AREA EXPANDING $\mathscr{C}^{1+\alpha}$ SUSPENSION SEMIFLOWS

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Abstract. We study a large class of suspension semiflows which contains the Lorenz semiflow. This is a class with low regularity (the return map is piecewise $\mathscr{C}^{1+\alpha}$ and the return time is piecewise $\mathscr{C}^{\alpha}$) and where the return time is unbounded. We establish the functional analytic framework which is typically employed to study rates of mixing. The Laplace transform of the correlation function is shown to admit a meromorphic extension to a strip about the imaginary axis. As part of this argument we give a new result, of independent interest, concerning the quasi-compactness of weighted transfer operators for piecewise $\mathscr{C}^{1+\alpha}$ expanding interval maps.

1. Introduction

Some dynamical systems exhibit very good statistical properties in the sense of, for example, exponential decay of correlation and the stability of the invariant measure under deterministic or random perturbations. Such properties have been shown for many discrete-time dynamical systems and more recently for some flows. Very strong results now exist for smooth contact Anosov flows [15, 27, 13, 39, 40, 17]. Good results also exist for suspension flows over uniformly-expanding Markov maps when the system is $\mathscr{C}^2$ or smoother [35, 11, 7]. The above are all rather smooth and regular systems and arguably not realistic or relevant in many physical systems. There are two important examples which come to mind: dispersing billiards [14] and the Lorenz flow [29]. The fine statistical properties of both these systems remain, to some extent, open problems. We therefore direct our interest to such systems with rather low regularity. Some recent progress includes the proof of exponential mixing for piecewise-cone-hyperbolic contact flows [10] and also for a class of three-dimensional singular flows [6].

This is our theme: To make progress on the understanding of the fine results on statistical properties of systems with low regularity. The primary motivation for this study is the Lorenz flow mentioned above. This is a smooth three-dimensional singular hyperbolic flow. Baladi [8] studied suspension semiflows which were inspired by Lorenz flows but required that the various quantities were of bounded variation where we require them to be Hölder (Hölder does not imply bounded variation; e.g. $x \mapsto x\sin(1/x)$). The work of Araújo and Varandas [6] proved exponential decay of correlation for a class of volume-expanding flows with singularities, a class which is inspired by the Lorenz flow. However their method required the existence of a $\mathscr{C}^2$ stable foliation for the flow. Unfortunately in the case of the Lorenz flow (for the classical parameters) even the weak stable foliation of the flow (this is the two-dimensional foliation consisting of leaves which are spanned by the flow direction and the stable direction) is merely $\mathscr{C}^{1+\alpha}$. Consequently there seems to be no hope of extending their strategy to the original problem. The problem of the stable foliations (of the return map) being merely $\mathscr{C}^{1+\alpha}$ for Lorenz like flows

Date: May 1, 2013.

It is a pleasure to thank Carlangelo Liverani for many helpful discussions and comments. Research partially supported by the ERC Advanced Grant MALADY (246953).

has been partially tackled by Galatolo and Pacifico [16] followed by Araújo, Galatolo and Pacifico [3] but results on decay of correlations are limited to the return map and not the flow. In this paper we make some progress in a complementary direction. There exist two popular strategies for approaching problems of this type: The first possible strategy is to construct an anisotropic Banach space in order to study the flow directly as was done for contact Anosov flows [27] and piecewise-cone-hyperbolic contact flows [10]. The second possible strategy is to study the *Lorenz semiflow* (details given in Section 3) which is given by quotienting along the stable manifolds of the flow. At this stage it is unclear how to construct the space required for the first possibility and we therefore consider the second. This however requires one to work with a system which is merely $\mathscr{C}^{1+\alpha}$. There are many issues involved in studying the quotient flow in this particular setting. The reader interested in this question should consult the discussion which is postponed until the end of Section 3 after we have introduced the pertinent details concerning Lorenz flows.

In this paper we focus on a particular class of semiflows which are suspensions over expanding interval maps. This class includes the Lorenz semiflows. They have low regularity in the following four ways:

(1) The expansion of the return map may be unbounded. I.e. the derivative of the return map blows up close to certain points of discontinuity. This bad distortion issue is seen in both billiard systems and the Lorenz flow.
(2) The reciprocal of the derivative of the return map is merely Hölder continuous.
(3) The return time function is unbounded. This is a direct result of the existence of fixed points of the flow. However in the case of certain suspension semiflows this has already been shown to not be a barrier to good statistical properties [11].
(4) The semiflow is merely area-expanding and not uniformly-expanding in the sense that it is not possible to define a forward invariant conefield which is uniformly transversal to the flow direction. This puts us in the category of singular hyperbolicity [31, 33].

In order to study the class of flows considered in this paper, and other systems which are the object of current research, it is crucial to understand whether these above issues are real barriers to good statistical properties or merely technical difficulties. On this issue we succeed in making some progress in the present work showing that the listed issues are not real barriers to the statistical properties, at least in this setting. For proving exponential decay of correlation for flows there is one particular established approach which involves studying the Laplace transform of the correlation function. We apply this strategy to our present setting and show that the Laplace transform of the correlation function admits a meromorphic extension into the left half plane. This fact is also of use when studying other statistical properties. In Section 2 we define precisely the class of semiflows we are interested in and state the results. In Section 3 we discuss Lorenz flows and demonstrate the connection with the class of semiflows we consider. In Section 4 we give a generalisation of the result of Keller concerning function spaces of *generalised bounded variation* [25] such that it is possible to apply to our present application. This is a new result for the essential spectral radius of such transfer operators for these piecewise $\mathscr{C}^{1+\alpha}$ expanding interval maps and the section is independent of the others. The reader interested in the comparison of this result to other related results for similar function spaces should consult the discussion at the beginning of Section 4. Section 5 contains the proof of the main result, reducing the problem to the study of certain weighted transfer operators and then using the results of Section 4.

2. Results

For our purposes we define a *suspension semiflow* to be the triple (Ω, f, τ): The set Ω is an open interval and $\{\omega_i\}_{i\in\mathcal{I}}$ is a finite or countable set of disjoint open sub-intervals which exhaust Ω modulo a set of zero Lebesgue measure; $f \in \mathscr{C}^1(\tilde{\Omega}, \Omega)$ (for convenience let $\tilde{\Omega} = \bigsqcup_{i\in\mathcal{I}} \omega_i$) is a bijection onto its image when restricted to each ω_i; $\tau \in \mathscr{C}^0(\Omega, \mathbb{R}_+)$ is such that $\int_\Omega \tau(x)\ dx < \infty$. In a moment we will add some stronger assumptions on the regularity of f and τ but we will never require f to be Markov. We call f the return map and τ the return time function. Let $\Omega_\tau := \{(x,s) : x \in \tilde{\Omega}, 0 \leq s < \tau(x)\}$ which we call the state space. For all $(x,s) \in \Omega_\tau$ and $t \in [0, \tau(x) - s]$ let

$$\Phi^t(x,s) := \begin{cases} (x, s+t) & \text{if } t < \tau(x) - s \\ (f(x), 0) & \text{if } t = \tau(x) - s. \end{cases} \tag{2.1}$$

Note that $\Phi^{u+t}(x,s) = \Phi^u \circ \Phi^t(x,s)$ for all u, t such that each term is defined. The flow is then defined for all $t \geq 0$ by assuming that this relationship continues to hold.

Now we define the class of suspension semiflows which we will study. Firstly we require that the return map is expanding, i.e. that[1] $\|1/f'\|_{\mathbf{L}^\infty(\Omega)} < 1$. We suppose that there exist some $\alpha \in (0,1)$ and $\sigma > 0$ such that the following three conditions hold. Firstly we must have some, albeit weak, control on the regularity. We assume that[2]

$$x \mapsto \frac{e^{z\tau(x)}}{f'(x)} \qquad \text{is } \alpha\text{-Hölder on } \tilde{\Omega} \text{ for each } \Re(z) \in [-\sigma, 0]. \tag{2.2}$$

Furthermore we must require sufficient expansion in proportion to the return time. We assume that

$$\sup_{i\in\mathcal{I}} \left(\|\tfrac{1}{f'}\|_{\mathbf{L}^\infty(\omega_i)} \right)^\alpha e^{\sigma \|\tau\|_{\mathbf{L}^\infty(\omega_i)}} < 1 \tag{2.3}$$

Finally, to deal with the possibility of a countable and not finite number of disconnected components of Ω, we assume that

$$\sum_{i\in\mathcal{I}} \|\tfrac{1}{f'}\|_{\mathbf{L}^\infty(\omega_i)} e^{\sigma \|\tau\|_{\mathbf{L}^\infty(\omega_i)}} < \infty. \tag{2.4}$$

Note that we never require any lower bound on τ. Let ν denote some f-invariant probability measure which is absolutely continuous with respect to Lebesgue on Ω. The existence of such a probability measure in this setting is known [25] but is also implied by the results of Section 4. For simplicity we assume that this absolutely continuous invariant probability measure is unique. It holds that $\mu := \nu \otimes \text{Leb} / \nu(\tau)$ is a Φ^t-invariant probability measure which is absolutely continuous with respect to Lebesgue on Ω_τ. Given $u, v : \Omega_\tau \to \mathbb{C}$ which are α-Hölder we define for all $t \geq 0$ the correlation

$$\xi(t) := \mu(u \cdot v \circ \Phi^t) - \mu(u) \cdot \mu(v).$$

[1] In general it is sufficient to suppose that there exists $n \in \mathbb{N}$ such that $\|1/(f^n)'\|_{\mathbf{L}^\infty(\Omega)} < 1$. In which case one simply considers the n^{th} iterate of the suspension flow and proceeds as before, although care must be taken with assumption (2.2).

[2] We say that some $\xi : \Omega \to \mathbb{C}$ is "α-Hölder on Ω" if there exists $H_\xi < \infty$ such that $|\xi(x) - \xi(y)| \leq H_\xi |x-y|^\alpha$ for all $x, y \in \Omega$ with the understanding that this inequality is trivially satisfied if $x \in \omega_i$, $y \in \omega_{i'}$, $i \neq i'$ since in this case $|x-y|$ is not finite. Note that H_ξ does not depend on i.

Main Theorem. *Suppose the suspension semiflow is as described above, in particular satisfying the assumptions* (2.2)*,* (2.3)*, and* (2.4)*. Then the Laplace transform of the correlation* $\widehat{\xi}(z) := \int_0^\infty e^{-zt}\,\xi(t)\,dt$ *admits a meromorphic extension to the set* $\{z \in \mathbb{C} : \Re(z) \geq -\sigma\}$*.*

The proof of this theorem is given in Section 5 and is based on the results of Section 4. The argument involves the usual method of *twisted transfer operators* but for this setting we require a generalisation of Keller's previous work [25] on $\mathscr{C}^{1+\alpha}$ expanding interval maps which is the content of Section 4.

Let us recall in detail some closely related results which were mentioned in the introduction. Baladi and Vallée [11] (argument later extended to higher dimensions by Avila, Yoccoz and Gouëzel [7]) studied suspension semiflows which had return maps which were Markov and also $\mathscr{C}^2$. They allowed the return time to be unbounded but only in a mild way as they required τ'/f' to be bounded. As part of the study of Lorenz-like flows Araújo and Varandas [6] studied suspension semiflows very similar to the present setting but had to additionally require that the return map was $\mathscr{C}^2$ rather than our weaker assumption of $\mathscr{C}^{1+\alpha}$. We therefore see that our setting is more general and sufficiently general to be used for the study of the Lorenz flow with the classical parameters (see Section 3). However in each of the above mentioned cases exponential decay of correlation is proven, a significantly stronger result than is proven in this present work. To obtain results on exponential decay of correlations would require a stronger estimate at one stage of the argument; this is the oscillatory cancelation argument as pioneered by Dolgopyat [15]. However it seems likely that such an estimate would require the return time function to be at least $\mathscr{C}^1$ and not just Hölder as in the setting of the above result. At the end of Section 3 we return to the discussion of this issue in the motivating case of the Lorenz flow.

3. Lorenz Semiflows

Introduced in 1963 as a simple model for weather, the Lorenz flow [29] is a smooth three-dimensional flow which, from numerical simulation, appeared to exhibit a robust chaotic attractor. In the late 1970s Afraĭmovič, Bykov and Silnikov [1] and Guckenheimer and Williams [19, 44] introduced a geometric model of the Lorenz flow and many years later, in 2002, Tucker [41] showed that the geometric Lorenz flow really was a representative model for the original Lorenz flow and hence showed that the Lorenz attractor really did exist. The attractor is zero volume (Hausdorff dimension ≈ 2.05 [18]), contains the fixed point at the origin and has a complex *Cantor book* structure [44]. For some history of the problem and an explanation of the physical relevance of the system see [43].

This flow has long proved elusive to thorough study. It is not uniformly hyperbolic. The class of *singular hyperbolic flows* was introduced and studied in the late 1990s by Morales, Pacifico, and Pujals [31, 33, 32]. This class of flows contains the uniform hyperbolic flows and also contains the Lorenz attractor. Whereas the uniformly hyperbolic flows are the flows which are structurally stable as shown by Hayashi [21, 22], the singular hyperbolic flows are the flows which are stably transitive. It is known that singular hyperbolic flows are chaotic in that they are expansive and admit an SRB measure [4]. Some further results are known limited to the particular case of the Lorenz attractor. It is know to be mixing [30] and that the Central Limit Theorem and Invariance Principle hold [24]. As mentioned earlier, a class of Lorenz-like flows has been shown to mix exponentially [6] although this result is limited to such flows which have $\mathscr{C}^2$ stable foliations, a property which cannot be expected to hold in general or for the original Lorenz flow.

Here we collect together some known facts [41, 5] in order to show that the Lorenz flow reduces to a suspension semiflow of the class introduced in Section 2. The Lorenz flow $\Phi^t : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the system of differential equations:

$$\begin{cases} \dot{x}_1 = -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 = r x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 = x_1 x_2 - b x_3 \end{cases}$$

where $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$ are the so-called *classical parameter values.* The flow is uniformly volume contracting and possesses three fixed points; in particular the origin is a fixed point of saddle type with one positive eigenvalue ζ_1 and two negative eigenvalues $-\zeta_2$, and $-\zeta_3$, where

$$\zeta_1 = (\sqrt{1201} - 11)/2 \approx 11.8,$$
$$\zeta_2 = (\sqrt{1201} + 11)/2 \approx 22.8, \quad \zeta_3 = 8/3 \approx 2.67.$$

Note that $0 < \zeta_3 < \zeta_1 < \zeta_2$. There is a forward invariant open set U which contains the origin but is bounded away from the other two fixed points. The set U is a torus of genus two, the holes centred around the two excluded fixed points. Eventually all trajectories enter this set. The maximal invariant set $\Lambda := \bigcap_{t \geq 0} \Phi^t U$ (the attractor) has zero volume due to the volume contraction of the flow and also contains the unstable manifold of the origin. There exists a one-dimensional stable foliation. In the available literature there is some confusion over the regularity of the various invariant foliations but is it universally agreed that the two dimensional weak-stable foliation (equivalently the stable foliation of the return map to a suitable Poincaré section) is differentiable with Hölder derivative and the stable foliation of the flow is at least Hölder. Let $\gamma \in (0, 1)$ be such that the weak stable foliation is $\mathscr{C}^{1+\gamma}$ and the stable foliation is $\mathscr{C}^{\gamma}$.

The $\mathscr{C}^{1+\gamma}$ regularity of the weak-stable foliation seems to be unavoidable. Note that if the Lorenz flow is sufficiently dissipative (the eigenvalues must satisfy $\zeta_2/\zeta_1 > \zeta_3/\zeta_1 + k$) then the stable manifolds for the return map would be $\mathscr{C}^k$ [5, §3.3.4.1]. Alves and Soufi [2] consider Lorenz-like flows where one may take $k = 2$ for their study of statistical stability. Unfortunately this is not the case for the Lorenz flow with the classical parameters.

Quotienting along the stable manifolds one may reduce the three dimensional flow to a suspension semiflow over a piecewise expanding map. This procedure is described in [30, 24]. By an appropriate choice of coordinates we may assume that the return map is defined on the interval $\Omega = (-1, 1)$ with a single point of discontinuity at the origin. Let $f : \Omega \setminus \{0\} \to \Omega$ denote the return map and let $\tau : \Omega \setminus \{0\} \to \mathbb{R}_+$ denote the return time.

The return map f is piecewise $\mathscr{C}^{1+\gamma}$ and is expanding for some iterate. Consequently there exists an invariant measure for the quotient map f which is absolutely continuous with respect to Lebesgue. This measure allows one to construct an SRB measure for the original flow. This process is described in [42, §7] (although it is there claimed that the density of the invariant measure for the quotient map is of bounded variation although it will merely be of generalised bounded variation).

Non-resonance of the eigenvalues, by Sternberg [37], means that it is possible to $\mathscr{C}^\infty$ linearise the flow in a neighbourhood of the singularity (actually it is possible to $\mathscr{C}^2$ linearise all close flows [24, Remark 2.1]). This allows precise estimates on the suspension semiflow. By [24, Proposition 2.6] we have the estimates

$$\tau(x) = -\tfrac{1}{\zeta_1} \ln |x| + \tau_0(x), \qquad f'(x) = |x|^{-(1-\beta)}\, g(x), \tag{3.1}$$

where $\beta = \zeta_3/\zeta_1 \in (0, 1)$, $\tau_0 \in \mathscr{C}^\gamma(\Omega)$, $g \in \mathscr{C}^{\beta\gamma}(\Omega)$, and $\inf_x g(x) > 0$. It is convenient to subdivide the set $\Omega \setminus \{0\}$ into small subintervals and so for each

$j \in \mathbb{N}$ let $\omega_j^+ := (e^{-(j+1)}, e^{-j})$, and $\omega_j^- := (-e^{-j}, -e^{-(j+1)})$. This is merely a means of having some form of weak distortion control on the combination of return map and return time, that in some sense the behaviour of the system is similar in each of the above defined intervals: Alternatively one could rephrase conditions (2.3) and (2.4). However we take advantage of the fact that we can cut the system wherever we like since discontinuities are allowed.

We must verify that the conditions (2.2), (2.3), and (2.4) are satisfied for this suspension semiflow. We choose $\alpha := \min\{\gamma\beta, (1-\beta)/(2-\beta)\}$ and $\sigma > 0$ such that

$$\sigma < \alpha\zeta_1(1-\beta), \tag{3.2}$$

the larger the better. Note this implies that $\alpha \in (0, \frac{1}{2})$ and that

$$\sigma \leq \zeta_1(1-\beta-\alpha). \tag{3.3}$$

Let $\Re(z) \in [-\sigma, 0]$. According to (3.1)

$$\frac{e^{-z\tau(x)}}{f'(x)} = |x|^{\frac{z}{\zeta_1}+1-\beta} g(x)^{-1} e^{-z\tau_0(x)}$$

and we know that $x \mapsto g(x)^{-1}e^{-z\tau_0(x)}$ is $\mathscr{C}^{\gamma\beta}$. Further note that $\Re(z/\zeta_1 + 1 - \beta) \geq -\sigma/\zeta_1 + 1 - \beta \geq \alpha$ by (3.3). Note that $y^\zeta - x^\zeta = \zeta \int_x^y s^{\zeta-1}\, ds$ for all $x, y \in \mathbb{R}$ and so for $y \geq x$ we have

$$y^\zeta - x^\zeta \leq |\zeta| \int_x^y s^{\Re(\zeta)-1}\, ds = \frac{|\zeta|}{\Re(\zeta)}(y^{\Re(\zeta)} - x^{\Re(\zeta)}).$$

Consequently $x \mapsto e^{z\tau(x)}/f'(x)$ is α-Hölder on $(0,1)$ and similarly on $(-1,0)$. We must now show that (2.3) and (2.4) also hold. By (3.1) we have the simple estimates (identical estimates hold for the ω_j^-)

$$\|\tfrac{1}{f'}\|_{\mathbf{L}^\infty(\omega_j^+)} \leq Ce^{-j(1-\beta)}, \qquad \|\tau\|_{\mathbf{L}^\infty(\omega_j^+)} \leq C + \tfrac{1}{\zeta_1}(j+1),$$

for some $C > 0$. This means that

$$\|\tfrac{1}{f'}\|^\alpha_{\mathbf{L}^\infty(\omega_j^+)} e^{\sigma\|\tau\|_{\mathbf{L}^\infty(\omega_i)}} \leq Ce^{-j[\alpha(1-\beta)-\sigma/\zeta_1]}e^{\sigma/\zeta_1}$$

(perhaps by increasing C) and by (3.2) we know that $\alpha(1-\beta) - \sigma/\zeta_1 > 0$. This means that (2.4) is satisfied and (2.3) is satisfied for all large j. Unfortunately we are not quite done since we have not shown that (2.3) is satisfied for all j and we have not shown that $\|1/f'\|_{\mathbf{L}^\infty(\Omega)} < 1$. We do know however that there exists $n \in \mathbb{N}$ such that $\|1/(f^n)'\|_{\mathbf{L}^\infty(\Omega)} < 1$. Consequently we instead consider the n^{th} iterate suspension semiflow. I.e. we consider the return map f^n and the return time $\tau + \tau \circ f + \cdots + \tau \circ f^{n-1}$. This does not change the flow we study, it is merely a choice of coding. Care must be taken by the Hölder continuity assumption (2.2). It is to be expected that this is now only satisfied by decreasing $\alpha > 0$ since it is the composition of Hölder continuous functions. We now fix this smaller value of α and choose $\sigma > 0$ corresponding smaller as required above (3.2). Condition (2.4) remains satisfied, but now for the new return map and new return time function with the obvious refinement of the partition. Condition (2.3) is still satisfied by the iterate for all but a finite number of terms. However since $\|1/(f^n)'\|_{\mathbf{L}^\infty(\Omega)} < 1$ it is possible to choose $\sigma > 0$ sufficiently small such that the condition is satisfied. The above estimates mean that the results of the main theorem of this paper applies to the Lorenz semiflows. It is likely that the choice of $\alpha \in (0,1)$ and $\sigma > 0$ by the above procedure is far from optimal. In principle the rigorous numerics approach of Tucker could be used to obtain values which were close to optimal.

We make a few more comments about this particular suspension semiflow. This suspension semiflow presents the difficulty that the return time is not bounded but moreover τ'/f' is not bounded and will not be bounded for any iterate. (Such a condition is crucially required in [11, 7].) This means that the semiflow is not uniformly expanding in the sense of there existing an invariant conefield, uniformly bounded away from the flow direction, inside of which there is uniform expansion. Lorenz semiflows as discussed above are our main application although we study a more general class of suspension semiflows.

A natural and important question is to what extent the meromorphic extension result of the previous section can be improved in the setting of the Lorenz attractor. In particular, how could one show exponential decay of correlation for the Lorenz attractor? As briefly mentioned earlier the remaining estimate required would be a stronger bound on the spectral radius of the twisted transfer operator (see Section 5) for large imaginary values in the weight of the transfer operator. This is the oscillatory cancelation argument of Dolgopyat [15]. The regularity of the return time function is key for such an argument. The result of this paper does not require the return time to be better than Hölder but it seems unlikely that the above mentioned oscillatory cancellation estimate could be proved with regularity less than $\mathscr{C}^1$. This is because the cancellation method works like an oscillatory integral with the return time appearing as one key part of the intergrand.

There is hope that the return function for the Lorenz semiflow is differentiable. As usual in such situations the regularity of the foliation depends on the balance between the expansion and contraction in the complementary invariant subspaces [23]. In many situations it cannot be hoped for invariant foliations to be better than Hölder [20]. For an Anosov flow it is easy to show that codimension one invariant foliations are $\mathscr{C}^{1+\alpha}$ but the foliation we are interested in is codimension two. For three dimensional contact Anosov flows one may show that the stable foliation of the flow is $\mathscr{C}^{1+\alpha}$ by using crucially the contact structure, this in particular includes the case of geodesic flows on surfaces of negative curvature.[3] The evidence is therefore not inspiring for our aim. However the Lorenz attractor does not preserve volume, it is actually highly dissipative. It seems possible that one can take advantage of this fact to give an improved estimate of the regularity of the stable foliation for the Lorenz attractor. This is the subject of ongoing work.

4. Generalised Bounded Variation

We must consider the weighted transfer operators associated to expanding maps of the interval which have countable discontinuities and for which the inverse of the derivative and the weighting are merely Hölder continuous. This means that we cannot study the transfer operator acting on any relatively standard spaces. One possibility is the generalised bounded variation introduced by Keller [25] and used for expanding interval maps. However he does not consider the case when there are countable discontinuities and also does not consider the case of general weights. Saussol [36] used the same spaces for multi-dimensional expanding maps and showed that a countable number of discontinuities (as opposed to finite) are allowable but again did not study a general classes of weights and furthermore required the derivative of the map to be bounded. These are the spaces we will use in this section. Although not proven in the above mentioned references, with delicate estimates these spaces, as we will prove in this section, are useful for our application. Recently several people have worked on possible alternatives for solving the problem at hand and similar problems. The results of this section are able

[3] In general the contact structure merely implies that the stable foliation enjoys the same regularity as the weak-stable foliation (similarly for the unstable). This means that for higher dimensional Anosov flows the contact structure is not sufficient to obtain $\mathscr{C}^{1+\alpha}$ regularity for the stable and unstable foliations. Nonetheless, using the contact structure, it is possible to carry out the oscillatory cancellation argument in such situations [27].

to favourably settle this question. In particular the following three options were studied. (1) Thomine [38] uses Sobolev space (with fractional exponent) and uses extensively complex interpolation between Banach spaces for many of the calculations. He is able to treat piecewise expanding maps of any dimension. The work is the natural restriction to expanding maps of the ideas used for hyperbolic maps by Baladi and Gouëzel [9]. Unfortunately the maps are required to have derivatives which are uniformly bounded from above and below. (2) Liverani introduced a norm [26] which he studied [28] in the context of piecewise expanding maps of the interval. There is a simple definition for the norm in terms of integrals against test functions that are Hölder continuous. He can study piecewise expanding maps of the interval which have a countable number of discontinuities and also allows the case where the derivative of the map blows up and the case where the weighting of the transfer operator is not bounded but there is a condition that links the Hölder regularity of the weighting in the transfer operator to the rate at which the derivative and weighting blows up. (3) The author also developed an alternative Banach space [12] for studying these problems. The Banach space is a very natural object in that it is equivalent to the space given by real interpolation between $\mathbf{L^1}$ and $\mathbf{BV}$. This has the benefit that it is very easy to work with. Also this approach allows the study of similar settings as the space of Liverani but unfortunately also suffers from the same limitations. Consequently each of these options suffers from some problem which prevents the use in this present setting without imposing undesirable further conditions on the semiflow we wish to study. One particular problem is that we cannot guarantee that the weighting is bounded (see Section 5) and consequently we cannot guarantee that the weighted transfer operator is bounded on $\mathbf{L^1}$. Keller's Banach space of *generalised bounded variation* [25] is contained within $\mathbf{L}^\infty$, a distinct difference to the available alternatives [12, 28]. This suggests the possibility that the transfer operator is bounded on this space even when not bounded on $\mathbf{L^1}$. In the remainder of this section we see that this speculation is shown to be correct.

4.1. **The Banach Space.** The following definitions are identical to [25] with minor changes of notation. For any interval $\mathcal{S}$ and $h : \mathcal{S} \to \mathbb{C}$ let

$$\operatorname{osc}[h, \mathcal{S}] := \operatorname{ess\,sup}\{|h(x_1) - h(x_2)| : x_1, x_2 \in \mathcal{S}\}$$

where the essential supremum is taken with respect to Lebesgue measure on $\mathcal{S}^2$. Let $B_\epsilon(x) := \{y \in \mathbb{R} : |x - y| \leq \epsilon\}$. If $\alpha \in (0, 1)$ and Ω is some finite or countable union of open intervals, let

$$|h|_{\mathfrak{B}_\alpha} := \sup_{\epsilon \in (0, \epsilon_0)} \epsilon^{-\alpha} \int_\Omega \operatorname{osc}[h, B_\epsilon(x) \cap \Omega] \; dx, \tag{4.1}$$

where $\epsilon_0 > 0$ is some fixed parameter. Hence let

$$\mathfrak{B}_\alpha := \left\{ h \in \mathbf{L^1}(\Omega) : |h|_{\mathfrak{B}_\alpha} < \infty \right\}.$$

The seminorm defined above will depend on $\epsilon_0 > 0$ although the sets $\mathfrak{B}_\alpha$ do not. It is known [25, Theorem 1.13] that this set is a Banach space when equipped with the norm

$$\|h\|_{\mathfrak{B}_\alpha} := |h|_{\mathfrak{B}_\alpha} + \|h\|_{\mathbf{L^1}(\Omega)},$$

that $\mathfrak{B}_\alpha \subset \mathbf{L}^\infty(\Omega)$, and that the embedding

$$\mathfrak{B}_\alpha \hookrightarrow \mathbf{L^1}(\Omega) \qquad \text{is compact.} \tag{4.2}$$

4.2. **Piecewise Expanding Transformations.** As before, we suppose that Ω is an open interval and $\{\omega_i\}_{i\in\mathcal{I}}$ is a finite or countable set of disjoint open subintervals which exhaust Ω modulo a set of zero Lebesgue measure (for convenience let $\tilde{\Omega} = \bigsqcup_{i\in\mathcal{I}} \omega_i$) and that we are given

$$f \in \mathscr{C}^1(\tilde{\Omega}, \Omega)$$

which is bijective when restricted to each ω_i. We further suppose that we are given $\xi : \Omega \to \mathbb{C}$ which we call the *weighting*. We require that

$$\|1/f'\|_{\mathbf{L}^\infty(\Omega)} \in (0,1), \tag{4.3}$$

furthermore that

$$\sum_{i\in\mathcal{I}} \|1/f'\|_{\mathbf{L}^\infty(\omega_i)} \|\xi\|_{\mathbf{L}^\infty(\omega_i)} < \infty, \tag{4.4}$$

and finally that $\frac{\xi}{f'} : \Omega \to \mathbb{C}$ is α-Hölder. I.e. there exists $H_\xi < \infty$ and $\alpha \in (0,1)$ such that

$$\left|\tfrac{\xi}{f'}(x) - \tfrac{\xi}{f'}(y)\right| \le H_\xi \left|x-y\right|^\alpha \qquad \text{for all } x, y \in \omega_i \text{ for each } i \in \mathcal{I}. \tag{4.5}$$

For convenience let $f_i : \omega_i \to \Omega$ denote the restriction of f to ω_i. As usual the weighted transfer operator is given, for each $h : \Omega \to \mathbb{C}$, by[4]

$$\mathscr{L}_\xi h(x) := \sum_{i\in\mathcal{I}} \left(\frac{\xi \cdot h}{f'}\right) \circ f_i^{-1}(x) \cdot \mathbf{1}_{f\omega_i}(x). \tag{4.6}$$

By (4.4) we know that $\mathscr{L}_\xi : \mathbf{L}^\infty(\Omega) \to \mathbf{L}^\infty(\Omega)$ is well defined even thought, since we do not require $\|\xi\|_{\mathbf{L}^\infty(\Omega)} < \infty$, we cannot guarantee that the operator is well defined on $\mathbf{L^1}(\Omega)$.

The purpose of this section is to prove the following new result which is a generalisation of the work of Keller [25] to the case of countable discontinuities and unbounded weightings.

Theorem 4.1. *Suppose the transformation $f : \Omega \to \Omega$ and the weighting $\xi : \Omega \to \mathbb{C}$ are as above and satisfy* (4.3)*,* (4.4) *and* (4.5)*. Then $\mathscr{L}_\xi : \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$ is a bounded operator with essential spectral radius not greater than*

$$\lambda := \sup_{i\in\mathcal{I}} \|1/f'\|^\alpha_{\mathbf{L}^\infty(\omega_i)} \|\xi\|_{\mathbf{L}^\infty(\omega_i)}.$$

By a standard argument (see for example [27, p.1281]) the essential spectral radius estimate of the above theorem follows from the compact embedding (4.2) and the Lasota-Yorke type estimate contained in the following theorem. In the case where $\|\xi\|_{\mathbf{L}^\infty(\Omega)} < \infty$ an elementary estimate shows that $\|\mathscr{L}_\xi\|_{\mathbf{L^1}(\Omega)} \le \|\xi\|_{\mathbf{L}^\infty(\Omega)}$ and so, once the essential spectral radius estimate has been shown, this implies that the spectral radius is not greater than $\|\xi\|_{\mathbf{L}^\infty(\Omega)}$.

Theorem 4.2. *Suppose that f and ξ are as per the assumptions of Theorem 4.1. Then for all $\delta > 0$ there exists $C_\delta < \infty$ such that*

$$\|\mathscr{L}_\xi h\|_{\mathfrak{B}_\alpha} \le (2+\delta)\lambda \left\|h\right\|_{\mathfrak{B}_\alpha} + C_\delta \|h\|_{\mathbf{L^1}(\Omega)} \qquad \textit{for all } h \in \mathfrak{B}_\alpha.$$

The remainder of this section is devoted to the proof of the above theorem. This estimate is an extension of the result of Keller [25] to our setting. The proof follows a similar argument to Keller's original with various additional complications, in particular because of the weighting ξ and the possibility that $\mathcal{I}$ is merely countable. As such we are forced to redo the proof but when possible we refer to the relevant theorems and lemmas which we can reuse.

[4]For any set, e.g. A, we let $\mathbf{1}_A$ denote the indicator function of that set.

4.3. **Proof of Theorem 4.2.** We may assume that $\delta \leq 1$. First $\epsilon_0 > 0$ must be carefully chosen and it is convenient to divide the index set as $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$. By (4.4) we may choose a finite set $\mathcal{I}_1 \subset \mathcal{I}$ such that

$$\sum_{i \in \mathcal{I}_2} \|1/f'\|_{\mathbf{L}^\infty(\omega_i)} \|\xi\|_{\mathbf{L}^\infty(\omega_i)} \leq \frac{\lambda\delta}{16}, \tag{4.7}$$

where $\mathcal{I}_2 := \mathcal{I} \setminus \mathcal{I}_1$. Let $\Gamma := 32\delta^{-1} + 2$. Choosing ϵ_0 sufficiently small we ensure that

$$|f\omega_i| \geq \epsilon_0 \Gamma \qquad \text{for all } i \in \mathcal{I}_1 \tag{4.8}$$

and that

$$\epsilon_0^\alpha \leq \frac{\delta\lambda}{8(8+\delta)H_\xi \Gamma}. \tag{4.9}$$

(The reason for this particular choice will subsequently become clear (4.18).) If $|f\omega_i| > 2\epsilon_0\Gamma$ for some $i \in \mathcal{I}_1$ we chop ω_i into pieces such that $\epsilon_0\Gamma \leq |f\omega_j| \leq 2\epsilon_0\Gamma$ for all the resulting pieces. If $|f\omega_i| > 2\epsilon_0\Gamma$ for some $i \in \mathcal{I}_2$ we chop ω_i into pieces as before but in this case we move the resulting pieces into the set $\mathcal{I}_1$. This means that the estimate (4.7) remains unaltered. Note that $\mathcal{I}_1$ may no longer be a finite set. To conclude we have arranged so that (4.7), (4.8), and (4.9) hold and furthermore that

$$|f\omega_i| \leq 2\epsilon_0\Gamma \qquad \text{for all } i \in \mathcal{I}. \tag{4.10}$$

Fix $h \in \mathfrak{B}_\alpha$. We start by noting that by the definition (4.1) of the seminorm and the definition (4.6) of the transfer operator

$$\begin{aligned} |\mathscr{L}_\xi h|_{\mathfrak{B}_\alpha} &= \sup_{\epsilon \in (0,\epsilon_0)} \epsilon^{-\alpha} \int_\Omega \operatorname{osc}\left[\mathscr{L}_\xi h, B_\epsilon(x) \cap \Omega\right] \, dx \\ &\leq \sup_{\epsilon \in (0,\epsilon_0)} \sum_i \epsilon^{-\alpha} \int_\Omega \operatorname{osc}\left[\left(\tfrac{\xi \cdot h}{f'}\right) \circ f_i^{-1} \cdot \mathbf{1}_{f\omega_i}, B_\epsilon(x) \cap \Omega\right] \, dx. \end{aligned} \tag{4.11}$$

To proceed we take advantage of several estimates which have already been proved elsewhere. Firstly by [25, Theorem 2.1] for each $i \in \mathcal{I}_1$, since $|f\omega_i| \geq (32\delta^{-1}+2)\epsilon_0$ by (4.8) and $|f\omega_i| \geq 4\epsilon_0$, we have that

$$\begin{aligned} &\int_\Omega \operatorname{osc}\left[\left(\tfrac{\xi \cdot h}{f'}\right) \circ f_i^{-1} \cdot \mathbf{1}_{f\omega_i}, B_\epsilon(x) \cap \Omega\right] \, dx \\ &\qquad \leq (2 + \tfrac{\delta}{4}) \int_{f\omega_i} \operatorname{osc}\left[\left(\tfrac{\xi \cdot h}{f'}\right) \circ f_i^{-1}, B_\epsilon(x) \cap f\omega_i\right] \, dx \\ &\qquad + \frac{\epsilon}{\epsilon_0} \int_{f\omega_i} \left|\tfrac{\xi \cdot h}{f'}\right| \circ f_i^{-1}(x) \, dx. \end{aligned} \tag{4.12}$$

For $i \in \mathcal{I}_2$ (where $|f\omega_i|$ may be small) we use the following, more basic estimate. By [36, Proposition 3.2 (ii)] for each i

$$\begin{aligned} \operatorname{osc}\left[\left(\tfrac{\xi \cdot h}{f'}\right) \circ f_i^{-1} \cdot \mathbf{1}_{f\omega_i}, B_\epsilon(x) \cap \Omega\right] &\leq \operatorname{osc}\left[\left(\tfrac{\xi \cdot h}{f'}\right) \circ f_i^{-1}, B_\epsilon(x) \cap f\omega_i\right] \cdot \mathbf{1}_{f\omega_i} \\ &\quad + 2\|\tfrac{\xi \cdot h}{f'}\|_{\mathbf{L}^\infty(\omega_i)} \mathbf{1}_{F_{i,\epsilon}}(x) \end{aligned}$$

where $F_{i,\epsilon}$ denotes the set of all points $x \in \Omega$ which are within a distance of ϵ of the end points of the interval $f\omega_i$. Since $|\int_{f\omega_i} \mathbf{1}_{F_{i,\epsilon}}(x) \, dx| \leq 2\epsilon$ the above implies that

$$\begin{aligned} &\int_\Omega \operatorname{osc}\left[\left(\tfrac{\xi \cdot h}{f'}\right) \circ f_i^{-1} \cdot \mathbf{1}_{f\omega_i}, B_\epsilon(x) \cap \Omega\right] \, dx \\ &\qquad \leq \int_{f\omega_i} \operatorname{osc}\left[\left(\tfrac{\xi \cdot h}{f'}\right) \circ f_i^{-1}, B_\epsilon(x) \cap f\omega_i\right] \, dx \\ &\qquad + 4\epsilon \|\xi\|_{\mathbf{L}^\infty(\omega_i)} \|h\|_{\mathbf{L}^\infty(\Omega)} \|1/f'\|_{\mathbf{L}^\infty(\omega_i)}. \end{aligned} \tag{4.13}$$

Note that the integral term in the middle line of the above equation is identical to the integral term of the middle line of (4.12). We also require the following basic estimate for the osc $[\cdot,\cdot]$ of a product.

Lemma 4.3. *Suppose $\mathcal{S} \subset \Omega$ is an interval, $g_1 : \mathcal{S} \to \mathbb{C}$, $g_1 : \mathcal{S} \to \mathbb{C}$ and $y \in \mathcal{S}$. Then*

$$\operatorname{osc}[g_1 \cdot g_2, \mathcal{S}] \leq |g_1(y)| \cdot \operatorname{osc}[g_2, \mathcal{S}] + 2\|g_2\|_{\mathbf{L}^\infty(\mathcal{S})} \cdot \operatorname{osc}[g_1, \mathcal{S}] .$$

Proof. Suppose $x_1, x_2, y \in \mathcal{S}$. It suffices to observe that

$$\begin{aligned}(g_1 \cdot g_2)(x_1) - (g_1 \cdot g_2)(x_2) &= g_1(y)\,(g_2(x_1) - g_2(x_2)) \\ &\quad + g_2(x_1)\,(g_1(x_1) - g_1(y)) + g_2(x_2)\,(g_1(y) - g_1(x_2)) . \quad \square\end{aligned}$$

This means in particular that (this is the term which appears in the middle lines of (4.12) and (4.13))

$$\begin{aligned}&\int_{f\omega_i} \operatorname{osc}\left[\left(\tfrac{\xi \cdot h}{f'}\right) \circ f_i^{-1}, B_\epsilon(x) \cap f\omega_i\right]\, dx \\ &\qquad \leq \int_{f\omega_i} \left|\tfrac{\xi}{f'}\right| \circ f_i^{-1}(x) \cdot \operatorname{osc}\left[h \circ f_i^{-1}, B_\epsilon(x) \cap f\omega_i\right]\, dx \\ &\qquad + 2\|h\|_{\mathbf{L}^\infty(\omega_i)} \int_{f\omega_i} \operatorname{osc}\left[\tfrac{\xi}{f'} \circ f_i^{-1}, B_\epsilon(x) \cap f\omega_i\right]\, dx.\end{aligned} \tag{4.14}$$

Recalling (4.11) and applying the estimates of (4.12), (4.13) and (4.14) we have

$$|\mathscr{L}_\xi h|_{\mathfrak{B}_\alpha} \leq \sup_{\epsilon \in (0,\epsilon_0)} \left(A_{1,\xi,h}(\epsilon) + A_{2,\xi,h}(\epsilon) + A_{3,\xi,h}(\epsilon) + A_{4,\xi,h}(\epsilon)\right), \tag{4.15}$$

where we have definded for convenience

$$\begin{aligned}A_{1,\xi,h}(\epsilon) &:= \epsilon^{-\alpha}(2 + \tfrac{\delta}{4}) \sum_{i \in \mathcal{I}} \int_{f\omega_i} \left|\tfrac{\xi}{f'}\right| \circ f_i^{-1}(x) \cdot \operatorname{osc}\left[h \circ f_i^{-1}, B_\epsilon(x) \cap f\omega_i\right]\, dx \\ A_{2,\xi,h}(\epsilon) &:= 2\epsilon^{-\alpha}(2 + \tfrac{\delta}{4}) \sum_{i \in \mathcal{I}} \|h\|_{\mathbf{L}^\infty(\omega_i)} \int_{f\omega_i} \operatorname{osc}\left[\tfrac{\xi}{f'} \circ f_i^{-1}, B_\epsilon(x) \cap f\omega_i\right]\, dx \\ A_{3,\xi,h}(\epsilon) &:= 4\epsilon^{1-\alpha} \|h\|_{\mathbf{L}^\infty(\Omega)} \sum_{i \in \mathcal{I}_2} \|1/f'\|_{\mathbf{L}^\infty(\omega_i)} \|\xi\|_{\mathbf{L}^\infty(\omega_i)} \\ A_{4,\xi,h}(\epsilon) &:= \frac{\epsilon^{1-\alpha}}{\epsilon_0} \sum_{i \in \mathcal{I}_1} \int_{f\omega_i} \left|\tfrac{\xi \cdot h}{f'}\right| \circ f_i^{-1}(x)\, dx.\end{aligned}$$

The remainder of the proof involves independently estimating each of these four terms.

We start by estimating $A_{1,\xi,h}(\epsilon)$. Let $\sigma_i := \|1/f'\|_{\mathbf{L}^\infty(\omega_i)} \in (0,1)$ by assumption (4.3). Since $f_i^{-1} B_\epsilon(x) \subseteq B_{\sigma_i \epsilon}(f_i^{-1} x)$ we have that

$$\begin{aligned}\operatorname{osc}\left[h \circ f_i^{-1}, B_\epsilon(x) \cap f\omega_i\right] &= \operatorname{osc}\left[h, f_i^{-1} B_\epsilon(x) \cap \omega_i\right] \\ &\leq \operatorname{osc}\left[h, B_{\sigma_i \epsilon}(y_i) \cap \omega_i\right]\end{aligned} \tag{4.16}$$

where $y_i := f_i^{-1} x$. We change variables in the integral and so

$$\begin{aligned}A_{1,\xi,h}(\epsilon) &\leq \epsilon^{-\alpha}(2 + \tfrac{\delta}{4}) \sum_{i \in \mathcal{I}} \int_{\omega_i} |\xi|\,(y_i) \cdot \operatorname{osc}\left[h, B_{\sigma_i \epsilon}(y_i) \cap \omega_i\right]\, dy_i \\ &\leq \epsilon^{-\alpha}(2 + \tfrac{\delta}{4}) \|\xi\|_{\mathbf{L}^\infty(\omega_i)} \int_\Omega \operatorname{osc}\left[h, B_{\sigma_i \epsilon}(y) \cap \Omega\right]\, dy \\ &\leq \sigma_i^\alpha (2 + \tfrac{\delta}{4}) \|\xi\|_{\mathbf{L}^\infty(\omega_i)} \,|h|_{\mathfrak{B}_\alpha} \leq (2 + \tfrac{\delta}{4}) \lambda\, |h|_{\mathfrak{B}_\alpha} .\end{aligned} \tag{4.17}$$

Now we estimate $A_{2,\xi,h}(\epsilon)$. By [25, Lemma 2.2] we have the estimate

$$\|h\|_{\mathbf{L}^\infty(\omega_i)} \leq \epsilon_0^{-1} \int_{\omega_i} \operatorname{osc}\left[h, B_{\epsilon_0}(x)\right] \, dx + |\omega_i|^{-1} \|h\|_{\mathbf{L}^1(\omega_i)}.$$

By assumption (4.5) we know that $\operatorname{osc}\left[\frac{\xi}{f'}, B_{\sigma_i\epsilon}(y_i) \cap \omega_i\right] \leq 2H_\xi \sigma_i^\alpha \epsilon^\alpha$ and so, changing variables as per (4.16), we have

$$\begin{aligned}\int_{f\omega_i} \operatorname{osc}\left[\tfrac{\xi}{f'} \circ f_i^{-1}, B_\epsilon(x) \cap f\omega_i\right] \, dx &\leq \int_{\omega_i} \operatorname{osc}\left[\tfrac{\xi}{f'}, B_{\sigma_i\epsilon}(y_i) \cap \omega_i\right] \, dy_i \\ &\leq 2 \, |\omega_i| \, H_\xi \sigma_i^\alpha \epsilon^\alpha.\end{aligned}$$

Combining the above two estimates means that

$$A_{2,\xi,h}(\epsilon) \leq 4(2 + \tfrac{\delta}{4}) H_\xi \sum_{i\in\mathcal{I}} \sigma_i^\alpha \left(\epsilon_0^{-(1-\alpha)} \, |\omega_i| \, \epsilon_0^{-\alpha} \int_{\omega_i} \operatorname{osc}\left[h, B_{\epsilon_0}(x)\right] \, dx + \|h\|_{\mathbf{L}^1(\omega_i)}. \right)$$

By the expanding assumption (4.3) and by (4.10) we know that $|\omega_i| \leq \sigma_i \, |f\omega_i| \leq 2\sigma_i\epsilon_0\Gamma$. Using also (4.9) this means that for all $i \in \mathcal{I}$

$$\begin{aligned}4(2 + \tfrac{\delta}{4}) H_\xi \sigma^\alpha \epsilon_0^{-(1-\alpha)} \, |\omega_i| &\leq 8(2 + \tfrac{\delta}{4}) H_\xi \|1/f'\|_{\mathbf{L}^\infty(\Omega)}^{1+\alpha} \epsilon_0^\alpha \Gamma \\ &\leq \tfrac{\delta}{4}\lambda.\end{aligned}$$

Consequently we have shown that

$$A_{2,\xi,h}(\epsilon) \leq \tfrac{\delta}{4}\lambda \, |h|_{\mathfrak{B}_\alpha} + 4(2 + \tfrac{\delta}{4}) H_\xi \|1/f'\|_{\mathbf{L}^\infty(\Omega)}^\alpha \|h\|_{\mathbf{L}^1(\Omega)}. \tag{4.18}$$

Now we estimate $A_{3,\xi,h}(\epsilon)$. Using again [25, Lemma 2.2] we have the estimate

$$\|h\|_{\mathbf{L}^\infty(\Omega)} \leq \epsilon_0^{-(1-\alpha)} \, |h|_{\mathfrak{B}_\alpha} + |\Omega|^{-1} \, \|h\|_{\mathbf{L}^1(\Omega)}. \tag{4.19}$$

This means that

$$A_{3,\xi,h}(\epsilon) \leq 4 \left(|h|_{\mathfrak{B}_\alpha} + \frac{\epsilon_0^{1-\alpha}}{|\Omega|} \|h\|_{\mathbf{L}^1(\Omega)} \right) \sum_{i\in\mathcal{I}_2} \|1/f'\|_{\mathbf{L}^\infty(\omega_i)} \|\xi\|_{\mathbf{L}^\infty(\omega_i)}.$$

By (4.7) we know that $\sum_{i\in\mathcal{I}_2} \|1/f'\|_{\mathbf{L}^\infty(\omega_i)} \|\xi\|_{\mathbf{L}^\infty(\omega_i)} \leq \frac{\lambda\delta}{16}$ and so

$$A_{3,\xi,h}(\epsilon) \leq \tfrac{\delta}{4}\lambda \, |h|_{\mathfrak{B}_\alpha} + \left(\frac{\epsilon_0^{1-\alpha}\delta}{4\,|\Omega|}\lambda \right) \|h\|_{\mathbf{L}^1(\Omega)}. \tag{4.20}$$

Now we estimate $A_{4,\xi,h}(\epsilon)$. Using again the assumption (4.4) we may choose a finite set $\mathcal{I}_3 \subset \mathcal{I}_1$ such that

$$\sum_{i\in\mathcal{I}_4} \|1/f'\|_{\mathbf{L}^\infty(\omega_i)} \|\xi\|_{\mathbf{L}^\infty(\omega_i)} \leq \frac{\delta\epsilon_0}{4}\lambda,$$

where $\mathcal{I}_4 := \mathcal{I}_1 \setminus \mathcal{I}_3$. We therefore estimate, using also a change of variables $y_i := f_i^{-1}x$,

$$\begin{aligned}A_{4,\xi,h}(\epsilon) &\leq \epsilon_0^{-\alpha} \sum_{i\in\mathcal{I}_1} \int_{f\omega_i} \left|\tfrac{\xi\cdot h}{f'}\right| \circ f_i^{-1}(x) \, dx \\ &\leq \epsilon_0^{-\alpha} \sum_{i\in\mathcal{I}_3} \int_{\omega_i} |\xi \cdot h| \, (y_i) \, dy_i + \epsilon_0^{-\alpha} \sum_{i\in\mathcal{I}_4} \|\frac{\xi}{f'}\|_{\mathbf{L}^\infty(\omega_i)} \|h\|_{\mathbf{L}^\infty(\Omega)}.\end{aligned}$$

Using (4.19) to estimate $\|h\|_{\mathbf{L}^\infty(\Omega)}$, this means that for all $\epsilon \in (0, \epsilon_0)$ we have

$$A_{4,\xi,h}(\epsilon) \leq \frac{\delta}{4}\lambda \, |h|_{\mathfrak{B}_\alpha} + \left(\epsilon_0^{-\alpha} \, |\Omega|^{-1} \sup_{i\in\mathcal{I}_3} \|\xi\|_{\mathbf{L}^\infty(\omega_i)} \right) \|h\|_{\mathbf{L}^1(\Omega)}. \tag{4.21}$$

Summing the estimates of (4.17), (4.18), (4.20), and (4.21) we have shown that

$$|\mathscr{L}_\xi h|_{\mathfrak{B}_\alpha} \leq (2 + \delta)\lambda \, |h|_{\mathfrak{B}_\alpha} + C_\delta \|h\|_{\mathbf{L}^1(\Omega)}$$

for all $h \in \mathfrak{B}_\alpha$, where

$$C_\delta := 4(2+\tfrac{\delta}{4})H_\xi \|\tfrac{1}{f'}\|^\alpha_{\mathbf{L}^\infty(\omega_i)} + \frac{1}{\epsilon_0^\alpha |\Omega|} \sup_{i\in\mathcal{I}_3} \|\xi\|_{\mathbf{L}^\infty(\omega_i)} + \frac{\lambda\epsilon_0^{1-\alpha}\delta}{4|\Omega|}.$$

This completes the proof of Theorem 4.2.

5. Twisted Transfer Operators

In this section we follow the standard "twisted transfer operator" approach to studying flows. We will take steps to allow the transfer operator results of the previous section to be applied to the original problem of the meromorphic extension of the correlation function. Throughout this section we suppose that we are given a suspension semiflow (Ω, f, τ) which satisfies the assumptions of the Main Theorem, in particular assumptions (2.2), (2.3), and (2.4). First we show that a condition named *exponential tails* in [7] holds also in this setting.

Lemma 5.1. $\int_\Omega e^{\sigma\tau(x)}\,dx < \infty$.

Proof. We estimate $\int_\Omega e^{\sigma\tau(x)}\,dx \leq \sum_{i\in\mathcal{I}} |\omega_i|\, e^{\sigma\|\tau\|_{\mathbf{L}^\infty(\omega_i)}}$. Since also we have that $|\omega_i| \leq \|1/f'\|_{\mathbf{L}^\infty(\omega_i)} |\Omega|$ then the supposition (2.4) implies the lemma. □

For all $t \geq 0$ let $A_t := \{(x,s) \in \Omega_\tau : s + t \geq \tau(x)\}$ and $B_t := \Omega_\tau \setminus A_t$. Hence we may write

$$\mu(u \cdot v \circ \Phi^t) = \mu(u \cdot v \circ \Phi^t \cdot \mathbf{1}_{A_t}) + \mu(u \cdot v \circ \Phi^t \cdot \mathbf{1}_{B_t}). \tag{5.1}$$

Exponential decay for the second term is simple to estimate.

Lemma 5.2. *Exists $C < \infty$ such that $|\mu(u \cdot v \circ \Phi^t \cdot \mathbf{1}_{B_t})| \leq C\,|u|_\infty\,|v|_\infty\, e^{-\sigma t}$ for all $u, v : \Omega_\tau \to \mathbb{C}$ bounded and $t \geq 0$.*

Proof. Since μ is given by a formula in terms of the measure ν which is absolutely continuous with respect to Lebesgue there exists $C < \infty$ such that, letting $D_t := \{x \in \Omega : \tau(x) - t > 0\}$, we have

$$\left|\mu(u \cdot v \circ \Phi^t \cdot \mathbf{1}_{B_t})\right| \leq C\,|u|_\infty\,|v|_\infty \int_\Omega (\tau(x) - t) \cdot \mathbf{1}_{D_t}(x)\,dx \tag{5.2}$$

for all $t \geq 0$. For $y \in \mathbb{R}$ we define $k(y)$ equal to y if $y \geq 0$ and equal to 0 otherwise. This definition means that $(\tau(x) - t) \cdot \mathbf{1}_{D_t}(x) \leq k(\tau(x) - t)$. Since $\ln y \leq y$ for all $y > 0$ it follows that $\ln(\sigma y) = \ln\sigma + \ln y \leq \sigma y$ and so $y \leq \sigma^{-1} e^{\sigma y}$ for all $y > 0$. The case $y \leq 0$ is simple and so we have shown that $k(y) \leq \sigma^{-1}e^{\sigma y}$ for all $y \in \mathbb{R}$. This means that

$$(\tau(x) - t) \cdot \mathbf{1}_{D_t}(x) \leq \sigma^{-1} e^{\sigma(\tau(x)-t)}, \qquad \text{for all } x \in \Omega.$$

We conclude using the above with (5.2) since $\int e^{\sigma\tau(x)}\,dx < \infty$ by Lemma 5.1. □

In order to proceed we must estimate the other term in (5.1) and so it is convenient to define

$$\rho(t) := \mu(u \cdot v \circ \Phi^t \cdot \mathbf{1}_{A_t}). \tag{5.3}$$

Note that $|\mu(u \cdot v \circ \Phi^t \cdot \mathbf{1}_{A_t})| \leq |u|_\infty\,|v|_\infty$ for all $t \geq 0$. For all $z \in \mathbb{C}$ such that $\Re(z) > 0$ we consider the Laplace transform of the above function

$$\hat\rho(z) := \int_0^\infty e^{-zt}\rho(t)\,dt. \tag{5.4}$$

Additionally for any $u : \Omega_\tau \to \mathbb{C}$ and $z \in \mathbb{C}$ let

$$\hat u_z(x) := \int_0^\infty e^{-zs} u(x,s)\,ds \tag{5.5}$$

for all $x \in \Omega$. Furthermore for all $n \in \mathbb{N}$ let $\tau_n := \sum_{k=0}^{n-1} \tau \circ f^k$. Since the invariant measure ν is absolutely continuous with respect to Lebesgue there exists a density $h_0 \in \mathbf{L^1}(\Omega)$ such that $\mu(\eta) = \int_\Omega \int_0^{\tau(x)} \eta(x,s)\ ds\ h_0(x)\ dx$ for all bounded $\eta : \Omega_\tau \to \mathbb{C}$. As in [34, 35, 11, 7] we have the following representation of the Laplace transform in terms of an infinite sum.

Lemma 5.3. *For all $z \in \mathbb{C}$ such that $\Re(z) > 0$ and all $|u|_\infty < \infty$, $|v|_\infty < \infty$,*

$$\hat{\rho}(z) = \sum_{n=1}^{\infty} \int_\Omega (h_0 \cdot \hat{u}_{-z} \cdot e^{-z\tau_n} \cdot \hat{v}_z \circ f^n)(x)\ dx.$$

Proof. Recall that $h_0 \in \mathbf{L^1}(\Omega)$ is the density of the f-invariant measure ν. For all $\Re(z) > 0$

$$\begin{aligned}\hat{\rho}(z) &= \int_0^\infty \int_\Omega \int_0^{\tau(x)} e^{-zt} u(x,s) v \circ \Phi^t(x,s) \mathbf{1}_{A_t}(x,s) h_0(x)\ ds\ dx\ dt \\ &= \sum_{n=1}^{\infty} \int_\Omega \int_0^{\tau(x)} \int_{\tau_n(x)-s}^{\tau_{n+1}(x)-s} e^{-zt} u(x,s) v \circ \Phi^t(x,s) h_0(x)\ dt\ ds\ dx.\end{aligned}$$

We change variables letting $t' = t - \tau_n(x) + s$ and note that when $t \in [\tau_n(x) - s, \tau_{n+1}(x) - s]$ then $\Phi^t(x,s) = (f^n x, t - \tau_n(x) + s)$. This means that

$$\begin{aligned}\hat{\rho}(z) = \sum_{n=1}^{\infty} \int_\Omega e^{-z\tau_n(x)} &\left(\int_0^{\tau(x)} e^{zs} u(x,s)\ ds \right) \\ &\times \left(\int_0^{\tau(f^n x)} e^{-zt'} v(f^n x, t')\ dt' \right) h_0(x)\ dx.\end{aligned}$$

Recalling the definition (5.5) for $\hat{u}_{-z}$ and $\hat{v}_z$ we conclude. □

We now relate the sum given by Lemma 5.3 to the *twisted transfer operators*. For all $z \in \mathbb{C}$ such that $\Re(z) \in [-\sigma, 0]$ let $\xi_z : \Omega \to \mathbb{C}$ be defined as

$$\xi_z := e^{-z\tau}. \tag{5.6}$$

We consider the map $f : \Omega \to \Omega$ with the weighting ξ_z. It is immediate that the assumptions imposed on the semiflow imply that the pair f and ξ_z satisfy the assumptions of Theorem 4.1. Consequently the transfer operator $\mathscr{L}_z : \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$ (for convenience we now write $\mathscr{L}_z$ for $\mathscr{L}_{\xi_z}$) and which is given by the formula

$$\mathscr{L}_z h(x) := \sum_{i \in \mathcal{I}} \left(\frac{e^{-z\tau} \cdot h}{f'} \right) \circ f_i^{-1}(x) \cdot \mathbf{1}_{f\omega_i}(x).$$

has essential spectral radius strictly less than 1. Let $\mathscr{B}(\mathfrak{B}_\alpha, \mathfrak{B}_\alpha)$ denote the space of bounded linear operators mapping $\mathfrak{B}_\alpha$ to $\mathfrak{B}_\alpha$.

Lemma 5.4. *The operator valued function $z \mapsto (\mathbf{id} - \mathscr{L}_z)^{-1} \in \mathscr{B}(\mathfrak{B}_\alpha, \mathfrak{B}_\alpha)$ is meromorphic on the set $\{z \in \mathbb{C} : \Re(z) \in [-\sigma, 0]\}$.*

Proof. We know that $\mathscr{L}_z \in \mathscr{B}(\mathfrak{B}_\alpha, \mathfrak{B}_\alpha)$ has essential spectral radius less than 1 for all $\Re(z) \in [-\sigma, 0]$ and so is of the form $\mathscr{L}_z = \mathscr{K}_z + \mathscr{A}_z$ where $\mathscr{K}_z$ is compact, the spectral radius of $\mathscr{A}_z$ is strictly less than 1 and $\mathscr{K}_z \mathscr{A}_z = 0$. Furthermore both $z \mapsto \mathscr{K}_z \in \mathscr{B}(\mathfrak{B}_\alpha, \mathfrak{B}_\alpha)$ and $z \mapsto \mathscr{A}_z \in \mathscr{B}(\mathfrak{B}_\alpha, \mathfrak{B}_\alpha)$ are holomorphic operator-valued functions. Note that

$$(\mathbf{id} - \mathscr{L}_z) = (\mathbf{id} - \mathscr{K}_z)(\mathbf{id} - \mathscr{A}_z).$$

and that $(\mathbf{id} - \mathscr{A}_z)$ is invertible. By the Analytic Fredholm Theorem $z \mapsto (\mathbf{id} - \mathscr{K}_z)^{-1}$ is meromorphic on the set $\{z \in \mathbb{C} : \Re(z) \in [-\sigma, 0]\}$. □

Lemma 5.5. *The operator valued function* $z \mapsto \sum_{n=1}^{\infty} \mathscr{L}_z^n \in \mathscr{B}(\mathfrak{B}_\alpha, \mathfrak{B}_\alpha)$ *is meromorphic on the set* $\{z \in \mathbb{C} : \Re(z) \in [-\sigma, 0]\}$.

Proof. We note that $\sum_{n=1}^{\infty} \mathscr{L}_z^n = (\mathbf{id} - \mathscr{L}_z)^{-1}\mathscr{L}_z$ and apply Lemma 5.4. □

Proof of The Main Theorem. By Lemma 5.2 it suffices to know that $\hat{\rho}$ admits the relevant meromorphic extension. Since, as usual for transfer operators, we have that

$$\int_\Omega \mathscr{L}_z^n h_1(x) \cdot h_2(x)\, dx = \int_\Omega h_1(x) \cdot h_2 \circ e^{-z\tau_n(x)} \circ f^n(x)\, dx$$

the formula for $\hat{\rho}(z)$ given by Lemma 5.3 means that

$$\hat{\rho}(z) = \sum_{n=1}^{\infty} \int_\Omega \mathscr{L}_z^n (h_0 \hat{u}_{-z})(x) \cdot \hat{v}_z(x)\, dx.$$

This equality was shown to hold for all $\Re(z) > 0$. But since the right hand side is meromorphic on the set $\{z \in \mathbb{C} : \Re(z) \in [-\sigma, 0]\}$ we have shown that the left hand side admits such an extension. □

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