Tropical Geometry

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Basic Notions Seminar

Trieste, 15 Juli 2020

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Algebraic varieties: zero sets of polynomials $F(x_1, \ldots, x_n)$ in $\mathbb{C}^n$
Replaced by simpler combinatorial objects

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Many problems in algebraic geometry can be translated into tropical geometry and then are easier to solve

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  physics
  economy (e.g. auctions)
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I first learned tropical geometry from Gathmann's notes [Ga] arXiv:math/0601322, (and consulted them for this talk)
also inspired by Atom lectures
https://sites.google.com/wisc.edu/atonlinemeetings/lecture-series
please watch if you want to know more, or read the book [Mac-S]

In algebraic geometry study algebraic varieties:
zero sets of polynomials

$$X = Z(F_1, \ldots, F_k) \subset \mathbb{A}^n = \mathbb{C}^n, \qquad F_i \in \mathbb{C}[x_1, \ldots, x_n]$$

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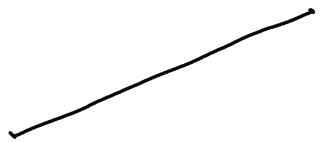
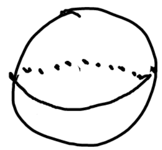
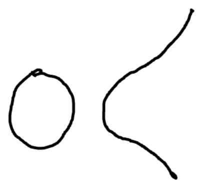
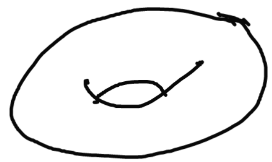
$Z(x) \subset \mathbb{A}^2$ line





$Z(y^2 - x(x-1)(x+1)) \subset \mathbb{A}^2$ elliptic curve





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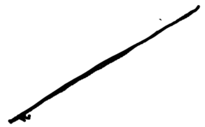

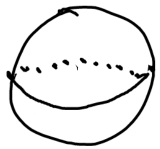

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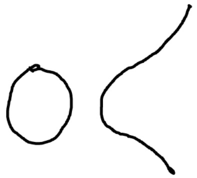

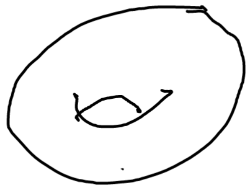

Algebraic geometry is based of algebra of polynomials,
therefore ultimately on the standard basic operations

- addition $a + b$ in $\mathbb{C}$
- multiplication $a \cdot b \in \mathbb{C}$

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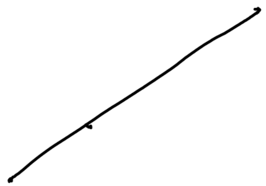

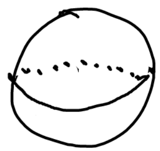

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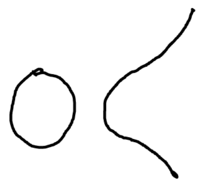

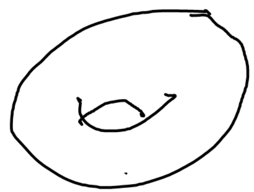

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The tropical semifield is $\overline{\mathbb{R}} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ with

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Thus we have

$$5 \oplus 6 = 6, \qquad 5 \odot 6 = 11$$

The tropical semifield

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This looks crazy, but is it almost a field:
Easy to check:

- $\oplus$ and $\odot$ are associative, commutative
- distributive law holds:
  $(a \oplus b) \odot c = \max(a, b) + c = max(a + c, b + c) =$
  $(a \odot c) \oplus (b \odot c)$
- $-\infty$ is neutral element of $\oplus$
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- $-a$ is $\odot$-inverse to $a \in \mathbb{R} = \overline{\mathbb{R}} \setminus \{-\infty\}$

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Only thing missing: elements $a \in \mathbb{R}$ have no $\oplus$-inverse:
there is no $b \in \overline{\mathbb{R}}$ with $max(a, b) = -\infty$

Tropical polynomials

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We write $x_i^m := \underbrace{x_i \odot \ldots \odot x_i}_{m \text{ times}} = mx_i$

For a multiindex $I = (i_1, \ldots, i_n)$ and $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_n})$ we write

$$\mathbf{x}^I := x_1^{i_1} \odot \ldots \odot x_n^{i_n} = (i_1 x_1 + \ldots + i_n x_n) = I \cdot \mathbf{x}$$

The *tropical polynomial* $f = \bigoplus_I a_I \odot \mathbf{x}^I, \quad a_I \in \overline{\mathbb{R}}$ is the function

$$f = \max_I (a_I + I \cdot \mathbf{x}) : \mathbb{R}^n \to \mathbb{R} : b \mapsto \max_I (a_I + I \cdot b)$$

Do not write terms with $a_I = -\infty$. Note $0 \odot x^I = x^I$

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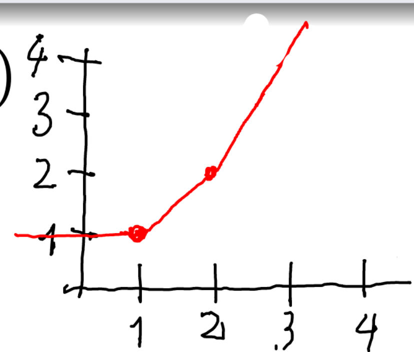
Example: $(-2) \odot x^2 \oplus x \oplus 1 = \max(1, x, 2x - 2)$



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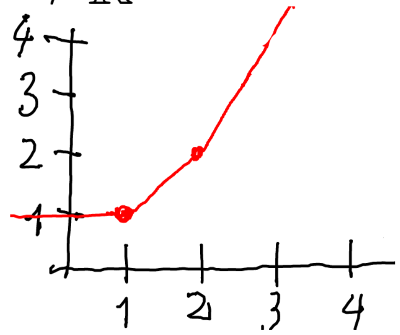

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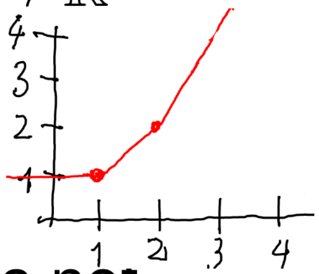

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the *tropical hypersurface* $Z(f)$ defined by f is the set of all $b \in \mathbb{R}^n$, such that the maximum $f(b)$ is obtained by at least two $\oplus$ monomials $a_I + I \cdot \mathbf{x}$.

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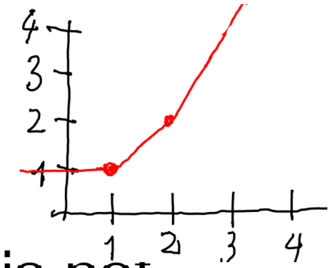

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Example: $Z((-2) \odot x^2 \oplus x \oplus 1) = \{1, 2\}$

Examples of tropical plane curves $Z(f) \subset \mathbb{R}^2$:

In each sector write which monomial achieves the maximum

Tropical line $Z(x \oplus y \oplus 0) = \max(x, y, 0)$

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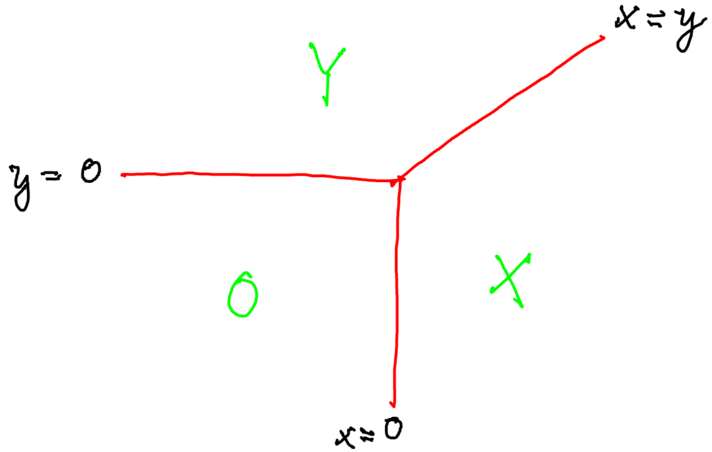
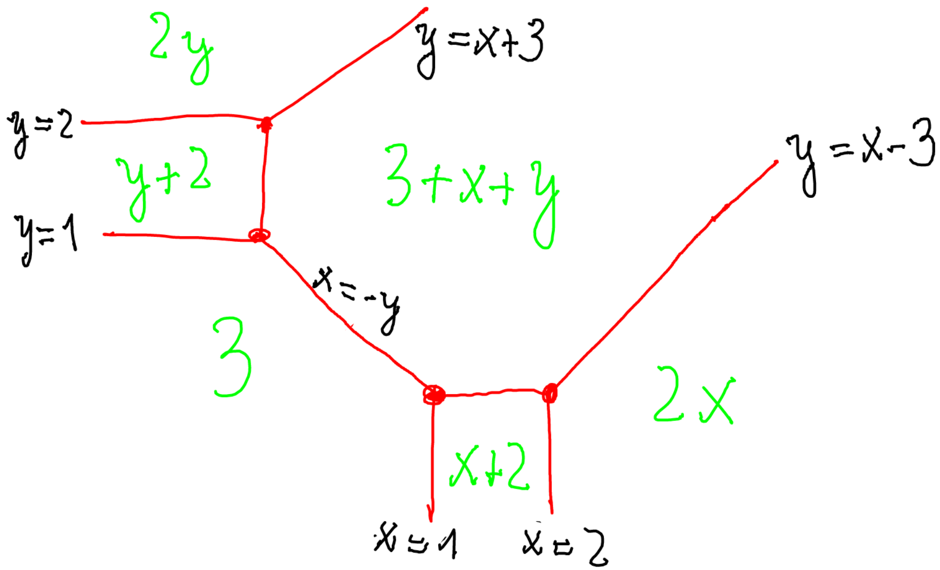


Tropical conic

$$x^2 \oplus y^2 \oplus 3 \odot x \odot y \oplus 2 \odot y \oplus 2 \odot x \oplus 3 = \max(2x, 2y, (x+y+3), x+2, y+2, 3)$$



Definition

Let $f_1, \ldots, f_r \in (\overline{\mathbb{R}}, \oplus, \odot)[x_1, \ldots, x_n]$ be tropical polynomials. The tropical algebraic set defined by $f_1, \ldots, f_r$ is

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We will in the rest of the talk specialize to plane tropical curves, i.e. tropical hypersurfaces in $\mathbb{R}^2$ given as corner sets of tropical polynomials in x, y.

Definition

The degree of a tropical polynomial $f = \bigoplus_{i,j} a_{i,j} \odot x^i \odot y^j$ is the maximum of the $i + j$ occuring
(i.e. where the coefficient is not $-\infty$, note that $x^i = 0 \odot x^i$)
A *plane tropical curve of degree* d is $Z(f) \subset \mathbb{R}^2$ for f a tropical polynomial in x, y of degree d

Definition

The tropical polynomial $f = \bigoplus_{i,j} a_{i,j} \odot x^i \odot y^j$ has degree d if d is the maximum of the $i + j$ occuring and $x^d, y^d, x^0 \odot y^0$ occur.
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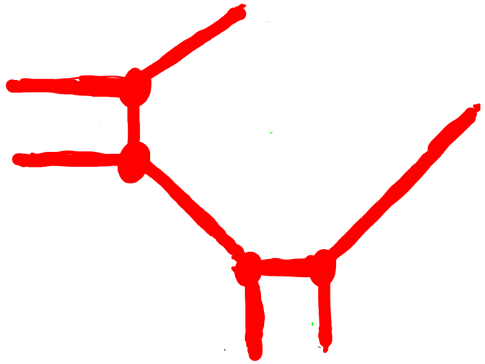
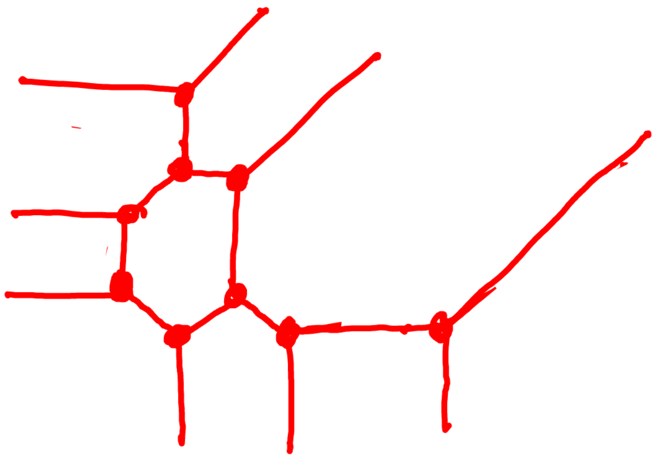
conic: $Z(x^2 \oplus y^2 \oplus 3 \odot x \odot y \oplus 2 \odot y \oplus 2 \odot x \oplus 3)$



cubic:



Plane tropical curves

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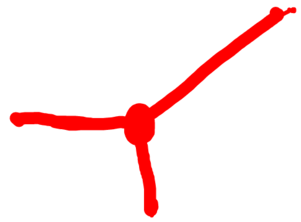

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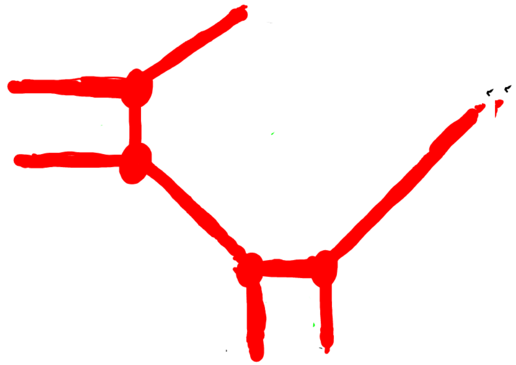

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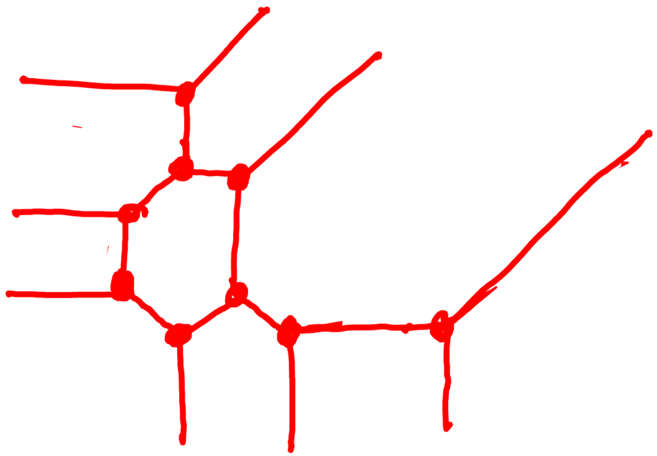

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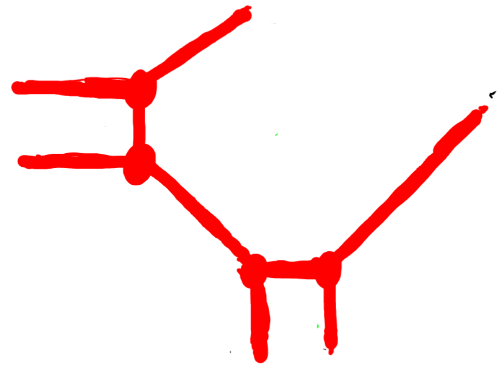

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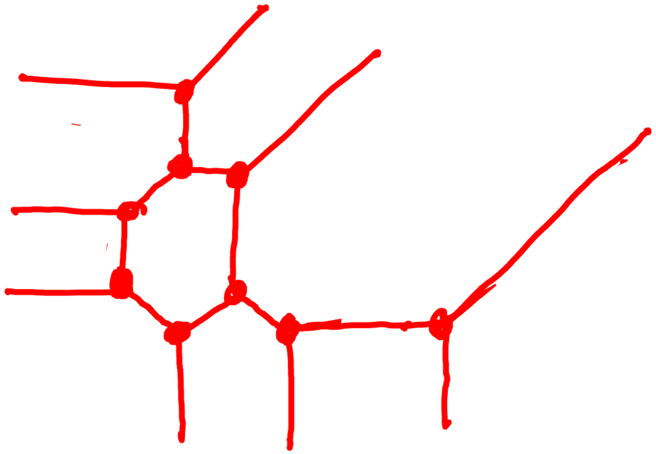


We observe that a plane tropical curve of degree d has d unbounded edges in each of the directions $(-1, 0)$, $(0, -1)$, $(1, 1)$. We will see later why this is true.

How does a plane tropical curve of degree d look like?

$\Gamma = Z(F)$ plane tropical curve of degree d

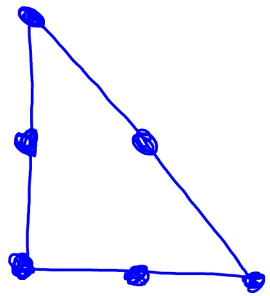
$$F = \max(a_i x + b_i y + c_i,\ i = 1, \dots, n)$$

$= \bigoplus_i c_i \odot x^{a_i} \odot y^{b_i}$

(a_i, b_i) are distinct integer points in triangle

including $(d,0)$ $(0,d)$ $(0,0)$

$$\Delta_d = \{(a,b) \in \mathbb{R}^2 \mid a \geq 0, b \geq 0, a+b \leq d\}$$



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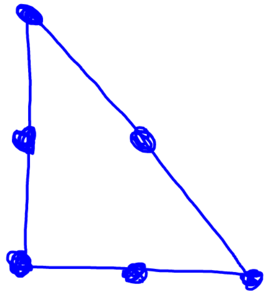

Δ_2

When $\exists (x, y) \in \mathbb{R}^2$, sth

$F(x, y) = a_i x + b_i y + c_i = a_j x + b_j y + c_j$ for some $i \neq j$

Γ has edge in dir. orth. to line $\overline{(a_i, b_i), (a_j, b_j)}$

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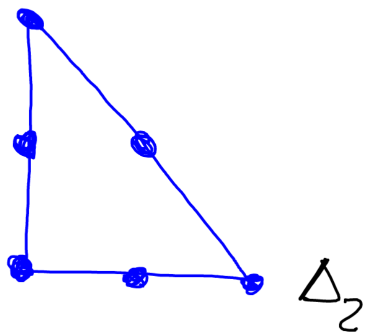
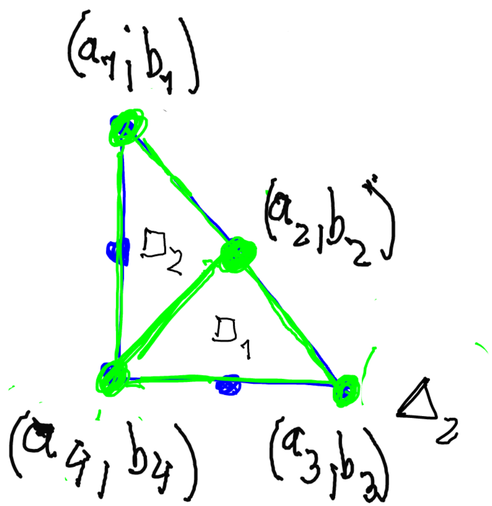


When $\exists (x, y) \in \mathbb{R}^2$, sth

$$F(x, y) = a_i x + b_i y + c_i = a_j x + b_j y + c_j \text{ for some } i \neq j$$



Γ has edge in dir. orth. to line $\overline{(a_i, b_i), (a_j, b_j)}$
The lines $\overline{(a_i, b_i), (a_j, b_j)}$ give subdivision of Δ_d
into polygons $\square_j$ with integer vertices

The shape of plane tropical curves

$\Gamma = Z(F)$, $F(x,y) = \max(a_i x + b_i y + c_i,\ i = 1, \dots, n)$
(a_i, b_i) are integer pts in $\Delta_d = \{(a,b) \in \mathbb{R}^2 \mid a \geq 0, b \geq 0, a+b \leq d\}$
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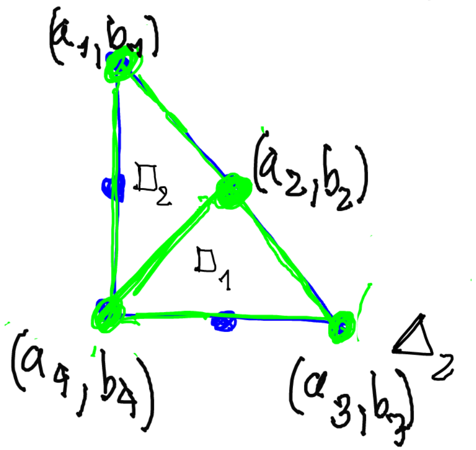

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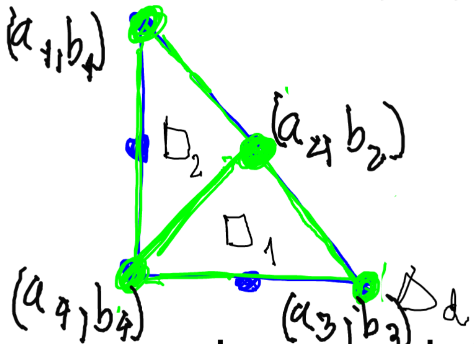


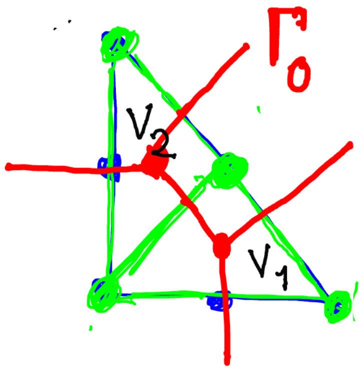
The condition that $x^d, y^d, x^0 \odot y^0$ occur in F implies that the edges of Δ_d are unions of some $\overline{(a_i,b_i),(a_j,b_j)}$

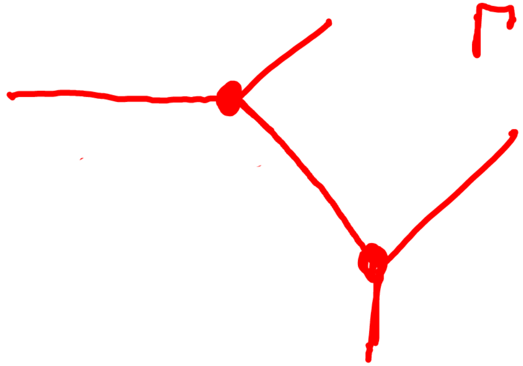
Choose one inner point V_j in each $\Box_j$

connect the V_j in adjacent $\Box_j$ by lines orthog. to $\overline{(a_i,b_i),(a_j,b_j)}$

Get graph Γ_0 w. vertices V_j and edges same slope and incidence as Γ

i.e. Γ_0 is equal to Γ except for lengths of the edges and the position of Γ in $\mathbb{R}^2$





Plane tropical curves as balanced graphs

$\Gamma = Z(\max(a_i x + b_i y + c_i,\ i = 1, \ldots, n))$
determines subdivision of Δ_d
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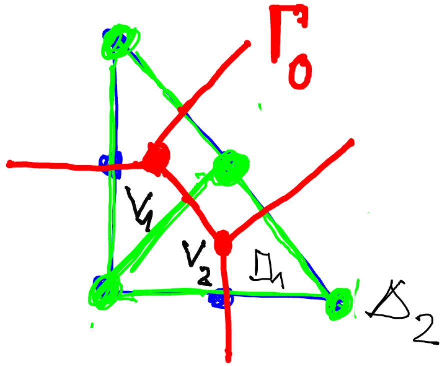
Edges orthog. to edges of $\square_j$ give graph
Γ_0 equal to Γ except for edge lenghts



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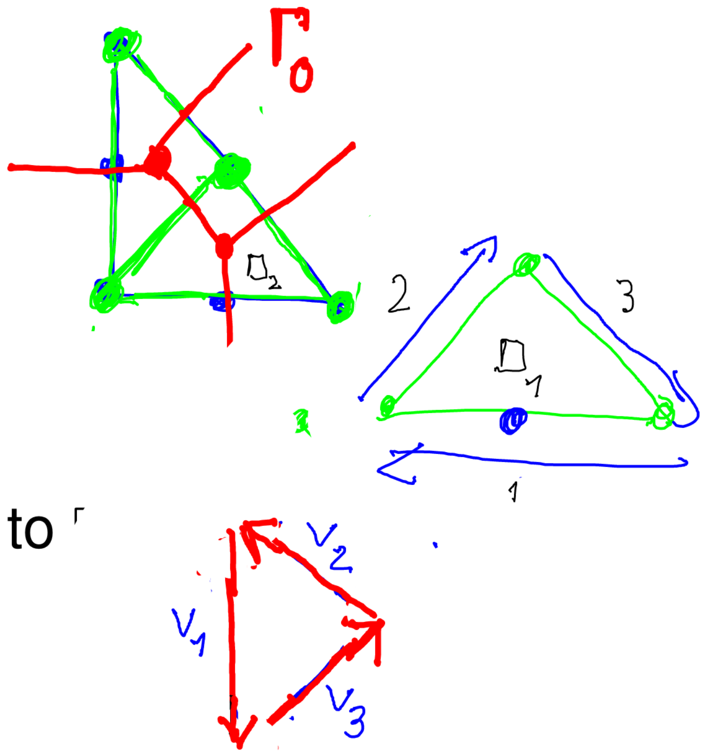
Let $\Box$ one of the polygons,
with verticies $(a_1, b_1), \ldots, (a_k, b_k)$ clockwise
$v_i = (b_i - b_{i+1}, a_{i+1} - a_i)$ is outer normal vector to
$(a_i, b_i) - (a_{i+1} b_{i+1})$ turned by 90 degrees)
Thus $v_1 + \ldots + v_k = 0$



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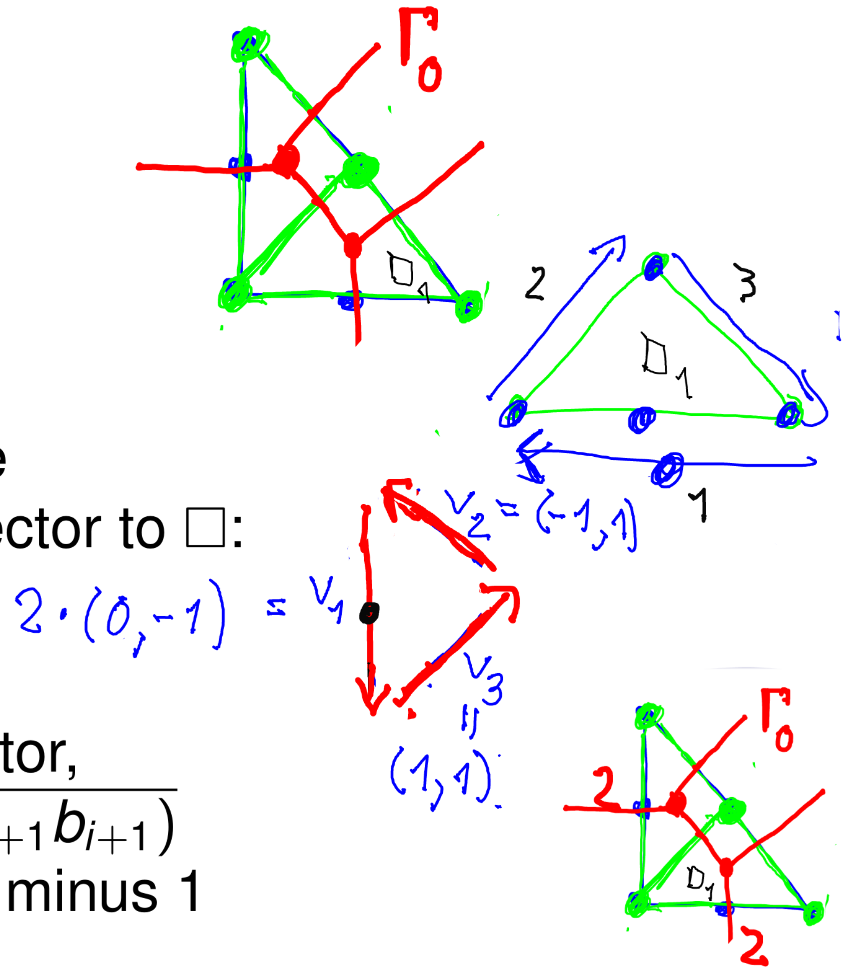
Write $v_i = w_i u_i$, with u_i primitive integer vector,
w_i :=multiplicity=lattice length of $\overline{(a_i, b_i), (a_{i+1} b_{i+1})}$
lattice length of side A=#integer points in A minus 1



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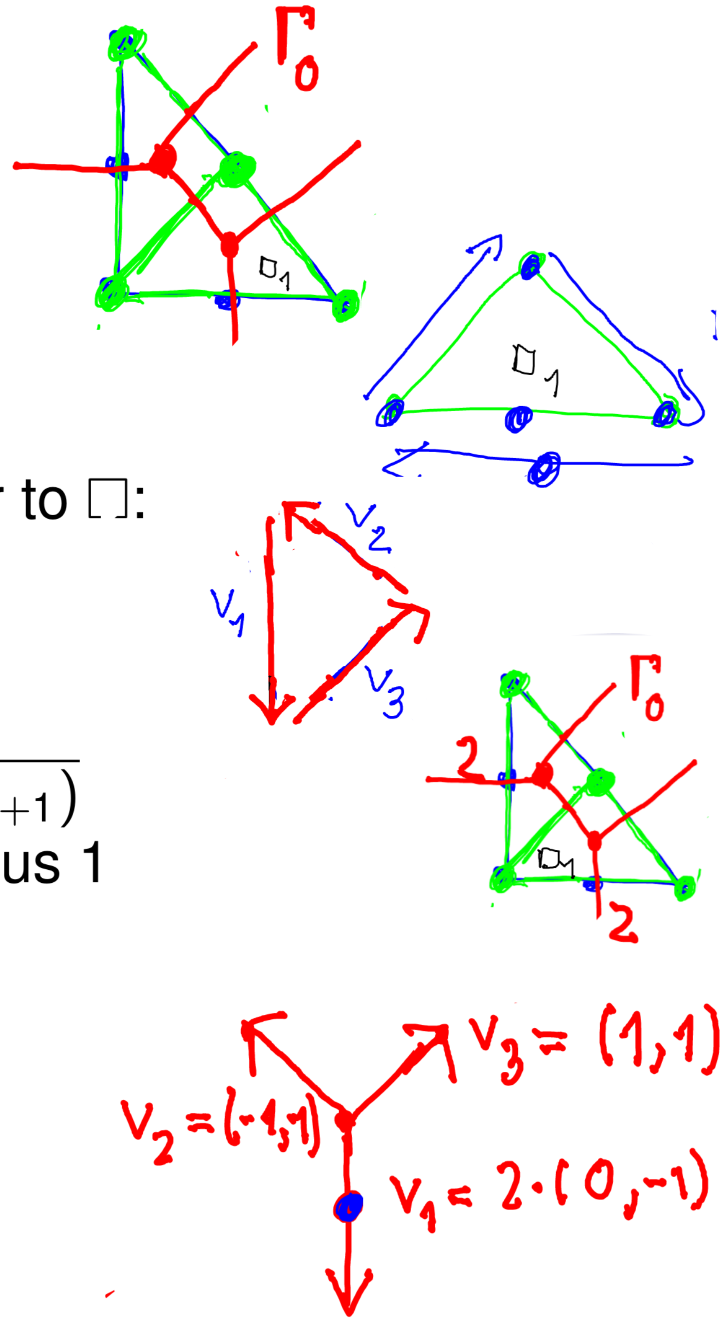
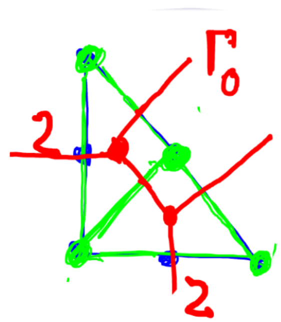
Balancing condition: for every vertex V of Γ
with outgoing edge vectors $v_i = w_i u_i$ have

$$\sum_{i=1}^{k} w_i u_i = 0$$



Edges of Δ_d are unions of some of segments $\overline{(a_i, b_i), (a_{i+1} b_{i+1})}$ thus Γ_0 has one (unbounded) edge through each of these segments.
Lattice length of edges of Δ_d is d
$\Longrightarrow$ d unbounded edges of Γ in each of the directions $(-1, 0)$, $(0, -1)$, $(1, 1)$ counted with weights



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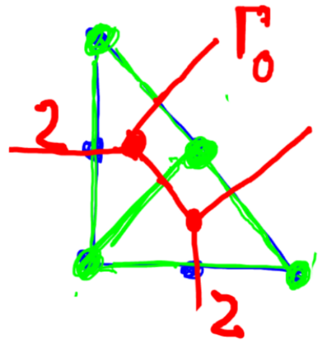
Mark each edge e by their weight $w(e) \in \mathbb{Z}_{>0}$.



Theorem

A plane tropical curve of degree d is a balanced weighted graph with edges of rational slopes with d (counted with weight) unbounded edges in directions $(-1, 0)$, $(0, -1)$, $(1, 1)$ satisfying at each vertex V the **balancing condition**

$$\sum_{i=1}^{k} w_i u_i = 0$$

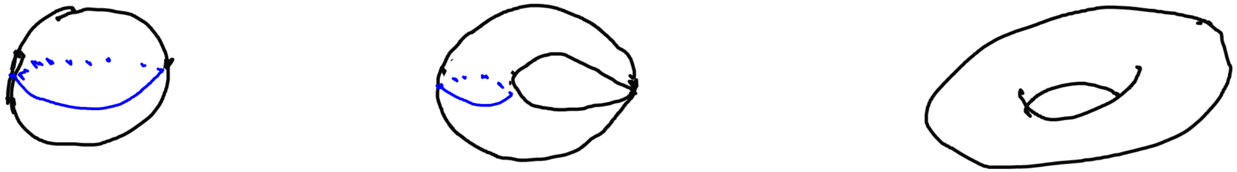
($v_i = w_i u_i$ outgoing edges at V, w_i weight, u_i primitive integer vector)

Genus of a plane tropical curve

Important topological invariant of a smooth projective algebraic curve C over $\mathbb{C}$: genus $g(C)$

Equals the number of handles of C, $g(C) = \frac{1}{2}dim(H_1(C, \mathbb{R}))$

For C singular $g(C)$ is genus of normalization (pull sing. apart)



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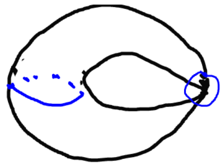
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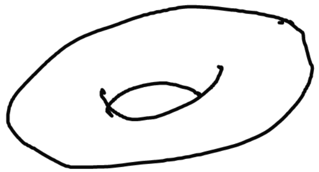
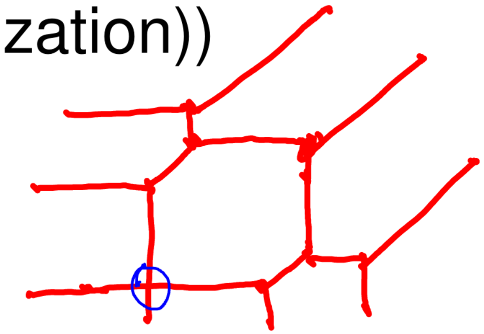

genus 1

The genus of a plane tropical curve Γ is $g(\Gamma) = dim(H_1(C, \mathbb{R}))$
=#indep. cycles in Γ. (cycles closed by intersect. edges do not count (normalization))


genus 0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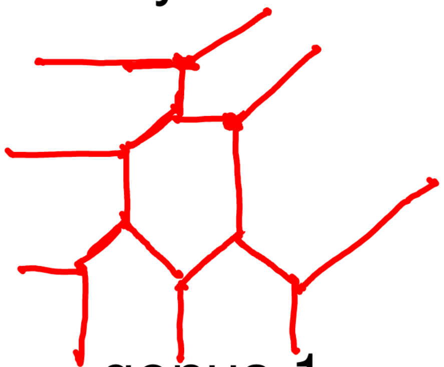
genus 1

Theorem

C smooth degree d complex curve in $\mathbb{P}^2$ then $g(C) = \frac{(d-1)(d-2)}{2}$

For C singular $g(C) < \frac{(d-1)(d-2)}{2}$

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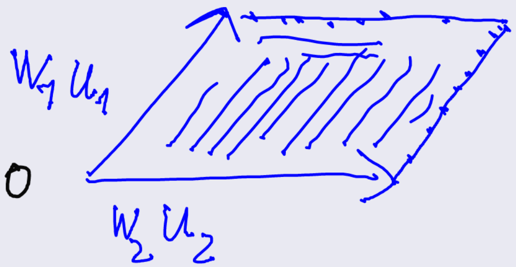
What is a smooth tropical curve?

Definition

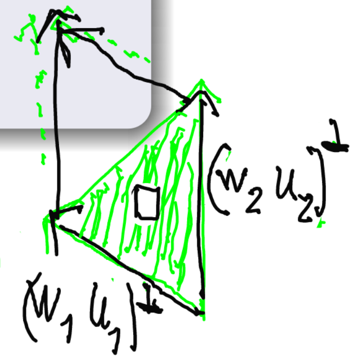
Let V be a trivalent vertex of plane tropical curve
u_1, u_2, u_3 unit outg. vec. along edges at V; weights w_1, w_2, w_3
Mikhalkin multiplicity of V is

$$m(V) = w_1 w_2 \left|\det(u_1, u_2)\right| \in \mathbb{Z}_{>0}$$



(area spanned by $w_1 u_1, w_2 u_2$)

$m(V) = 2\,\text{area}(\square)$

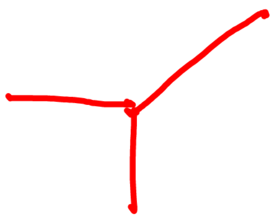


For V trivalent vertex $m(V) = w_1 w_2 |\det(u_1, u_2)|$

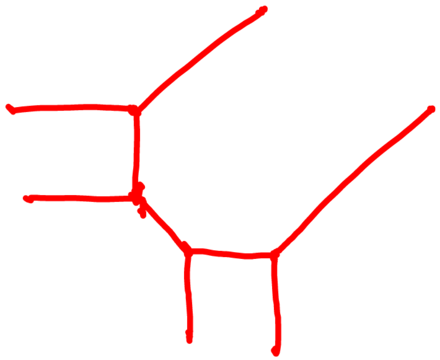
Definition

A plane tropical curve Γ of degree d is called nonsingular if

- Γ is trivalent
- for every vertex V of Γ we have $m(V) = 1$



line



nonsingular conic

Theorem

For Γ *smooth plane degree* d *tropical curve* $g(\Gamma) = \frac{(d-1)(d-2)}{2}$
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Show first part (second part exercise):
$\Gamma_0 := \#$vertices of Γ, $\Gamma_1 = \#$ bounded edges of Γ
The genus of Γ is

$$g(\Gamma) = 1 + \Gamma_1 - \Gamma_0 \tag{1}$$

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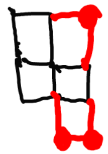
adding a cycle means adding k edges and k-1 vertices

$$g(\Gamma) = 1 + \Gamma_1 - \Gamma_0 \quad (1)$$



Γ has $3d$ unbounded edges, every vertex is trivalent, every bounded edge connects 2 vertices, thus

$$3d + 2\Gamma_1 = 3\Gamma_0 \quad (2)$$

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Δ_d has area $\frac{d^2}{2}$. Triangle corresp. to each vertex V has area $\frac{1}{2}m(V) = \frac{1}{2}$, thus

$$\Gamma_0 = d^2 \tag{3}$$

Combining (1), (2), (3) gives $g(\Gamma) = \frac{(d-1)(d-2)}{2}$.

Enumerative geometry of curves:
Count curves satisfying suitable conditions
These questions has applications e.g. in string theory.

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Space of (singular) curves in $\mathbb{P}^2$ of degree d and genus g,
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Thus $\exists$ finitely many degree d genus g curves through $3d + g - 1$ general points in $\mathbb{P}^2$. What is this number?

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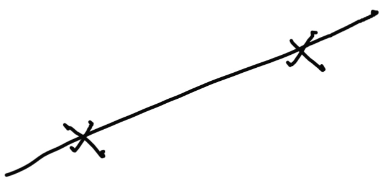
Definition

The *Severi degree* $N_{d,g}$ is the number of degree d genus g curves in $\mathbb{P}^2$ through $3d+g-1$ general points in $\mathbb{P}^2$ (independent of the choice of points).

e.g. $N_{1,0}=1$ (there is one line through 2 points)



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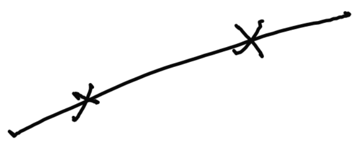

To determine Severi degrees one uses advanced tools of algebraic geometry to study parameter spaces of curves [C-H]

Instead count tropical curves

For instance we see: though 2 general points in $\mathbb{R}^2$ there is a unique tropical line.

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Mikhalkin multiplicity of V is $m(V) = w_1 w_2 |\det(u_1, u_2)|$ (area spanned by $w_1 u_1, w_2 u_2$)

Assume Γ is trivalent. The Mikhalkin multiplicity of Γ is

$$m(\Gamma) = \prod_{V \text{ vertex of } \Gamma} m(V) \in \mathbb{Z}_{>0}$$

Definition

Given $3d + g - 1$ general points in $\mathbb{R}^2$, there are finitely many degree d genus g tropical curves Γ through the p_i, all trivalent
The tropical Severi degree is

$$N_{d,g}^{\text{trop}} = \sum_{\Gamma} m(\Gamma) = \sum_{\Gamma} \prod_{V \text{ vertex of } \Gamma} m(V).$$

(sum over degree d genus g tropical curves through the p_i.

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Theorem (Mik1)

Severi degrees and tropical Severi degrees agree:

$$N_{d,g} = N_{d,g}^{trop}$$

This allows us to do enumerative geometry of curves via combinatorics

Counting tropical curves is much easier than counting complex curves, still the combinatorics is complicated
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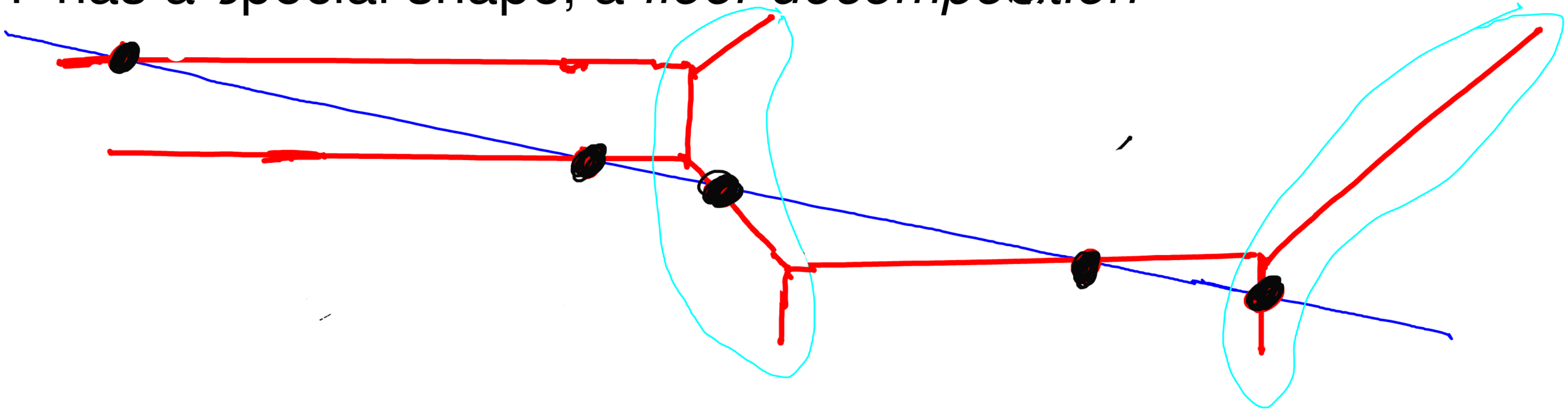
Floor diagrams are tool to simplify the task

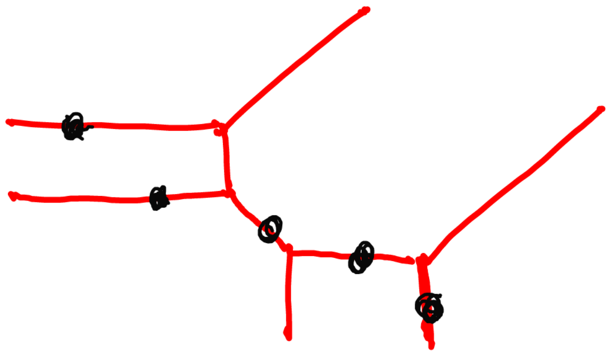
Count degree d genus g tropical curves Γ through $3d + g - 1$ general points in $\mathbb{R}^2$

Choose these points to lie on a line of extremely small irrational slope, stretched out widely (horizontally stretched)
Then Γ has a special shape, a *floor decomposition*



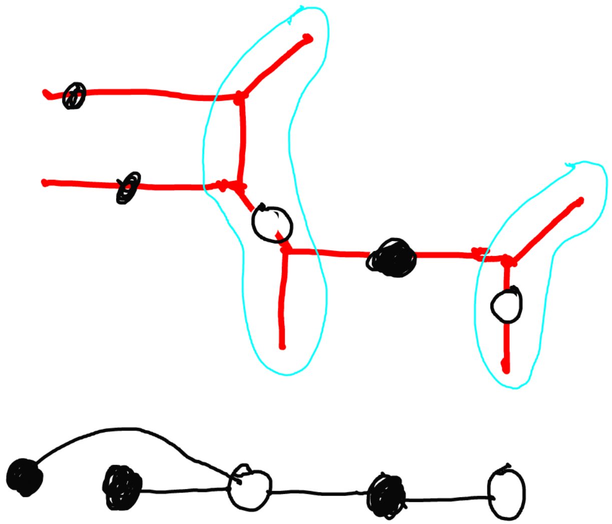
A horizontal edge of C is called an **escalator**
A connected component of closure of complement of escalators in Γ is called a **floor**.
The following properties hold:

1. Every floor and every escalator contains precisely one marked point.
2. Only the escalators can have weights different from 1
3. any vertex v has multiplicity $m(v) = 1$, unless it is adjacent to am escalator e, in which case the multiplicity is $m(v) = w(e)$.



To Γ tropical curve through horizontally stretched conf. of points associate marked floor diagram.

escalators: horizontal segments of Γ
floors: conn. comp. of complem. of escalators. One marked point on every floor and escalator
Floor diagram: black vertex for escalator white vertex for floor

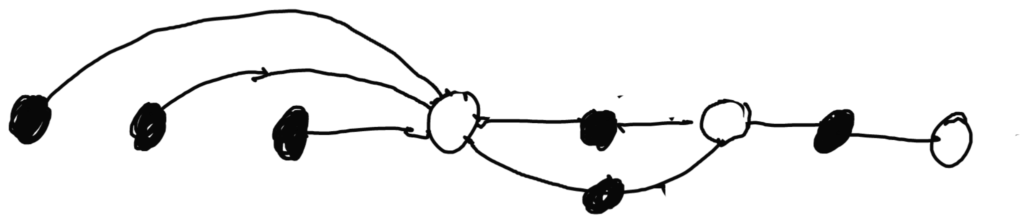


connect if escalator connects to floor
keep weight of escalator

To count tropical curves we can just count floor diagrams

Description of floor diagrams

1. Every bounded edge connects a black and a white vertex
2. Every unbounded edge connects to a black vertex
3. every black vertex is connected to two edges, one incoming (i.e. from left), one outgoing, both of the same weight.
4. white vertices v can have several incoming and outgoing edges with $\sum_{e-incoming} w(e) - \sum_{e-outgoing} w(e) = +1$



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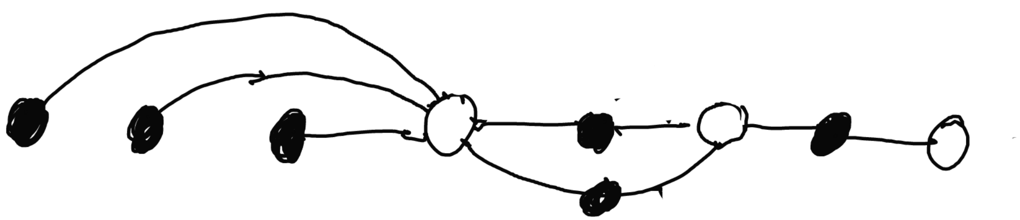
4. white vertices v can have several incoming and outgoing edges with $\sum_{e-incoming} w(e) - \sum_{e-outgoing} w(e) = -1$

A floor diagram of degree d and genus g (i.e. of degree d genus g tropical curve)

has d incoming edges of weight 1, no outgoing edges and g cycles



Floor diagrams

Put $m(\Lambda) := \prod_{e \text{ edges}} w(e)$

$$N^{trop}_{d,g}(y) = \sum_{\text{floor diagrams } \Lambda \text{ of degree } d \text{ and genus } g} m(\Lambda)$$

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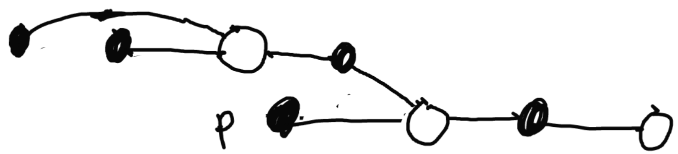
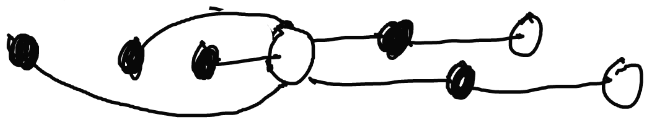
Example: rational cubics

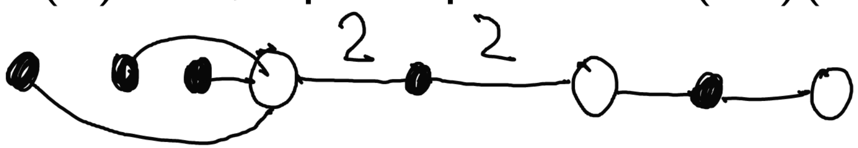
We use floor diagrams to compute $N_{3,0}$, the number of genus 0 cubics through 8 general points



$m(\Lambda) = 1$, point positions $p = 1, 2, 3, 4, 5$ count 5



$m(\Lambda) = 1$, point positions (56)(78), (57)(68) (58)(67) count 3



$m(\Lambda) = 2 \times 2 = 4$ point positions fixed count 4

$\Longrightarrow N_{3,0} = 5 + 3 + 4 = 12$

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