

On non-linear susceptibility in supercooled liquids

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Abstract. In this paper, we discuss theoretically the behaviour of the four-point non-linear susceptibility and its associated correlation length for supercooled liquids close to the mode-coupling instability temperature T_c . We work in the theoretical framework of the glass transition as described by mean-field theory of disordered systems, and the hypernetted-chain approximation. Our results give an interpretation framework for recent numerical findings on heterogeneities in supercooled liquid dynamics.

1. Introduction

Recently, a lot of attention has been devoted to understanding the nature of dynamical heterogeneities in supercooled liquids [1–7]. Many numerical experiments have found long-lived dynamical structures which are characterized by a typical length and a typical relaxation time which depend on the values of the external parameters (temperature and density). A way to quantify these dynamical heterogeneities is in terms of the four-point density function, and its associated non-linear susceptibility, which show power-law behaviour as one approaches the mode-coupling temperature T_c from above. In this paper we review the details of the theoretical calculations of this function put forward in [8–10] and discuss some new results.

At the glass transition one observes freezing of density fluctuations. The function

$$g_2(x) = \langle (\rho(x+y) - \bar{\rho})(\rho(y) - \bar{\rho}) \rangle \quad (1)$$

is often regarded as the Edwards–Anderson order parameter signalling the onset of glassiness. It is therefore quite natural to try to interpret the dynamical heterogeneities and the correlation length in terms of fluctuations of the order parameter, and study the four-point function

$$g_4(x) = \langle [(\rho(x+y) - \bar{\rho})(\rho(y) - \bar{\rho})]^2 \rangle - \langle (\rho(x+y) - \bar{\rho})(\rho(y) - \bar{\rho}) \rangle^2 \quad (2)$$

and its related non-linear susceptibility $\chi_4 = \int dx g_4(x)$. To our knowledge the first proposal to study the four-point function to identify a growing correlation length in structural glasses was in [11] in the context of a numerical study of a Lennard-Jones liquid. There, no sign of growing correlation was found, probably because of the insufficient thermalization. However, more accurate measurements [4, 5, 9, 12] show that there is a dynamical correlation length which grows as T_c is approached.

Here we would like to investigate theoretically the behaviour of this function in the context of the picture of the glass transition that comes out from the study of disordered mean-field models [13], and from some approximation scheme for molecular liquids [14].

In mean-field disordered systems one finds that, on decreasing the temperature from the liquid phase, two different transitions appear: a dynamical transition at a temperature

T_c , and a static (Kauzmann-like) transition at a lower temperature T_K . At the dynamical transition T_c , identified with the mode-coupling theory [15] transition temperature, equilibrium density fluctuations freeze and ergodicity breaks down. Below that temperature, the Boltzmann distribution is decomposed into an exponentially large number of ergodic components $e^{N\Sigma(T)}$. $\Sigma(T)$, the logarithm of the number of these components, is the configurational entropy, which decreases with decreasing temperature, and the ‘static’ transition signals the point where $\Sigma(T_K) = 0$. Dynamically, a non-zero Edwards–Anderson order parameter signals freezing.

As has been remarked many times, this theory misses the existence of local activated processes which restore ergodicity below T_c . These can be included phenomenologically to complete the picture. We will suppose that the ergodic components which the ideal theory predicts below T_c become in real systems metastable states (or quasistates), capable of confining the system for some large, but finite times on given portions of the configuration space. The inclusion of activated processes, although done by hand, has far-reaching consequences.

The foundation of the notion of quasistates is based on the timescale separation (as can be seen in the shape of the structure function), which allows one to consider ‘fast’ degrees of freedom quasi-equilibrated, before the ‘slow’ degrees of freedom can move. So, this notion applies below as well as above T_c , where the two-step relaxation is predicted even by the ideal theory. This point has been recently stressed in [16] in a different context. Both above and below T_c , we can talk of quasistates in which the system equilibrates almost completely before relaxing further. The typical lifetime of the quasistate will be of the order of the alpha relaxation time τ_α .

Our basic observation is that within the theoretical framework described, the quasistates correspond to highly correlated regions of the configuration space; typical configurations belonging to the same quasistate would appear to be highly correlated. On the other hand, configurations belonging to distant quasistates, such as the ones which correspond to large time separation $t \gg \tau_\alpha$, show typically low correlations.

We argue, then, that the dynamical correlation length and susceptibility observed in the simulations referred to above can be estimated by the corresponding quantities within a quasistate. On the other hand, the long-time limits of the same quantities, i.e. the values reached for times much larger than the lifetime of the quasistates, correspond to maximally distant quasistates. This predicts maximal fluctuations and heterogeneity on a timescale of the order of τ_α .

2. How to compute quasistate averages: ‘recipes for metastable states’

In this section, we address the question of how to compute correlation functions within single quasistates, reviewing some ‘recipes’ that were put forward in [17, 18]. Let us consider the case of mean-field spin-glass models below T_c , where there is true ergodicity breaking and the quasistates are true ergodic components. Suppose the temperature to be above T_K so that the configurational entropy $\Sigma(T) > 0$. Given any local observable $A(x)$, its Boltzmann average can be decomposed as

$$\langle A(x) \rangle_{\text{Boltzmann}} = \sum_{\alpha} w_{\alpha} \langle A(x) \rangle_{\alpha} \quad (3)$$

where the index α runs over all the $e^{N\Sigma}$ states; the weights of the different states w_{α} would all be of the same order, $w_{\alpha} \approx \exp(-N\Sigma(T))$. In the following we will be interested in computing space averages (correlation functions) among local observables, $\int dx \langle A(x) \rangle \langle B(x+y) \rangle$. If by $\langle \cdot \rangle$ we mean a Boltzmann average, we can expand each of the two averages according to (3)

and find that

$$\int dx \langle A(x) \rangle \langle B(x+y) \rangle = \int dx \sum_{\alpha, \beta} w_\alpha w_\beta \langle A(x) \rangle_\alpha \langle B(x+y) \rangle_\beta \quad (4)$$

which, due to the fact that the number of ergodic components is exponentially large, is dominated by the terms in the double sum with $\alpha \neq \beta$.

Our major interest will be in computing instead averages of the kind

$$\int dx \sum_{\alpha} w_\alpha \langle A(x) \rangle_\alpha \langle B(x+y) \rangle_\alpha$$

i.e. correlation functions in a particular ergodic component. To this end, we can use a *conditional* Boltzmann prescription [17, 18], where one fixes a reference configuration $Y = \{y_1, \dots, y_N\}$, and only the configurations $X = \{x_1, \dots, x_N\}$ that are similar enough to the reference configuration are given a non-vanishing weight.

Let us consider as a measure of similarity among two configurations X and Y the following function, which we call the overlap:

$$q(X, Y) = \int dx dy (\rho_X(x) - \bar{\rho})(\rho_Y(y) - \bar{\rho})w(|x - y|) \quad (5)$$

where:

- $\rho_Z(z)$ ($Z = X, Y$) is the microscopic density corresponding to the configuration Z :
 $\rho_Z(z) = \sum_i \delta(z_i - z)$;
- the function $w(r)$ is a short-range sigmoid (or step) function such that if r_0 denotes the typical radius of the particles, $w(r)$ is close to 1 for $r \leq ar_0$ and close to zero otherwise. The value of $a = 0.3$ gives a measure of overlap that is not too sensitive to small atomic displacements.

Notice that, with this definition, $q(X, Y)$ is maximal if $X = Y$, while it is equal to zero if X and Y are uncorrelated. Notice also that one can write $q(X, Y) = \sum_{i,j} w(|x_i - y_j|)$, a form which is manifestly invariant under permutations of the particles.

Suppose now that we fix a reference configuration Y , chosen with Boltzmann probability at temperature T , and consider the conditional probability

$$P_q(X|Y) = \frac{e^{-\beta H(X)} \delta(q(X, Y) - q)}{Z_q(Y)} \quad (6)$$

where the constrained partition function is

$$Z_q(Y) = \int dX e^{-\beta H(X)} \delta(q(X, Y) - q). \quad (7)$$

As Y is by hypothesis an equilibrium configuration, it will belong to some quasistate α , so, if we choose q as the typical overlap among configurations in this quasistate (with probability one, almost all configurations have the same overlap), i.e. the Edwards–Anderson parameter of the state q_{EA} , we will be able to compute the quasistate averages: given two observables $A(X)$ and $B(X)$ we can write

$$\sum_{\alpha} w_\alpha \langle A \rangle_\alpha \langle B \rangle_\alpha = \int dY \frac{e^{\beta H(Y)}}{Z} A(Y) \int dX \frac{e^{-\beta H(X)} \delta(q(X, Y) - q)}{Z_q(Y)} B(X). \quad (8)$$

Notice that if, on the other hand, in (6) we were to choose q as the typical overlap among different quasistates, the constraint would be completely irrelevant and we could get the Boltzmann average (4).

Notice that the overlap that we consider is a masked integral of the density–density correlation function among the two configurations X and Y . We are interested in studying the fluctuation of this quantity, which is the following integral of the four-point function:

$$\begin{aligned} \chi_4 = \beta(\overline{\langle q \rangle^2} - \overline{\langle q \rangle}^2) &= \int dx dy dz dr w(x-y)w(z-r) \\ &\times \frac{\overline{\langle (\rho_X(x) - \bar{\rho})(\rho_Y(y) - \bar{\rho}) \rangle \langle (\rho_Y(z) - \bar{\rho})(\rho_Z(r) - \bar{\rho}) \rangle}}{\overline{\langle (\rho_X(x) - \bar{\rho})(\rho_Y(y) - \bar{\rho}) \rangle} \overline{\langle (\rho_X(z) - \bar{\rho})(\rho_Y(r) - \bar{\rho}) \rangle}} \end{aligned} \quad (9)$$

where we have denoted with the angular brackets the conditional average with the distribution (6), and with the bar the average over the canonical distribution of the reference configuration Y .

Within this formalism, the generating functional of the correlation functions is the constrained free energy

$$V(q) = -(\beta/N) \int dY \frac{e^{\beta H(Y)}}{Z} \log(Z_q(Y)). \quad (10)$$

This function has been computed in various models having a glass transition, including mean-field disordered models and simple liquids in the HNC approximation, all giving consistent results [17, 18]. The shape of V as a function of q allows one to distinguish between the liquid and glass phase. We show the potential in figure 1 for hard spheres in the HNC approximation.

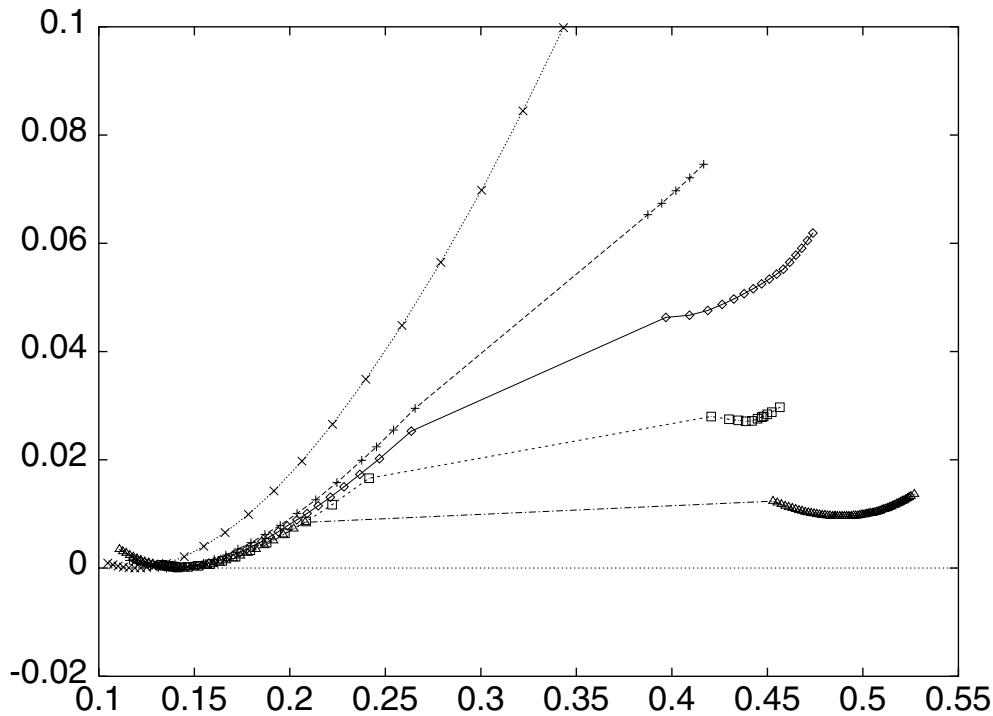


Figure 1. The potential for HNC hard spheres at several densities. We show the high- and low- q regions of the potential. The lines joining them are just guides for the eyes. Lower curves correspond to higher densities. At low density the potential is convex. The appearance of a secondary minimum signals the breaking of ergodicity, with exponentially many states.

In the liquid phase at high temperature, the potential is convex with a unique minimum for $q = 0$ which corresponds to the typical overlap among random liquid configurations. Lowering the temperature, the potential loses convexity, until, when T_c is reached, it develops a secondary minimum, at a high value of q . The height of the secondary minimum with respect to the first one is related to the configurational entropy by $V_{sec} - V_{pri} = T\Sigma(T)$, which vanishes at T_K .

A remarkable fact that has often been discussed [17] is that while the properties of the low- q minimum reflect the properties of the full Boltzmann average, the properties of the high- q minimum reflect the properties of averages in a single ergodic component.

In the shape of the potential, the MC transition appears as a spinodal point, and as such it has a divergent susceptibility. In fact, general relations in the effective-potential theory imply that the susceptibility is given just by the inverse curvature of the potential in the minimum, i.e. $\chi_4 = 1/V''(q)|_{\text{secondary minimum}}$. This quantity diverges for $T \rightarrow T_c$, which, in turn, implies the divergence of the spatial range of the correlations. Generically, in all the models studied, the slope of the inflection point vanishes linearly for $T \rightarrow T_c$, implying $\chi_4(T) \simeq |T - T_c|^{-\gamma}$ with a mean-field exponent $\gamma = 1/2$. We notice that χ_4 computed at the primary minimum represents the Boltzmann average of the order parameter fluctuations and is completely regular at T_c .

In figure 2 we see the susceptibility computed by this procedure (circles), which shows a divergence at T_c .

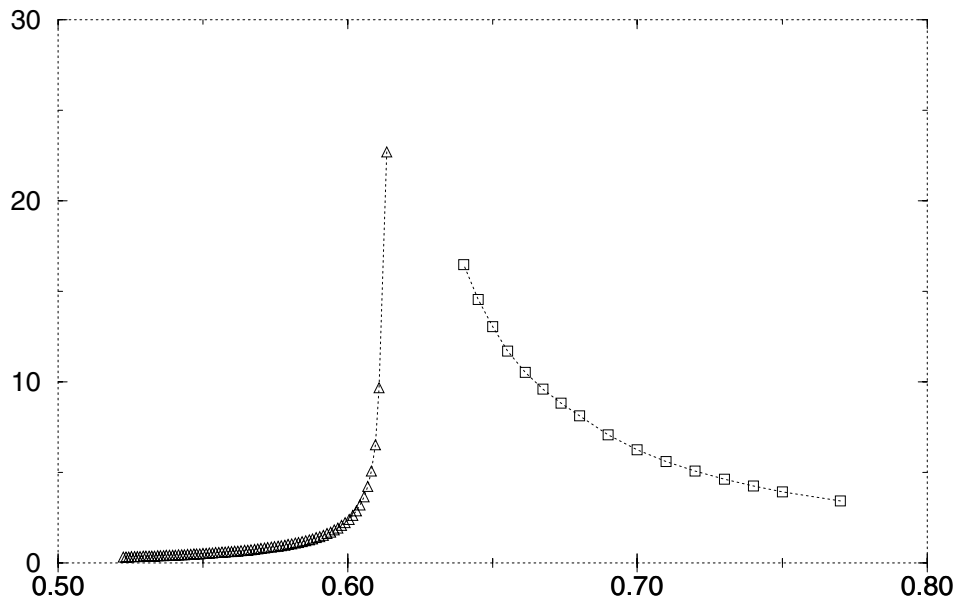


Figure 2. The susceptibility of the metastable states for the p -spin model with $p = 3$. Here $T_c = 0.612$. The low-temperature data come from the potential theory. The high-temperature data come from the dynamical equations.

3. A dynamical approach

The idea of considering a system coupled with a reference configuration can also be used in dynamics to compute the time-dependent susceptibility. In this context it is convenient

to couple with the initial configuration X_0 . Consider a system at equilibrium at time zero with respect to the Hamiltonian $H(X)$ which evolves for positive times with the modified Hamiltonian

$$H_{tot}(X) = H(X) - \epsilon q(X, X_0). \quad (11)$$

For small ϵ , linear response theory at equilibrium implies that

$$\chi_4(t) = \beta(\langle q(X_t, X_0)^2 \rangle - \langle q(X_t, X_0)q(X_0, X_0) \rangle) = \frac{\partial \langle q(X_t, X_0) \rangle}{\partial \epsilon}. \quad (12)$$

The problem of studying the evolution of a system with Hamiltonian (11) can in principle be addressed within any dynamical approximation scheme (e.g. MCT).

However, for the time being we have only addressed the problem in the context of the p -spin model, which, for all present purposes, should capture the essential features of the function $\chi_4(t)$. Clearly, as the model is completely lacking in any spatial structure, in order to infer from the behaviour of χ_4 something about a correlation length we need to resort to equation (9).

The p -spin model [19] describes N interacting variables S_1, \dots, S_N (spins) on the sphere $\sum_i S_i^2 = N$, with Hamiltonian $H = \sum_{i_1 < \dots < i_p} J_{i_1 \dots i_p} S_{i_1} \dots S_{i_p}$ where the couplings are random independent Gaussian variables with zero mean and variance $J^2 = p!/(2N^{p-1})$. The appropriate measure of overlap for this system is $q(S, S') = (1/N) \sum_i S_i S'_i$. For this model it is customary to consider Langevin dynamics, which in our case will be implemented using the Hamiltonian $H_{tot}(S) = H(S) - \epsilon q(S, S_0)$, where $S_0 = S(t=0)$ is an equilibrium initial condition. We have

$$\frac{dS_i(t)}{dt} = -\frac{\partial H(S(t))}{\partial S_i} + \epsilon S_i(0) + \eta_i(t) \quad (13)$$

where $\eta_i(t)$ is a white noise with amplitude $2T$ and $\mu(t)$ is a Lagrange multiplier which ensures that the spherical constraint is obeyed at all times.

Using standard manipulations based on the Martin–Siggia–Rose functional integral, one can write a self-consistent equation for a single spin which, using the notation $f(q) = 1/2q^p$, reads

$$\frac{dS(t)}{dt} = -\mu(t)S(t) + \int_0^t ds f''(C(t, s))R(t, s)S(s) + \beta f'(C(t, 0))S(0) + \epsilon S(0) + \xi_i(t) \quad (14)$$

where $\xi(t)$ is a coloured Gaussian noise with variance

$$\langle \xi(t)\xi(s) \rangle = f'(C(t, s)) + 2T\delta(t-s) \quad (15)$$

where C and R are the correlation and response functions of the system, to be determined self-consistently from $C(t, s) = \langle S(t)S(s) \rangle$, $R(t, s) = \langle \delta S(t)/\delta \xi(s) \rangle$. The detailed derivation of (14) is rather standard (see e.g. [20]) and we do not reproduce it here.

From (14), taking the correlations with $S(s)$ and $\xi(s)$ one can derive equations for C and R , which read, for $t > s$,

$$\begin{aligned} \frac{\partial C(t, s)}{\partial s} &= -\mu(t)C(t, s) + \int_0^t du f''(C(t, u))R(t, u)C(u, s) \\ &\quad + \int_0^s du f'(C(t, u))R(s, u) + \beta f'(C(t, 0))C(s, 0) + \epsilon C(s, 0) \\ \frac{\partial R(t, s)}{\partial s} &= -\mu(t)R(t, s) + \int_s^t du f''(C(t, u))R(t, u)R(u, s). \end{aligned} \quad (16)$$

Together with the equation specifying the time dependence of $\mu(t)$:

$$\begin{aligned} \mu(t) = & \int_0^t du f''(C(t, u))R(t, u)C(u, t) \\ & + \int_0^t du f'(C(t, u))R(t, u) + \beta f'(C(t, 0))C(t, 0) + \epsilon C(t, 0) + T \end{aligned} \quad (17)$$

equations (16), (17) form a complete set, which can be solved numerically to derive the value of $\chi_4(t)$ as

$$\chi_4(t) = dC(t, 0)/d\epsilon. \quad (18)$$

With a simple step-by-step integration [21] we could reach times of the order of 1000. This allowed us, in the case where $p = 3$, to compute the function $\chi_4(t)$ down to temperature $T = 0.7$, compared with a critical temperature $T_d = 0.612$. More sophisticated algorithms (see the contribution of Latz to this Special Issue) will allow one in the near future to approach the critical temperature much more closely. Our results for the function $\chi_4(t)$ are displayed in figure 3 for various temperatures. We see that χ has a maximum which becomes higher and higher as the temperature is lowered, and is pushed towards larger and larger times. This is the behaviour that is seen qualitatively in the numerical simulations [9, 12].

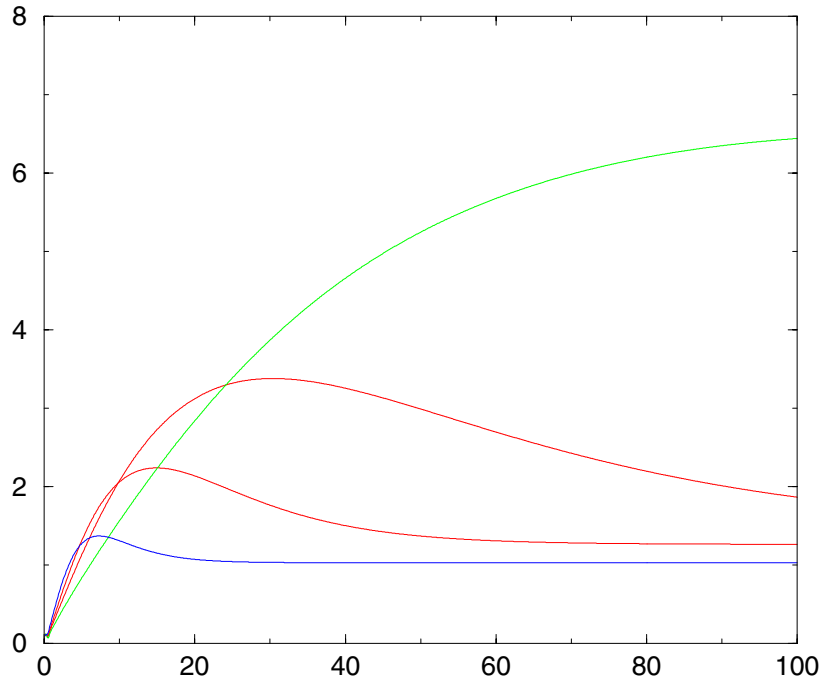


Figure 3. The dynamical susceptibility for the p -spin model ($p = 3$) as a function of time for several temperatures; the lower the maximum, the higher the temperature. From bottom to top, $T = 1.0, 0.9, 0.8, 0.7$.

As shown in reference [9], we define t^* as the time at which χ^* is maximum; we find that $\chi_4^* = \chi_4(t^*)$ exhibits a divergence at T_c , as presented in figure 2. Both quantities behave as powers of $T - T_c$: $t^* \sim (T - T_c)^{-\alpha}$; $\chi_4^* \sim (T - T_c)^{-\gamma}$. A best fit gives the values $\gamma = 0.52 \pm 0.02$, $\alpha = 1.1 \pm 0.1$.

4. Conclusions

In this paper we have reviewed the analysis of the non-linear susceptibility in supercooled liquids and glasses that comes from the mean-field theory of disordered systems, and liquid models in the HNC approximation.

The theory predicts that while the long-time, equilibrium susceptibility remains finite and is regular at all temperatures, the finite-time susceptibility displays a maximum as a function of time which becomes higher and higher and displaced to larger and larger times for temperatures close to T_c . This behaviour is a consequence of the critical character of the mode-coupling-like dynamical transition predicted by the ideal theory described in this paper. For real systems one can expect a similar behaviour, but with a rounding off of the divergence.

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