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THE KAC LIMIT FOR DILUTED SPIN GLASSES

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We study diluted spin glass models in arbitrary dimension, where each spin interacts with a finite number of other spins chosen at random with a probability decaying to zero over some distance γ^{-1}. For systems with pairwise interactions we show that the infinite-volume free energy converges to that of the mean-field Viana–Bray model,[1] in the Kac limit $\gamma \to 0$. For p-spin like models we get only one bound: the free-energy is bounded from above by the one of the mean-field diluted p-spin.

1. Introduction

Kac models[2] are a classical tool of Mathematical Statistical Physics to show the connection between systems with finite range interactions and their Mean Field counterparts. In these models one considers variables interacting through a potential of growing range γ^{-1} and intensity decreasing in such a way as to keep finite the total interaction strength of one particle with the surrounding environment. The pioneering analysis of Lebowitz and Penrose,[3] dating to almost forty years ago, in the case of ordered discrete and continous systems, showed the convergence of free-energy to the mean-field value in the Kac limit $\gamma \to 0$. This result is at the basis of rigorous expansions (see, for instance, Refs. 4 and 5) around $\gamma = 0$ which have led to the proof of the existence of phase transitions with non-ergodic low temperature phases for systems with large but finite interaction range. The analogous of the Lebowitz and Penrose analysis for the disordered spin glasses case has only very recently been achieved,[6,7] thanks to the use of simple interpolation methods which have already proved to be exremely powerful in the study of mean field spin glass systems, for instance in Refs. 8 and 9.

In the present work, we extend the result of Ref. 7 to the case of diluted spin glass models. Diluted disordered models have recently received a lot of attention, both in connection to physics and to computer science. They provide a class of models, in principle solvable within mean field theory, in which the number of interactions per spin remains finite, even in the thermodynamic limit, while the interacting spins are chosen at random in the whole system.

It is interesting that this result shows clearly that mean field behaviour can arise in finite range systems not only when one lets every degree of freedom interact with a large number of other degrees of freedom in the system, that is with the "effective field" generated by them, but also when one allows spins to interact at arbitrarily large distances. The feature common to both cases, indeed, is that short loops of interactions are negligible.

2. Definition of the Model and Results

We introduce here a finite-dimensional, Kac generalization of the diluted p-spin model defined on a random Poissonian hypergraph. For later convenience let us define the system on the cube Λ of size L and volume $L^d = N$ belonging to the d-dimensional lattice Z^d.

The usual (mean field) diluted p-spin model is defined by a system of $N = |\Lambda|$ Ising spins $\sigma_i = \pm 1$ interacting through the Hamiltonian

$$H_\Lambda(\sigma, \alpha) = \sum_{\mu=1}^{M} J^\mu \sigma_{i_1^\mu} \cdots \sigma_{i_p^\mu} \tag{1}$$

which depends on the quenched couplings J^μ which we choose to be IID with symmetric distribution, and on the random p-uples of indices $\{i_1^\mu, \ldots, i_p^\mu\}$ which we extract as IID for different μ with uniform probability

$$Q_\Lambda(\{i_1^\mu, \ldots, i_p^\mu\}) = \frac{1}{N^p} \chi_\Lambda(\{i_1^\mu, \ldots, i_p^\mu\}), \tag{2}$$

χ_Λ being the indicator function of the set Λ. Finally we choose the number of interactions $M = \alpha N$ to be proportional to the volume.

A finite dimensional generalization of the model can be defined keeping the same form (1) for the Hamiltonian, and modifying the law of probability for the choice of the indices involved in the interactions to make it depend on their mutual distance:

$$P_\gamma(\{i_1^\mu, \ldots, i_p^\mu\}) = \frac{A_N}{N} \gamma^{d(p-1)} \phi(\{\gamma |i_l^\mu - i_n^\mu|\}) \chi_\Lambda(\{i_1^\mu, \ldots, i_p^\mu\}). \tag{3}$$

A_N is a constant, close to 1 for large N, which ensures normalization. The function ϕ depends on the mutual distances between the spins and decays to zero when any of them diverges, and is normalized so that

$$\sum_{\{i_2, \ldots, i_p\} \in Z^{d-1}} \phi(\{|i_l^\mu - i_n^\mu|\}) = 1.$$

Note that, as in the mean field model, the average number of interactions per spin is $p\alpha$. This model was introduced in Ref. 6 in the case $p = 2$.

The existence of the infinite volume free-energy for the diluted model with indices distribution (2) has been recently proven for all temperatures in Ref. 10. On the other hand, it is not difficult to extend standard arguments[11] to show that the same is true for the finite dimensional case. In this paper we will prove that in the Kac limit $\gamma \to 0^+$, to be performed after the thermodynamic limit $N \to \infty$, the free-energy of the finite dimensional model is bounded from above by the one of the mean-field model. In the case $p = 2$, this complements the opposite bound of Ref. 6, and therefore fully characterizes the free-energy in the Kac limit.

Let us call $f(\alpha, \beta, \gamma)$ the free-energy of the model (3) and $f_{MF}(\alpha, \beta)$ the one of the mean-field model (2). Then, the following holds:

Theorem

If p is even, then

$$\lim_{\gamma \to 0} f(\alpha, \beta, \gamma) \le f_{MF}(\alpha, \beta) \,. \tag{4}$$

In order to prove the bound, we interpolate between the model in the domain Λ and a system made of a collection of many independent mean-field subsystems of volume $|\Omega| = \ell^d$. As in Ref. 3, we choose

$$\ell \ll \gamma^{-1} \ll L \,, \tag{5}$$

and in the end we let the three lengths diverge in this order. Let us divide the box Λ into sub-cubes Ω_n, $n = 1, \dots, K = N/|\Omega|$, and introduce the interpolating model defined by a Hamiltonian of the kind (1), where however the random indices in the interactions are chosen, for $t \in [0, 1]$, with probability

$$P^{(t)}(\{i_1^\mu, \dots, i_p^\mu\}) = t P_\gamma(\{i_1^\mu, \dots, i_p^\mu\}) + \frac{(1-t)}{K} \sum_n Q_{\Omega_n}(\{i_1^\mu, \dots, i_p^\mu\}) \,. \tag{6}$$

The form of this interpolation is particularly useful to obtain the inequality (4). Indeed, if we consider the t-dependent free-energy of this interpolating model, for the boundary values $t = 0, 1$ one has:

$$\frac{1}{|\Lambda|} E \ln Z_\Lambda(0) = \frac{1}{|\Omega|} E \ln Z_\Omega^{MF}(\alpha, \beta) \tag{7}$$

and

$$\frac{1}{|\Lambda|} E \ln Z_\Lambda(1) = \frac{1}{|\Lambda|} E \ln Z_\Lambda^{(\gamma)}(\alpha, \beta, \gamma) \,, \tag{8}$$

respectively.

As we show below, one has

$$\lim_{\gamma \to 0} \lim_{L \to \infty} \frac{d}{dt} \frac{1}{|\Lambda|} E \ln Z_\Lambda(t) \ge 0 \tag{9}$$

for $0 \le t \le 1$. This inequality can be integrated on t between 0 and 1, giving

$$-\beta \lim_{\gamma \to 0} f(\alpha, \beta, \gamma) \ge \frac{1}{|\Omega|} E \ln Z_{\Omega}^{MF}(\alpha, \beta) . \tag{10}$$

Letting then $|\Omega| \to \infty$, one finally finds the desired result.

The computation of the t-derivative of the free-energy is analogous to the computation performed in Ref. 10, and for this reason we do not give all details, but rather sketch the main points. Denoting by $\langle \cdots \rangle_{-\mu}$ the Boltzmann average in absence of the clause numbered as μ and $E_{-\mu}$ the disorder average on all quenched variables but the ones appearing in the μth interaction, we find

$$\frac{d}{dt}\frac{1}{N} E \ln Z_{\Lambda}(t) = \frac{1}{N} \sum_{\mu} \sum_{\{i_1^\mu, \ldots, i_p^\mu\}} \left(P_\gamma(\{i_1^\mu, \ldots, i_p^\mu\}) - \frac{1}{K} \sum_n Q_{\Omega_n}(\{i_1^\mu, \ldots, i_p^\mu\}) \right)$$
$$\times E_{J^\mu} E_{-\mu} \log(1 + \tanh(\beta J^\mu) \langle \sigma_{i_1^\mu} \cdots \sigma_{i_p^\mu} \rangle_{-\mu}) . \tag{11}$$

Now we perform two operations: first we expand the logarithm in Eq. (11) in a convergent series of $\tanh(\beta J)$, then we break the sum on the lattice sites grouping the terms in the different boxes Ω_n. Observing that all interaction terms indexed by μ give the same contribution, one has

$$-\alpha \sum_{\{n_1, \ldots, n_p\}}^{1,K} \sum_{\{i_1 \in \Omega_{n_1}, \ldots, i_p \in \Omega_{n_p}\}} \left(P_\gamma(\{i_1, \ldots, i_p\}) - \frac{1}{K} \sum_n Q_{\Omega_n}(\{i_1, \ldots, i_p\}) \right)$$
$$\times \sum_{k=1}^{\infty} \frac{1}{2k} E_J(\tanh^{2k}(\beta J^\mu)) E_{-1}(\langle \sigma_{i_1} \cdots \sigma_{i_p} \rangle_{-1})^{2k} . \tag{12}$$

Notice that a lower bound (12) can be obtained taking for each group of boxes $\{n_1, \ldots, n_p\}$ the maximum value of the probability density P_γ:

$$\phi^+_{\{n_1, \ldots, n_p\}} = \max_{i_1 \in \Omega_{n_1}, \ldots, i_p \in \Omega_{n_p}} P_\gamma(\{i_1, \ldots, i_p\}) .$$

For each term of order $2k$ of the series, we introduce replicas with spin degrees of freedom $\sigma_i^1, \ldots, \sigma_i^{2k}$ and the partial multi-overlaps in the nth box

$$q_n^{(2k)} = \frac{1}{|\Omega|} \sum_{i \in \Omega_n} \sigma_i^1 \cdots \sigma_i^{2k} , \tag{13}$$

and get

$$\frac{d}{dt}\frac{1}{N} E \ln Z_{\Lambda}(t)$$
$$\ge -\alpha \sum_{k=1}^{\infty} \frac{1}{2k} E_J(\tanh^{2k}(\beta J^\mu))$$
$$\times \left(\sum_{\{n_1, \ldots, n_p\}}^{1,K} \phi^+_{\{n_1, \ldots, n_p\}} E_{-1} \langle q_{n_1}^{(2k)} \cdots q_{n_p}^{(2k)} \rangle_{-1} - \frac{1}{K} \sum_n E_{-1} \langle (q_n^{(2k)})^p \rangle_{-1} \right) . \tag{14}$$

At this point, we use the simple but useful inequality

$$p\, x_1 \cdots x_p \leq x_1^p + \cdots + x_p^p \,, \tag{15}$$

which holds for all values of x provided that p is even, and the property:

$$\lim_{\gamma \to 0} \sum_{n_2,\ldots,n_p (\neq n)}^{1,K} \phi^+_{\{n,n_2,\ldots,n_p\}} = \frac{1}{\Omega} \,, \tag{16}$$

which is a simple consequence of the fact that in the Kac limit ϕ^+ varies slowly on space, to find the desired result. Note that this is the only point where we need to assume that p is even.

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