Glassy Transition and Aging in a Model Without Disorder

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We study the off-equilibrium relaxational dynamics of the Amit-Roginsky ϕ^3 field theory, for which the mode coupling approximation is exact. We show that complex phenomena such as aging and ergodicity breaking are present at low temperature, similar to what is found in long range spin glasses. This is an example of how the mode coupling theory of the structural glass transition can be generalized to off-equilibrium situations.

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Recently there has been important progress in the understanding of the off-equilibrium dynamics of disordered systems. Although aging phenomena were observed in spin glasses several years ago [1] and studied by several groups in spin glasses [2] and many other different disordered systems, only quite recently has a dynamical mean field theory (DMFT) for nonequilibrium phenomena, based on microscopic spin glass models, been put forward [3–5]. Among the most interesting features which emerge from the theory are the violations at low temperature of time translation invariance (TTI) and of the fluctuation-dissipation theorem (FDT), even for very large times, implying breaking of ergodicity. This breaking of ergodicity cannot be described simply as a separation of the phase space into different ergodic components within which the system thermalizes as in usual broken-symmetry phases. Rather, the distribution function continues to spread over larger regions of configuration space as time goes on. This spreading gets slower and slower but never stops; thus the system never comes to equilibrium, and its properties depend on the time elapsed since the quench to low temperature.

Far from being confined to the realm of disordered systems, aging phenomena have been observed in a number of different systems without disorder, in experiments [6] and numerical simulations [7].

At present there is a great deal of theoretical research on spin-glass-like phenomena in nonrandom systems [8–10]. In [8,9] a mechanism has been proposed in which an effective disorder can be induced in *a priori* uniform systems (although with "complicated" interactions among the variables). In this Letter we focus on the dynamical mean field theory for glasses, usually called mode coupling theory (MCT), showing how it is possible to extend it in order to include nonequilibrium phenomena.

It has been noted [11] that the mean field dynamical equations of many disordered mean field models have, in the high temperature phase, a structure which is surprisingly similar to that of the equation of the mode coupling theory often used to describe the structural glass transition [12,13]. The mode coupling equations have been derived from molecular or hydrodynamical models based on an approximation in which vertex corrections are neglected in the perturbation expansion of the self-energy [14]. Systematic use is made of the hypothesis that the system under study is at thermal equilibrium. On the other hand, there is experimental evidence that real glasses are out of equilibrium [13]; thus it is highly desirable to extend the theory to include nonequilibrium phenomena.

This Letter is a step in that direction. Technically, we construct a theory in which we do not assume TTI or the FDT *ab initio* in the derivation of the equations. This could be done systematically in all models for which MCT can be derived by a field theoretical expansion. We have chosen here to study a simple model without quenched disorder, where the mode coupling approximation is exact. This is a "spherical" version of the ϕ^3 field theory originally proposed by Amit and Roginsky [15] to study the critical properties of the three-state Potts model. We will find deep connections with the dynamics of the spherical p-spin glass model [3,11,16] for $p = 3$. At high temperature, supposing TTI and the FDT one gets the usual MCT equations, similar to those first proposed by Leutheusser to study the glassy transition. At low temperature there is a phase where ergodicity is broken in a way now known from mean field spin glass dynamics.

Amit and Roginsky (AR) considered an $O(3)$ symmetric ϕ^3 field theory where the basic fields transform under rotation according to representations of high angular momentum quantum number l. They found that in the limit of large l the field theoretical perturbative expansion can be resummed. Recently the same method has been exploited in two interesting papers [17], to find soluble limits of nonlinear stochastic equations. The AR method is applicable any time the nonlinearity is represented by a bilinear term. In this Letter we discuss for simplicity the dynamics of the original AR model. In a forthcoming paper [18] we will discuss the case of the fluctuating compressible hydrodynamics of Das and Mazenko [14], which, although formally much more complicated than the present model, gives rise to equations with the same structure.

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The AR model is defined by the $O(3)$ invariant Lagrangian

$$\mathcal{L}[\psi,\psi^*] = \sum_{m=-l}^{l} \psi_m^*(x)[\mu^2 - \nabla^2]\psi_m(x) + \frac{g}{3!}\sqrt{N} \sum_{m_1,m_2,m_3} \begin{pmatrix} l & l & l \\ m_1 & m_2 & m_3 \end{pmatrix} [\psi_{m_1}(x)\psi_{m_2}(x)\psi_{m_3}(x) + \text{c.c.}], \quad (1)$$

with $N = 2l + 1$, and ψ_m transforms according to the irreducible N-dimensional representation of $O(3)$. In the following we will denote by $\mathcal{L}_I[\psi,\psi^*]$ the cubic part of the Lagrangian. The coefficients $\left(\begin{smallmatrix} l & l & l \\ m_1 & m_2 & m_3 \end{smallmatrix}\right)$ are the Wigner $3j$ symbols. To have a nonvanishing interaction term l must be even. Amit and Roginsky have shown that in the limit $l \to \infty$ vertex corrections in the perturbation expansion can be neglected. Thus, defining the equilibrium correlation function $G(x,y) = \sum_m \langle \psi_m(x)\psi_m^*(y)\rangle_{\text{Gibbs}}$, the perturbative series can be resummed, and the Dyson equation of the model in real space is simply

$$G^{-1}(x,y) = G_0^{-1}(x,y) - \frac{g^2}{2}[G(x,y)]^2, \quad (2)$$

where $G_0^{-1}(x,y) = \delta(x-y)(\mu^2 - \nabla^2)$.

We will consider a spherical variation of the AR model, imposing the constraint $\sum_m \psi_m(x)\psi_m^*(x) = N\tilde{q}$. In this situation, μ^2 can be viewed as a Lagrange multiplier which enforces the constraint. Equation (2) for $x = y$ (appropriately regularized if $D \geq 2$) should in this case be read as an equation for μ^2. It is possible to see *a posteriori* that the spherical constraint is necessary to obtain glassy behavior.

We study the relaxational dynamics of this model, neglecting any spatial dependence of the field ψ_m [we drop the x dependence and the gradient term in (1)]. This "zero-dimensional limit" considerably simplifies the analysis but, as we shall see, does not prevent glassy behavior in the model.

We consider the Langevin equation

$$\frac{d}{dt}\psi_m(t) = -\mu^2(t)\psi_m(t) - \frac{\partial \mathcal{L}_I}{\partial \psi_m^*} + \eta_m(t), \quad (3)$$

where η_m is a white noise with correlation function $\langle \eta_m(t)\eta_n^*(s)\rangle = 2T\delta_{mn}\delta(t-s)$, and T is the temperature. We will see *a posteriori* that it is possible to choose μ as a function of t to impose the spherical constraint $C(t,t) = \tilde{q}$ at all times. In order to generate the perturbation expansion of Eq. (3), we consider the Martin-Siggia-Rose generating functional

$$Z[h] = \int \mathcal{D}\psi\,\mathcal{D}\psi^* \left\langle \prod_{m,t} \delta\left[\frac{d}{dt}\psi_m(t) + \frac{\partial \mathcal{L}_t}{\partial \psi_m^*} - \eta_m(t)\right]\right\rangle \exp\left[\int dt \sum_m h_m^*(t)\psi_m(t)\right] J[\psi]. \quad (4)$$

$J[\psi]$ is the functional determinant of $d/dt + \partial^2\mathcal{L}_t/\partial\psi_m\partial\psi_m^*$ which, with the Ito convention, is equal to 1, and we denoted by $\mathcal{L}_t$ the time dependent "Lagrangian" $\mathcal{L}_t = \mu^2(t)\sum_m \psi_m^*\psi_m + \mathcal{L}_I$. We now follow a standard procedure [19] to prove that the structure of the diagrammatic expansion for dynamics is the same as that for statics. We introduce an auxiliary field $i\hat{\psi}_m(t)$, to get $Z = \int \mathcal{D}\psi\,\mathcal{D}\psi^*\,\mathcal{D}\hat{\psi}\,\mathcal{D}\hat{\psi}^* \exp(-\int dt\, i\hat{\psi}_m^*[\dot{\psi}_m - Ti\hat{\psi}_m + \partial\mathcal{L}/\partial\psi_m^*])$. For simplicity we have set $h_m = 0$. This will not affect our subsequent manipulations. We then introduce an even Grassman variable θ, and the "superfield" $\phi_m(t,\theta) = \psi_m(t) + \theta i\hat{\psi}_m(t)$. In terms of these, Z reads

$$Z = \int \mathcal{D}\phi\,\mathcal{D}\phi^* \exp\left(-\int dt \int d\theta \left\{\sum_m \phi_m^*(t,\theta)G_0^{-1}\phi_m(t,\theta) + \mathcal{L}_I[\phi,\phi^*]\right\}\right), \quad (5)$$

where G_0 is defined by $G_0^{-1} = (1 - \theta\partial/\partial\theta)\partial/\partial t - T\partial/\partial\theta + \mu^2(t)$. The time integration here starts from an initial time $t = 0$, where the system is in a random initial condition. This corresponds to the experimental situation of a sudden quench of the system from very high temperature at the initial time.

The interaction term in (5) is represented by the same function $\mathcal{L}_I$ as in statics, but its arguments are now the superfields ϕ and ϕ^* rather than just ψ and ψ^*. Thus the structure of the dynamical perturbation expansion in terms of the superfield is formally identical to that of statics in terms of the field. We define at this point the correlator

$$\begin{aligned} G(t,\theta;t',\theta') &= \frac{1}{N}\sum_m \langle \phi_m(t,\theta)\phi_m^*(t',\theta')\rangle \\ &= \frac{1}{N}\sum_m \{\langle \psi_m(t)\psi_m^*(t')\rangle + \theta\langle i\hat{\psi}_m(t)\psi_m^*(t')\rangle + \theta'\langle \psi_m(t)i\hat{\psi}_m^*(t')\rangle + \theta\theta'\langle i\hat{\psi}_m(t)i\hat{\psi}_m^*(t')\rangle\}. \end{aligned} \quad (6)$$

We see that G codes for the correlation function $C(t,t') = (1/N)\sum_m \langle \psi_m(t)\psi_m^*(t')\rangle$ and the response function $r(t,t') = (1/N)\sum_m \langle \psi_m(t)i\hat{\psi}_m^*(t')\rangle = \langle \delta\psi_m(t)/\delta\eta_m(t')\rangle$, while the conservation of probability implies $(1/N)\sum_m \langle i\hat{\psi}_m(t)i\hat{\psi}_m^*(t')\rangle = 0$. Causality and the Ito convention imply $r(t,t') = 0$ for $t' > t$ and $r(t^+,t) = 1$. In terms of G_0 and G the Dyson equation of the theory has the same form as (2), replacing x and y, respectively, by (t,θ) and (t',θ'). The appearance of C and r only in the combination given by G is a consequence of the gradient character of the Langevin equation.

Inverting Eq. (2), multiplying it by G, and disentangling the superfield notation, we get coupled equations for the correlation and the response function; for $t > t'$ we have

$$\frac{\partial C(t,t')}{\partial t} = -\mu^2(t)C(t,t') + \frac{1}{2}g^2\left\{\int_0^t ds\, 2C(t,s)r(t,s)C(t',s) + \int_0^{t'} ds\, C^2(t,s)r(t',s)\right\},$$
$$\frac{\partial r(t,t')}{\partial t} = -\mu^2(t)r(t,t') + \frac{1}{2}g^2\int_0^t ds\, 2C(t,s)r(t,s)r(s,t'), \tag{7}$$

while for arbitrary $\mu^2(t)$, $C(t,t)$ satisfies

$$\frac{1}{2}\frac{dC(t,t)}{dt} = -\mu^2(t)C(t,t) + \frac{1}{2}g^2\int_0^t ds\, 3C^2(t,s)r(t,s) + T. \tag{8}$$

As mentioned above, we can choose $\mu(t)$ to satisfy $C(t,t) = \tilde{q}$ at all times. We recognize in (7) and (8) well studied equations in mean field spin glass theory: They are identical to those found in the dynamics of the so-called *spherical p-spin model* [11] for $p = 3$. This is defined by the Hamiltonian

$$H = -\sum_{i<j<k} J_{ijk}S_iS_jS_k, \tag{9}$$

where the "spins" S's are real variables satisfying the constraint $\sum_{i=1}^N S_i^2 = N\tilde{q}$, and the couplings J_{ijk} are independent Gaussian variables, symmetric under the interchange of any pair of indices, with zero mean and variance $\overline{J_{ijk}^2} = g^2/3N^2$. It is interesting to note that this model displays a glassy behavior analogous to that of the Potts glass [20].

The equivalence for large N of the $O(3)$ symmetric vertex with a random one in dynamics had already been noted in [17] and studied in detail in [21] for the turbulence problem. It is remarkable that the AR model, introduced originally as a soluble limit of the Potts model, is dynamically equivalent to the disordered model which displays glassy behavior analogous to that of the Potts glass [20].

Note that Eqs. (7) and (8) are causal first order integro-differential equations which admit a unique solution for finite times.

In a seminal paper [3] Cugliandolo and Kurchan have found an asymptotic solution to the equation of the p-spin model, showing the existence of a low temperature phase in which TTI and the FDT are violated even in the infinite time limit. The explicit numerical integration of equations with the same structure obtained for a related model [4] supports the asymptotic analysis of [3], which we review briefly here in the present context. At high temperature, the asymptotic dynamics is completely characterized by the two functions $C_{\rm as}(\tau) = \lim_{t\to\infty} C(t+\tau, t)$ and $r_{\rm as}(\tau) = \lim_{t\to\infty} r(t+\tau, t)$ related by the FDT, and such that $\lim_{\tau\to\infty} C_{\rm as}(\tau) = 0$. The long time limit of μ is given by $\mu^2 = (g^2/2T)\tilde{q}^2 + T/\tilde{q}$. This is just the zero-dimensional limit of Eq. (2) for the statics. Accordingly the "energy" of the system, $E = \langle \mathcal{L}_I(t)\rangle = -(g^2/3T)\int_0^t ds\, C^2(t,s)r(t,s)$, tends to the static value $E_{\rm static} = -g^2\tilde{q}^3/3T$. The set of equations (7) and (8) reduces to a single one for, e.g., the response function, almost identical to the one originally proposed by Leutheusser [12].

Below the critical temperature $T_c = g(\tilde{q}/2)^{3/2}$ such a regime still exists for $t, t' \to \infty$ with τ/t and $\tau/t' \to 0$, but with $\lim_{\tau\to\infty} C_{\rm as}(\tau) = q \neq 0$. In addition, there is an aging regime in which TTI and the FDT are violated. In this regime the limit $t, t' \to \infty$ is taken fixing the ratio $h(t')/h(t) = e^{-\sigma}(\sigma \geq 0)$. [$h(t)$ is an increasing function which the present theory is not able to predict [4,5]. On a numerical basis it was claimed in [3] that $h(t) = t$, leading to the suggestive result $C(t,t) = C(t'/t)$.] Here one has

$$C(t,t') = \hat{C}(\sigma),$$
$$Tr(t,t') = u\frac{\partial C(t,t')}{\partial t'} = -u\left(\frac{d\ln[h(t')]}{dt'}\right)\frac{\partial \hat{C}(\sigma)}{\partial \sigma}. \tag{10}$$

Continuity between the homogeneous and aging regimes implies $\hat{C}(0) = q$. The value of q, like that of many other quantities, is independent of the function h (see the discussion in [4,5]), and is specified at low temperature by the largest root of the equation $q = T^2/g^2(\tilde{q} - q)^2$, while μ^2 and u are given by $\mu^2 = g^2(\tilde{q}^2 - q)/2T + T/(\tilde{q} - q)$ and $u = (\tilde{q} - q)/q$. It is worth noticing that the former equations can be found in the replica formalism for the p-spin glass by imposing the condition of marginal stability [11]. The energy contains contributions from both the asymptotic and aging regions of times, and tends to the limit $E_{\rm Dyn} = -g^2[\tilde{q}^3 - (1-u)q^3]/3T$, different from its static counterpart. Figure 1 shows the results of numerical integration of Eqs. (7) and (8) at a temperature where aging occurs.

We have seen that the dynamics of the spherical AR model displays glassy behavior, with the sharp phase transition from ergodic to nonergodic behavior. For a correct picture of the model at low temperature, a set of coupled equations for the correlation and response functions have to be solved. It has often been discussed in the literature whether glasses undergo a real phase transition, or whether a better picture is one of a progressive freezing [13,14]. MCT equations which fit better with the latter scenario have been proposed [13,14]. Even though in this case one cannot expect aging behavior for infinite times no matter how the limit is taken, the relaxation to equilibrium can be extremely slow. On an experimental time scale aging phenomena (or possibly "interrupted aging") can be present, and the use of the off-equilibrium equations can be more appropriate than that of the asymptotic one. This phenomenon occurs in the present model for temperatures larger than but close to T_c.

A natural generalization of the MCT equation (2), and consequently of (7) and (8), is obtained by replacing the nonlinear term by some generic "self-energy" function $\Sigma(G(x,y))$. This can be introduced on a purely

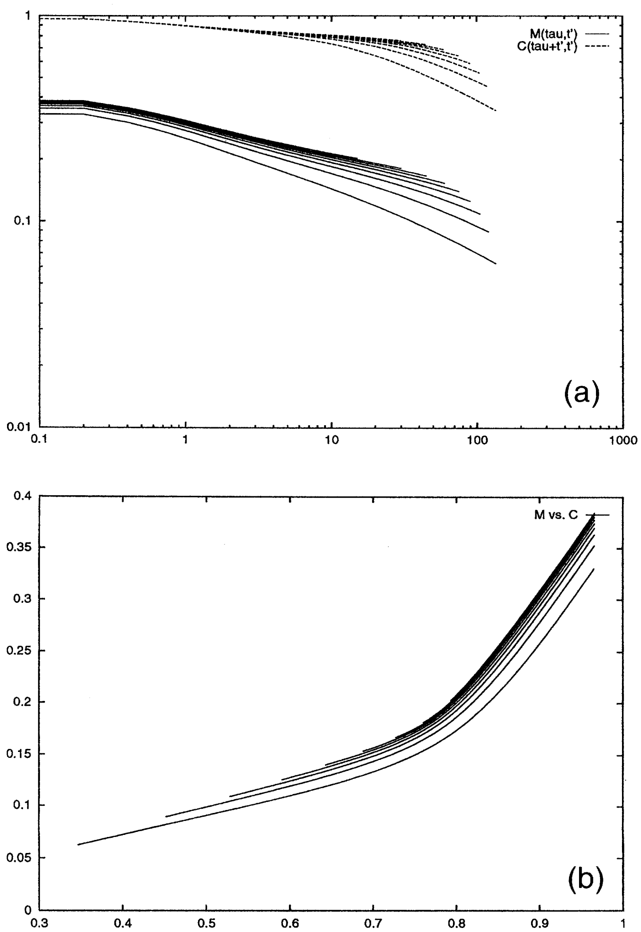

FIG. 1. A sketch of the correlation and response functions for $g = \tilde{q} = 1$, $T = 0.2 < T_c$. (a) Normalized correlation function $C(t_w + \tau, t_w)/\tilde{q}$ and integrated response function $m(t_w + \tau, t_w) = (T/\tilde{q}) \int_0^{t_w} r(t_w + \tau, s) ds$ as functions of τ, for different waiting times t_w. If TTI and the FDT held, these functions would be equal up to a constant shift and independent of t_w. (b) m plotted against C for different waiting times. For large values of C (short times), the FDT holds, and the curves approach a straight line of unit slope which intersects the C axis at q. The deviation from this line at smaller C (long times) indicates off-equilibrium behavior. In the region $C < q$ the behavior predicted by the asymptotic solution would be a straight line with slope u, starting from the origin. The times displayed in the figure are very short compared to the ones needed to see the asymptotic behavior.

phenomenological ground, as it has been done by Götze in the equilibrium case [22], or derived from some microscopic theory. Examples of equations of this form have been analyzed extensively in [4,5] in the context of mean field disordered systems. Depending on the form of Σ, the long time aging regime can assume different forms. In the model discussed here, we have found the simplest among the scenarios proposed in [4,5]. Further work is necessary to determine the most appropriate form of Σ to interpret real glass experiments. Our results, however, open the door for the systematic study, starting from a mean field level, of aging effects in real glasses.

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