1st Term

1. Real Analysis, I
2. Abstract Algebra
3. Topology
4. Complex Analysis
5. Ordinary Differential Equations

2nd Term

6. Differential Geometry
7. Functional Analysis, I

OPTION 1

8. Real Analysis, II
9. Functional Analysis, II

OPTION 2

10. Algebraic Geometry
11. Algebraic Topology
1. REAL ANALYSIS, I (30 hours)

The dual of \( L^p(\mathbb{R}) \).
Weak convergence in \( L^p(\mathbb{R}) \): Riemann-Lebesgue Lemma.

Many examples and exercises are given for every new concept. Three homeworks are corrected and the results are discussed in the classroom. The final examination is a written examination with 3-4 exercises.

2. ABSTRACT ALGEBRA (30 hours)

PART I: Groups


PART II: Rings and Fields

(A) Rings:
Definition of ring, first properties and examples. Zero divisors, integral domains, units of a ring and fields. Homomorphisms of rings, ideals, quotient rings, homomorphism theorem.
Prime ideals; maximal ideals.
Polynomial rings, Euclidean algorithm (division with rest), Euclidean rings, principal ideal domains.
The quotient field of an integral domain.
Prime elements, irreducible elements, unique factorization domains,
Eisenstein’s criterion for irreducibility of polynomials.

(B) Modules:
Definition of modules, submodules, homomorphisms. Free modules and bases.
Diagonalization of matrices over Euclidean rings. Generators and relations for modules,

References:
M. Artin: Algebra (This is what the students can use for further reading).
I.N. Herstein: Topics in Algebra. This reference is the closest to the course. Lecture notes will be provided.

3. TOPOLOGY (30 hours)

(a) Topological Spaces and Continuous Functions: Topological spaces; basis and subbasis for a topology; the order topology; the product and box topology; the subspace topology; closed sets and limit points; continuous functions; the metric topology; the quotient topology.

(b) Connectedness and Compactness: Connected and path-connected spaces; connected and compact sets in the real line; limit point compactness; compactness in metric spaces.

(c) Contability and Separation Axioms: The countability axioms; the separation axioms; Urysohn lemma; Urysohn metrization theorem; Tietze extension theorem.

(d) Basics of the Fundamental Group and Covering Spaces: Homotopy and homotopy of paths; the fundamental group; covering spaces; the fundamental group of the circle and some applications.

(Basically, the contents of chapters 2, 3, 4 and a small part of chapter 8 of Munkres's text.)


4. COMPLEX ANALYSIS (30 hours)

1. Analytic functions
c) Mobius transformations: group and geometric properties, cross ratio.
d) Elementary functions (some rational functions, exponent, trigonometric functions, logarithm). Multivalued functions, branch point.

2. Properties of Analytic functions

2.1. Integral
a) Curvilinear integral: main properties;
b) Cauchy's Integral Theorem and its generalizations;
c) Cauchy's Integral Formula, Liouville's Theorem; Morera's Theorem.

2.2. Taylor series
a) Weierstrass Theorem on uniform convergence sequences of analytic functions;
b) Expansion of analytic function in Taylor series, Cauchy inequalities;
c) Isolatedness of zeroes of analytic function, Uniqueness Theorem;
d) Maximum modulus principle.

2.3. Laurent series and singular points.
a) Expansion in Laurent series;
b) Classification of isolated singularities of Analytic Functions, entire and meromorphic functions;
c) Residues and application to calculation of integrals, Jordan Lemma;
d) Argument principle, Rouche's theorem, Fundamental theorem of algebra, Open mapping principle;
e) Maximum modulus principle and its corollaries.

3. Further results on analytic functions
a) Schwarz's Lemma, Compactness principle (Montel theorem), Hurwitz theorem; Riemann Mapping theorem (only the idea of the proof), Symmetry principle;
Depend on the time which will remain:
b) The notion of analytic extension and Riemann surface;
c)(optional) Expansion of entire and meromorphic functions (Mittag-Leffler theorem and Weierstrass theorem on expansion into the infinite product).

5. Selected Topics in ORDINARY DIFFERENTIAL EQUATIONS (7.5 hours)
The Cauchy problem.
Qualitative properties of the solutions.
Planar systems and phase plane analysis.
Dependence on the initial conditions.
Stability of equilibria (linear and nonlinear case).

6. DIFFERENTIAL GEOMETRY (30 hours)
The course will treat curves and surfaces in three dimensional euclidean space. The aim will be to do some elementary geometry while at the same time working with careful definitions. We will follow Do Carmo's book up to Chapter 4.
- smooth curve, arc-length, tangent, curvature torsion. The fundamental theorem: curvature and torsion determine a curve up to a rigid motion.
- regular surface. Critical point, critical value of a function. Implicit function theorem (special case), change of parameters, tangent plane, first fundamental form, area.
- Gauss map, Gauss curvature, Mean curvature, Minimal surfaces.
- Isometries, Conformal maps, Gauss Theorem, Parallel transformation, Geodesics, The exponential map, geodesic polar coordinates, Gauss-Bonnet theorem.
7. FUNCTIONAL ANALYSIS, I (30 hours)

Part I

Normed Linear Spaces; Hilbert Spaces; Linear Maps; Hahn Banach Theorem (Analytic and Geometric Forms); Uniform Boundedness Principle; Open Mapping and Closed Graph Theorems; Weak and Weak* Topologies; Reflexive Spaces; the Banach Alaoglu Theorem; the Theorem of Kakutani characterizing reflexive spaces; Uniformly Convex Spaces; Milman-Pettis Theorem; Eberlein-Smul'yian Theorem; Separable Spaces.


Part II

1. Introduction: resolvent, spectrum and its properties for bounded operators (compactness, spectral radius, nonemptiness).

2. Spectral theory of compact operators with application to integral equations: 
a) General properties of compact operators: the set of compact operators is closed, it is the right and left ideal in the space of all bounded operators;
b) Approximation by operators of finite range, the Fredholm alternative, spectrum of a compact operator (Riesz-Schauder Theorem), application to the integral operators;
c) Self-adjoint compact operators, completeness of the basis of the eigenvectors (the Hilbert-Schmidt Theorem), the Courant-Fischer Minimax Principle and its consequences;
d) Application to the Schturm-Liouville problem.

Recommended literature:
1. J. Dieudonne, Foundations of Modern Analysis, chapter 11;
2. F. Riesz, B. Szokefalvi-Nagy, Functional analysis, chapter VI;
3. A. N. Kolmogorov, S. V. Fomin, Elements of the theory of functions and functional analysis, chapters IV, x 5 and x 6, chapter IX;

8. REAL ANALYSIS, II (15 hours)

Classes of subsets: semi-ring, ring field (algebra), sigma-ring, sigma-field (Borel field), monotone class.
Set functions:additive, sigma-additive, measure, sigma-finite. outer measure. Continuity and extension of set functions.
Hahn and Jordan decompositions.
Lebesgue and Lebesgue-Stieltjes measures.
Simple and measurable functions, Borel measurability.
9. FUNCTIONAL ANALYSIS, II  (30 hours)

1a) Function spaces.
Spaces of continuously differentiable functions; test functions; smooth partition of unity, convolution, mollifiers. Density of smooth functions in $L^p$ spaces. Distributions; definition and basic properties, support of a distribution, differentiation, multiplication. Sobolev spaces $W^{1,p}$ and $W_0^{1,p}$; definition, basic properties and characterization; completeness, separability, reflexivity. Functions of bounded variation on real intervals, Lipschitz continuity and relation with Sobolev functions. Chain rule. Extension theorems, approximation of Sobolev functions by smooth functions, Stampacchia theorem, Sobolev embedding theorem, Hölder continuity and Morrey embedding theorem, Rellich theorem. Sobolev spaces of higher order. Traces.

1b) Elliptic partial differential equations.

Parts of the course not completely treated in the textbook will be covered by lecture notes.

10. ALGEBRAIC GEOMETRY  (30 hours)


II. Functions and morphisms: Regular functions on affine and quasiprojective varieties, morphisms, characterization of morphisms of affine and projective varieties, examples. Segre embedding and products of quasiprojective varieties. Separatedness, completeness, proof that projective varieties are complete with applications, rational maps.

III. Dimension and nonsingularity.
Study of finite morphisms, definition of dimension, theorem on dimension of a hypersurface. Dimension of intersection of two varieties in affine and projective space, relation of dimension to transcendence degree.
Tangent space, singular and nonsingular points of a variety, Bertini's theorem, study of nonsingular curves, divisors on curves.

11. ALGEBRAIC TOPOLOGY  (15 hours)


Degree of a map. Applications of homology: No retraction theorem, Fixed-point theorem, Hairy ball theorem, Jordan-Brouwer separation theorem, Invariance of domain theorem.


MAIN TEXTBOOK:


SUPPLEMENTARY TEXTBOOKS:


June 2007