/i^*T_M] \subset_{closed} [E_1/E_0]$$

Assume $X$ can be embedded into smooth scheme $M$.
Let $C_{X/M}$ normal cone. Then $C_{X/M} \subset_{closed} \mathcal{N}_{X/M}$.
Intrinsic normal cone: $\mathfrak{C}_X := [C_{X/M}/i^*T_M]$
$E^\bullet$ obstruction theory implies $[\mathcal{N}_{X/M}/i^*T_M] \subset_{closed} [E_1/E_0]$

$$\mathfrak{C}_X := [C_{X/M}/i^*T_M] \subset_{closed} [E_1/E_0]$$

Let $\pi : E_1 \to [E_1/E_0]$. Put $C := \pi^{-1}(\mathfrak{C}_X)$, cone in $E_1$.
$s_0 : X \to E_1$ zero section.

- virtual fundamental class

$$[X]^{\mathrm{vir}} := s_0^*([C]) \in A_d(X)$$

Assume $X$ can be embedded into smooth scheme $M$.
Let $C_{X/M}$ normal cone. Then $C_{X/M} \subset_{closed} \mathcal{N}_{X/M}$.
Intrinsic normal cone: $\mathfrak{C}_X := [C_{X/M}/i^*T_M]$
$E^\bullet$ obstruction theory implies $[\mathcal{N}_{X/M}/i^*T_M] \subset_{closed} [E_1/E_0]$

$$\mathfrak{C}_X := [C_{X/M}/i^*T_M] \subset_{closed} [E_1/E_0]$$

Let $\pi : E_1 \to [E_1/E_0]$. Put $C := \pi^{-1}(\mathfrak{C}_X)$, cone in $E_1$.
$s_0 : X \to E_1$ zero section.

- virtual fundamental class

$$[X]^{\mathrm{vir}} := s_0^*([C]) \in A_d(X)$$

- virtual structure sheaf

$$\mathcal{O}_X^{\mathrm{vir}} := [Ls_0^*\mathcal{O}_C] \in K_0(X)$$

$X$ nonsingular, $E$ vector bundle of rk $= r$ on $X$,
view $X = Z(s_0)$, $s_0 =$ zero section
Expected dimension: $d := dim(X) - r$
Obstruction theory: $[E^\vee \xrightarrow{0} \Omega_X]$
$[X]^{\mathrm{vir}} = c_r(E)$,
$\mathcal{O}_X^{\mathrm{vir}} = [\Lambda^r(E^\vee) \xrightarrow{0} \Lambda^{r-1}(E^\vee) \xrightarrow{0} \ldots \xrightarrow{0} E^\vee \xrightarrow{0} \mathcal{O}_X]$.

$K^0(X)$ :=Groth. group of vector bundles
$K_0(X)$ :=Groth. Group of coh. sheaves
Let $V$ vector bundle, Chern roots $x_1, \dots, x_r$,

$$\mathrm{ch}(V) := \sum_{i=1}^{r} e^{x_i} \in A^*(X)$$

defined on $K^0(X)$ by $\mathrm{ch}(V_1 - V_2) = \mathrm{ch}(V_1) - \mathrm{ch}(V_2)$,

$$\mathrm{td}(V) := \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}} \in A^*(X)^\times$$

defined on $K^0(X)$ by $\mathrm{td}(V_1 - V_2) = \mathrm{td}(V_1)/\mathrm{td}(V_2)$.

For $V \in K^0(X)$ virtual holomorphic Euler characteristic

$$\chi^{\mathrm{vir}}(X, V) := \chi(X, V \otimes \mathcal{O}_X^{\mathrm{vir}})$$

Let $V \in K^0(X)$

**Theorem (virtual Grothendieck-Riemann-Roch)**

*$f : X \to Y$ proper morphism, $Y$ nonsingular.*

$$\mathrm{ch}(f_*(V \otimes \mathcal{O}_X^{\mathrm{vir}})) \cdot \mathrm{td}(T_Y) \cap [Y] = f_*(\mathrm{ch}(V) \cdot \mathrm{td}(T_X^{\mathrm{vir}}) \cap [X]^{\mathrm{vir}})$$

Let $V \in K^0(X)$

**Theorem (virtual Grothendieck-Riemann-Roch)**

*$f : X \to Y$ proper morphism, $Y$ nonsingular.*

$$\mathrm{ch}(f_*(V \otimes \mathcal{O}_X^{\mathrm{vir}})) \cdot \mathrm{td}(T_Y) \cap [Y] = f_*(\mathrm{ch}(V) \cdot \mathrm{td}(T_X^{\mathrm{vir}}) \cap [X]^{\mathrm{vir}})$$

This is not the best possible version: Working on an extension, where also $Y$ is only virtually smooth.

Let $V \in K^0(X)$

**Theorem (virtual Grothendieck-Riemann-Roch)**

*$f: X \to Y$ proper morphism, $Y$ nonsingular.*

$$\mathrm{ch}(f_*(V \otimes \mathcal{O}_X^{\mathrm{vir}})) \cdot \mathrm{td}(T_Y) \cap [Y] = f_*(\mathrm{ch}(V) \cdot \mathrm{td}(T_X^{\mathrm{vir}}) \cap [X]^{\mathrm{vir}})$$

This is not the best possible version: Working on an extension, where also $Y$ is only virtually smooth.

**Corollary (virtual Hirzebruch-Riemann-Roch)**

$$\chi^{\mathrm{vir}}(X, V) = \int_{[X]^{\mathrm{vir}}} \mathrm{ch}(V) \cdot \mathrm{td}(T_X^{\mathrm{vir}})$$

## virtual Riemann-Roch

Let $V \in K^0(X)$

**Theorem (virtual Grothendieck-Riemann-Roch)**

*$f : X \to Y$ proper morphism, $Y$ nonsingular.*

$$\mathrm{ch}(f_*(V \otimes \mathcal{O}_X^{\mathrm{vir}})) \cdot \mathrm{td}(T_Y) \cap [Y] = f_*(\mathrm{ch}(V) \cdot \mathrm{td}(T_X^{\mathrm{vir}}) \cap [X]^{\mathrm{vir}})$$

This is not the best possible version: Working on an extension, where also $Y$ is only virtually smooth.

**Corollary (virtual Hirzebruch-Riemann-Roch)**

$$\chi^{\mathrm{vir}}(X, V) = \int_{[X]^{\mathrm{vir}}} \mathrm{ch}(V) \cdot \mathrm{td}(T_X^{\mathrm{vir}})$$

**Corollary (weak virtual Serre duality)**

$$\chi^{\mathrm{vir}}(X, V) = (-1)^d \chi^{\mathrm{vir}}(X, V^\vee \otimes K_X^{\mathrm{vir}})$$

Use singular Riemann-Roch (Fulton Chap. 18)
There exists $\tau_X : K_0(X) \to A_*(X)$, s.th.

1. for $V \in K^0(X)$, $\mathcal{F} \in K_0(X)$,

$$\tau_X(V \otimes \mathcal{F}) = \mathrm{ch}(V) \cap \tau_X(\mathcal{F}),$$

2. for $f : X \to Y$ proper,

$$f_* \circ \tau_X = \tau_X \circ f_* : K_0(X) \to A_*(Y)$$

Use singular Riemann-Roch (Fulton Chap. 18)
There exists $\tau_X : K_0(X) \to A_*(X)$, s.th.

1. for $V \in K^0(X)$, $\mathcal{F} \in K_0(X)$,
$$\tau_X(V \otimes \mathcal{F}) = \mathrm{ch}(V) \cap \tau_X(\mathcal{F}),$$
2. for $f : X \to Y$ proper,
$$f_* \circ \tau_X = \tau_X \circ f_* : K_0(X) \to A_*(Y)$$

With this reduce to the following: Let $p : C \to X$ projection. Then
$$\tau_C(\mathcal{O}_C) = p^*(\mathrm{td}(E_0)) \cap [C]).$$

Show this by deformation to the normal cone.

Let $E$ vector bundle of rank $r$. Put

$$\Lambda_t E := \sum_{i=0}^{r} \Lambda^i E t^i \in K^0(X)[t], \quad S_t E := \sum_{i\geq 0} S^i E t^i \in K^0(X)[[t]],$$

Easy: $S_t E = 1/\Lambda_{-t} E$, get
$\Lambda_t : K^0(X) \to K^0(X)[[t]], \Lambda_t(E - F) = \Lambda_t E / S_{-t} E.$

Let $E$ vector bundle of rank $r$. Put

$$\Lambda_t E := \sum_{i=0}^{r} \Lambda^i E t^i \in K^0(X)[t], \quad S_t E := \sum_{i\geq 0} S^i E t^i \in K^0(X)[[t]],$$

Easy: $S_t E = 1/\Lambda_{-t} E$, get
$\Lambda_t : K^0(X) \to K^0(X)[[t]], \Lambda_t(E - F) = \Lambda_t E / S_{-t} E.$
Virtual $i$-forms: $\Omega_X^{i,\mathrm{vir}} := \mathrm{Coeff}_{t^i} \Lambda_t \Omega_X^{\mathrm{vir}}.$

Let $E$ vector bundle of rank $r$. Put

$$\Lambda_t E := \sum_{i=0}^{r} \Lambda^i E t^i \in K^0(X)[t], \quad S_t E := \sum_{i\geq 0} S^i E t^i \in K^0(X)[[t]],$$

Easy: $S_t E = 1/\Lambda_{-t} E$, get
$\Lambda_t : K^0(X) \to K^0(X)[[t]], \Lambda_t(E - F) = \Lambda_t E / S_{-t} E$.
Virtual $i$-forms: $\Omega_X^{i,\mathrm{vir}} := \mathrm{Coeff}_{t^i} \Lambda_t \Omega_X^{\mathrm{vir}}$.

## Definition

Virtual $\chi_{-y}$-genus:

$$\chi_{-y}^{\mathrm{vir}}(X) := \chi^{\mathrm{vir}}(X, \Lambda_{-y}\Omega_X^{\mathrm{vir}}) = \sum_{n\geq 0} (-1)^n \chi(X, \Omega_X^{i,\mathrm{vir}}) \in \mathbb{Z}[[y]],$$

$\chi_{-y}^{\mathrm{vir}}(X, V) := \chi^{\mathrm{vir}}(X, V \otimes \Lambda_{-y}\Omega_X^{\mathrm{vir}})$. Will show $\chi_{-y}^{\mathrm{vir}}(X) \in \mathbb{Z}[y]$.
virtual Euler characteristic $e^{\mathrm{vir}}(X) := \chi_{-1}^{\mathrm{vir}}(X)$.

**Theorem**

1. $\chi^{\mathrm{vir}}_{-y}(X) \in \mathbb{Z}[y]$, *of degree* $d$, *equivalently* $\chi(X, \Omega_X^{i,\mathrm{vir}}) = 0$ *for* $i > d$.
2. $\chi^{\mathrm{vir}}_{-y}(X) = y^d \chi^{\mathrm{vir}}_{-1/y}(X)$, *equivalently* $\chi(X, \Omega_X^{i,\mathrm{vir}}) = (-1)^d \chi(X, \Omega_X^{d-i,\mathrm{vir}})$
3. $e^{\mathrm{vir}}(X) = \int_{[X^{\mathrm{vir}}]} c_d(T_X^{\mathrm{vir}})$.

## Theorem

1. $\chi^{\mathrm{vir}}_{-y}(X) \in \mathbb{Z}[y]$, *of degree* $d$, *equivalently* $\chi(X, \Omega_X^{i,\mathrm{vir}}) = 0$ *for* $i > d$.
2. $\chi^{\mathrm{vir}}_{-y}(X) = y^d \chi^{\mathrm{vir}}_{-1/y}(X)$, *equivalently* $\chi(X, \Omega_X^{i,\mathrm{vir}}) = (-1)^d \chi(X, \Omega_X^{d-i,\mathrm{vir}})$
3. $e^{\mathrm{vir}}(X) = \int_{[X^{\mathrm{vir}}]} c_d(T_X^{\mathrm{vir}})$.

**Sketch of proof:** Apply Riemann-Roch theorem: $x_1, \ldots, x_n$ Chern roots of $E_0$, $u_1, \ldots, u_m$ Chern roots of $E_1$, $d = n - m$

$$\chi^{\mathrm{vir}}_{-y}(X) = \int_{[X]^{\mathrm{vir}}} \prod_{i=1}^{n} \frac{x_i(1 - ye^{-x_i})}{1 - e^{-x_i}} \prod_{j=1}^{m} \frac{1 - e^{-u_j}}{u_j(1 - ye^{-u_j})}$$

Computing modulo classes of degree $> d$, the integrand becomes a polynomial in $(1 - y)$.

A family of proper virtually smooth schemes is $\pi : \mathcal{X} \to B$ proper, $B$ nonsingular, with relative perfect obstruction theory. Then all fibres are virtually smooth.

A family of proper virtually smooth schemes is $\pi : \mathcal{X} \to B$ proper, $B$ nonsingular, with relative perfect obstruction theory. Then all fibres are virtually smooth.

**Theorem**

*Let $\mathcal{X} \to B$ family of proper virtually smooth schemes.*
*Let $V \in K^0(\mathcal{X})$.*
*Then $\chi^{\mathrm{vir}}(X_b, V|_{X_b})$ is locally constant on $B$.*

A family of proper virtually smooth schemes is $\pi : \mathcal{X} \to B$ proper, $B$ nonsingular, with relative perfect obstruction theory. Then all fibres are virtually smooth.

**Theorem**

*Let $\mathcal{X} \to B$ family of proper virtually smooth schemes. Let $V \in K^0(\mathcal{X})$. Then $\chi^{\mathrm{vir}}(X_b, V|_{X_b})$ is locally constant on $B$.*

**Corollary**

*$\chi^{\mathrm{vir}}_{-y}(X)$, and $e^{\mathrm{vir}}(X)$ are deformation invariants.*

A family of proper virtually smooth schemes is $\pi : \mathcal{X} \to B$ proper, $B$ nonsingular, with relative perfect obstruction theory. Then all fibres are virtually smooth.

**Theorem**

*Let $\mathcal{X} \to B$ family of proper virtually smooth schemes.*
*Let $V \in K^0(\mathcal{X})$.*
*Then $\chi^{\mathrm{vir}}(X_b, V|_{X_b})$ is locally constant on $B$.*

**Corollary**

$\chi^{\mathrm{vir}}_{-y}(X)$, *and* $e^{\mathrm{vir}}(X)$ *are deformation invariants.*

In particular if $B$ connected and $X_0$ smooth of dimension $d$ then $\chi^{\mathrm{vir}}(X_b, V|_{X_b}) = \chi(X_0, V|_{X_0})$, $\chi^{\mathrm{vir}}_{-y}(X_b) = \chi_{-y}(X_0)$, $e^{\mathrm{vir}}(X_b) = e(X_0)$.

**Fultons Chern class:** If $X \subset_{closed} M$, and $M$ nonsingular

$$c_F(X) := c(T_M|_X) \cap s(X, M) \in A_*(X)$$

(indep. of $M$). Generalization of $c(T_X)$ for singular schemes (related to $c_{SM}(X)$, which satisfies $\deg(c_{SM}(X)) = e(X)$).

**Fultons Chern class:** If $X \subset_{closed} M$, and $M$ nonsingular

$$c_F(X) := c(T_M|_X) \cap s(X, M) \in A_*(X)$$

(indep. of $M$). Generalization of $c(T_X)$ for singular schemes (related to $c_{SM}(X)$, which satisfies $\deg(c_{SM}(X)) = e(X)$).

## Corollary

1. *If $X$ is lci, then $e^{\mathrm{vir}}(X) = \deg(c_F(X))$.*
2. $\deg(c_F(X))$ *is a deformation invariant of proper lci schemes.*

## Definition

For $F \in K^0(X)$ put

$$\mathcal{E}(F) := \bigotimes_{n \geq 0} \left( \Lambda_{-yq^n} F^\vee \otimes \Lambda_{-y^{-1}q^n} F \otimes S_{q^n}(F \oplus F^\vee) \right)$$

The virtual elliptic genus is

$$Ell^{\mathrm{vir}}(X; z, \tau) := y^{-d/2} \chi^{\mathrm{vir}}_{-y}(X, \mathcal{E}(T^{\mathrm{vir}}_X)), \quad q = e^{2\pi i \tau}, y = e^{2\pi i z}$$

## Definition

For $F \in K^0(X)$ put

$$\mathcal{E}(F) := \bigotimes_{n\geq 0} \left(\Lambda_{-yq^n}F^\vee \otimes \Lambda_{-y^{-1}q^n}F \otimes S_{q^n}(F \oplus F^\vee)\right)$$

The virtual elliptic genus is

$$Ell^{\mathrm{vir}}(X;z,\tau) := y^{-d/2}\chi^{\mathrm{vir}}_{-y}(X,\mathcal{E}(T_X^{\mathrm{vir}})), \quad q = e^{2\pi i\tau}, y = e^{2\pi iz}$$

## Theorem

*Assume $c_1(K_X^{\mathrm{vir}}) = 0$, then*
*$Ell^{\mathrm{vir}}(X;z,\tau)$ is a weak Jacobi form of weight 0 and index $d/2$.*

This means it behaves like a theta function:
modular in $\tau$, elliptic in $z$.

Proof is similar to the standard case.
Apply Riemann-Roch and make explicit calculations with the Chern roots.

In work in progress with Nakajima, Mochizuki and Yoshioka, this will be applied to invariants of moduli spaces of vector bundles on surfaces.

Let $(X, H)$ projective surface

$$M_H^X(c_1, c_2) = \{H\text{-stable rank 2 sheaves}\}$$

$\mathcal{E} \to X \times M$ universal sheaf, $L \in H_2(X)$,

$$\mu(L) := \left(c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2\right)/L \in H^2(M).$$

Donaldson invariants: $\Phi^H_{X,c_1}(L^d) := \int_{[M]^{\mathrm{vir}}} \mu(L)^d$.

Let $\overline{L}$ line bundle on $M$ with $c_1(\overline{L}) = \mu(L)$ (determinant bundle).
$K$-theory Donaldson invariant: $\chi^{\mathrm{vir}}(M, \overline{L})$.

$M^X_H(c_1, c_2)$ depends on $H$ via system of walls and chambers in ample cone $C_X$

## Definition

$\xi \in H^2(X, \mathbb{Z})$ defines wall of type $(c_1, c_2)$ if

1. $\xi \equiv c_1 \mod 2H^2(X, \mathbb{Z})$
2. $4c_2 - c_1^2 + \xi^2 \geq 0$

The wall is

$$W^\xi := \{H \in C_X \mid H \cdot \xi = 0\}$$

Chambers=connected components of $C_X \setminus$ walls
$M^H_X(c_1, c_2)$ and invariants constant on chambers, change when $H$ crosses wall (i.e. $H_- \to H_+$ with $H_-\xi < 0 < H_+\xi$)

**Theorem**

1. *If* $p_g(X) > 0$*, then* $\Phi^H_{X,c_1}(L^d)$ *does not change under wallcrossing.*
2. *If* $p_g(X) = 0$*, explicit generating function for wallcrossing in terms of modular forms.*

(1) is well-know from gauge theory, algebraic proof due to Mochizuki.
(2) Proven by G-Nakajima-Yoshioka, in case moduli spaces are smooth, results of Mochizuki imply it for virtual case.

**Theorem**

1. *If $p_g(X) > 0$, then $\chi^{\mathrm{vir}}(M_H^X(c_1, c_2), \overline{L})$ does not change under wallcrossing.*
2. *If $p_g(X) = 0$, explicit generating function for wallcrossing in terms of elliptic functions.*

## Theorem

1. *If $p_g(X) > 0$, then $\chi^{\mathrm{vir}}(M_H^X(c_1, c_2), \overline{L})$ does not change under wallcrossing.*
2. *If $p_g(X) = 0$, explicit generating function for wallcrossing in terms of elliptic functions.*

(1) Follows from virtual Riemann-Roch and part (1) of previous theorem.
(2) Proven by G-Nakajima-Yoshioka, in case moduli spaces are smooth, virtual Riemann-Roch and results of Mochizuki allow to adapt argument for virtual case.

**Theorem**

1. *If $p_g(X) > 0$, then $\chi^{\mathrm{vir}}(M_H^X(c_1, c_2), \overline{L})$ does not change under wallcrossing.*
2. *If $p_g(X) = 0$, explicit generating function for wallcrossing in terms of elliptic functions.*

(1) Follows from virtual Riemann-Roch and part (1) of previous theorem.
(2) Proven by G-Nakajima-Yoshioka, in case moduli spaces are smooth, virtual Riemann-Roch and results of Mochizuki allow to adapt argument for virtual case.
**Note:** Uses moduli stacks of sheaves with fixed determinant, which are $\mu_r$-gerbes, extend virtual Riemann-Roch to gerbes.

**Theorem**

1. *If $p_g(X) > 0$, then $\chi^{\mathrm{vir}}(M_H^X(c_1, c_2), \overline{L})$ does not change under wallcrossing.*
2. *If $p_g(X) = 0$, explicit generating function for wallcrossing in terms of elliptic functions.*

(1) Follows from virtual Riemann-Roch and part (1) of previous theorem.
(2) Proven by G-Nakajima-Yoshioka, in case moduli spaces are smooth, virtual Riemann-Roch and results of Mochizuki allow to adapt argument for virtual case.
**Note:** Uses moduli stacks of sheaves with fixed determinant, which are $\mu_r$-gerbes, extend virtual Riemann-Roch to gerbes.

**Note:** If $c_2 \gg 0$, then $M_H^X(c_1, c_2)$ has the expected dimension and therefore is lci. Then

$$\chi(M_H^X(c_1, c_2), \overline{L}) = \chi^{\mathrm{vir}}(M_H^X(c_1, c_2), \overline{L})$$

**Theorem**

1. *If* $p_g(X) > 0$, *then* $e^{\mathrm{vir}}(M_H^X(c_1, c_2))$ *does not change under wallcrossing.*
2. *If* $p_g(X) = 0$, *explicit generating function for wallcrossing in terms of modular forms.*

Compute $e^{\mathrm{vir}}(M)$ as $\int_{[M]^{\mathrm{vir}}} c_d(T_M^{\mathrm{vir}})$.
Apply again results of Mochizuki.

**Note:** If $c_2 \gg 0$, and thus $M_H^X(c_1, c_2)$ is lci, then

$$e^{\mathrm{vir}}(M_H^X(c_1, c_2)) = \deg(c_F(M_H^X(c_1, c_2))).$$