The Standard Model of Particle Physics Introductory Lecture August 3rd 2014

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African School of Fundamental Physics and Applications 2014, Dakar, Senegal



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## Particles, the Universe and the LHC









Large Hadron Collider

All known elementary particles

An introduction and overview of particle physics. Also: why and what the LHC is for. Two Mysteries: Dark Matter and the Higgs Boson



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## Atoms and the Periodic Table



## Particles, Matter and the Universe

We, and the matter which makes up all of the Earth is made of atoms

The planets and stars also ALL seem to be made of atoms

The galaxy contains hundreds of billions (i.e.  $\sim 10^{11}$ ) of stars which are basically alike: so much of the galaxy is also be made of atoms.



Remarkably, all of the galaxies (and there are  $\sim 10^{11}$  of them) also look quite similar!

Hence, we learn that much of the matter of the entire Universe is made of atoms

We will ADD some very important details to this picture later.



## Abundance of Elements in the Universe



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## **The Structure of Atoms**

The structure and regularity of the periodic table suggests that atoms are made of even simpler objects

Electrons were discovered by Thompson in so-called cathode ray tubes around 1897.A wire filament with a current passing through was seen to emit particles with negative charge. This showed that atoms contained "electrons".  $\alpha$ -particles (now known to be Helium nuclei) were discovered as radiation emitted by various Rutherford used beams of  $\alpha$ -particles scattered off Gold to prove that atoms "must" contain a very dense nucleus

The  $\alpha$ -particles were deflected at large angles, proving that atoms had "structure" - dense nucleus.

Shortly after this, the Bohr model of the atom (protons and neutrons in a dense nucleus, surrounded by a cloud of electrons) was "established".



## **The Atom**



- The atom consists of a very dense nucleus surrounded by a "cloud of electrons"
- Atoms have sizes of order  $10^{-11}$  to  $10^{-10}$  m.
- Nucleus is  $10^{-4}$  times smaller!
- Protons are positively charged. Neutrons are neutral.
- Electrons have *minus* the charge of protons but are *much* lighter ( $m_e \sim m_p/2000$ )
- But are protons, neutrons Kings and electrons fundamental

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## **Charged particles in Electric and Magnetic Fields**



- Electric current is just moving charge
- You can create a magnetic field by coiling a current carrying wire
- By the same token, a magnetic field exerts a force on charged particles
- Strong electric and magnetic fields can be used to accelerate charged particles to high energies!
- This is how particle accelerators work: you will learn this in week 3!



## The galaxy is a particle collider!



- The galaxy has a strong magnetic field and charged particles!
- A high energy particle is created in a star and brought to Earth by the galactic magnetic field
- It strikes the upper atmosphere of Earth
- Multiple interactions occur between the particle and the atoms in the atmosphere
- Creates a cosmic ray "showe KING'S LONDON



## The galaxy as a particle collider!



- This is a particular example when the original particle is a proton
- This example produces pions, photons, muons, electrons and neutrinos!
- A muon from a cosmic ray like this just passed through your body!
- In fact, muons and pions were discovered in cosmic rays



## The Particle Zoo

- As a result of cosmic ray and ground-based particle collider experiments we now know that protons and neutrons are actually made of more fundamental particles called quarks and gluons
- We also know that protons and neutrons have many 'cousins' which are also made of quarks and gluons. Hundreds of these have been discovered over the past six decades.
- Neutrons, when not bound inside atoms, actually decay into a proton, an electron and a neutral (almost massless) particle called a neutrino.



• The most remarkable fact, though, is that *all of these hundreds of particles and their properties are* **precisely** *described by a very simple model.* 



## The Standard Model of Particle Physics



The entire periodic table can be explained by just the first and last columns! Last column are the force carrying particles There are three "families" of quarks and leptons. Why not just one? Why these particular masses? Are there more than just these particles?



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## Mass and The Higgs Boson. Original Slide from 2011. See next slide for update

- There is one particle predicted by the Standard Model which has not yet been discovered
- This is the so-called Higgs boson, after Peter Higgs who made significant contributions to this part of the Standard Model
- According to the model, it is the Higgs boson which is responsible for giving the quarks and leptons their mass
- The heavier the particle, the stronger its interactions with the Higgs particle, so the top quark has the strongest interaction with the Higgs, the electron the weakest one.
- One of the main reasons for the LHC physics programme is to find the Higgs boson, or prove that it doesn't exist!
- It's discovery would shed enormous light on the nature and origin of mass and "complete" the story of the Standard



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## Mass and The Higgs Boson. Slide updated 2013

- There is one additional particle predicted by the Standard Model which was not yet mentioned.
- This is the so-called Higgs boson, after Peter Higgs who made significant contributions to this part of the Standard Model in 1964
- According to the model, it is the Higgs boson which is responsible for giving the quarks and leptons their mass
- The heavier the particle, the stronger its interactions with the Higgs particle, so the top quark has the strongest interaction with the Higgs, the electron the weakest one.
- One of the main reasons for the LHC physics programme was to find the Higgs boson, or prove that it doesn't exist!



• It's discovery would shed enormous light on the nature and origin of mass and "complete" the story of the Standard



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## The LHC and The Higgs Boson

- Much data from the LHC collected in 2011 and 2012
- The first evidence for the existence of the Higgs boson has emerged
- A new particle has been discovered
- So far its properties are remarkably consistent with the Higgs boson
- More data is being collected and analysed to determine if the Standard Model is correct
- These are very exciting times. More about the LHC and the Higgs boson later in the course.



## **Dark Matter**

- Einstein's theory of gravity has been tested with remarkable accuracy in various systems within the galaxy
- But, if you estimate the mass of galaxies by the number of atoms that they contain, there's a huge problem
- Galaxies of that mass would not move in the way that the galaxies have been observed to be moving
- There is now very strong evidence that there is additional, "dark" matter in the Universe, beyond protons, neutrons, electrons, photons and neutrinos
- In fact there seems to be about five times more dark matter than "atomic matter"!



 There are some plausible arguments that particles made of dark matter could be produced directly at the LHC (more on K this later)

## **Cosmic History**



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## The Large Hadron Collider

#### • The basic idea is to :

- accelerate beams of protons to as high energy as possible
- smash them into each other as often as possible
- Create new particles like the Higgs boson or dark matter
- Detect these particles in "particle detectors" which surround the "collisions"
- The production of "new particles" is rare, so the more collisions we have the better
- The LHC ran successfully in 2011 and 2012. Will restart in 2015 at even higher energy!





## The LHC Accelerator complex



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## The LHC Accelerator complex

About 1600 superconducting magnets (most weigh more than 20 tonnes!) used to accelerate the proton beams About 100 tonnes of liquid helium required to keep the magnets at operating temperature o minus 271 Centigrade The LHC is the coolest place in the Universe! The magnetic field is about 8 Tesla (about  $10^5$ times that of the Earth!)

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## The LHC beam and Energy

- Each proton beam is made of bunches of protons. Up to 2808 bunches with 10<sup>11</sup> protons each. Energy of each proton 7000 GeV.
- Note: One GeV = about  $1.6 \times 10^{-10}$  Joules and 1 J = 1 Watt second.
- Energy per beam =  $2808 \times 10^{11} \times 7000 \text{ GeV}$ =  $2808 \times 10^{11} \times 7000 \times 1.6 \times 10^{-10} \text{J}$  = 362 MJ
- Equivalent to 87 Kilograms of TNT, which has about  $10^{16}$  times as many protons as the LHC beam!
- Equivalent to the Kinetic Energy of a small aircraft carrier moving at 40 km per hour!
- The energy in the magnets is about 10<sup>10</sup> Joules. (A typical house in Europe consumes about 2000 Joules per second on average).



## **A Particle Detector**



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## **A Particle Detector**





## **Particle Detection in ATLAS**





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## **Processes and Production Rates at the LHC**





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## **Producing Higgs Bosons with the LHC**



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## Many ideas to be explored.....

- Theoretical physicists have come up with many ideas for physics beyond the Standard Model
- These predict new particles and phenomena for the LHC:
- Supersymmetry predicts a new particle for every Standard Model particle eg a super-electron (or selectron).
- The LHC may discover extra dimensions. These are predicted by string theory and could lead to the discovery of tiny black holes!
- So far, these ideas have not been discovered at the LHC, so limits on such models have been obtained!



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# The Standard Model of Particle Physics





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## The Standard Model:

- is a mathematical model for describing the behaviour of the elementary particles: quarks, leptons, gauge bosons and the Higgs boson
- is the most precisely tested scientific model every constructed it has passed thousands of tests and most of its predictions have now been verified
- is based on **symmetry principles**: gauge invariance and Lorentz invariance (special relativity)
- is based on wave equations: Dirac, Maxwell, Yang-Mills, Klein-Gordon which all generalise the Schrodinger equation
- each of the particles obeys one of these equations and particle interactions are "potential terms" in the equations



 All interactions are determined from the symmetries except their strengths (coupling constants) and masses which are determined from experimental data e.g. α the fine structure constant or m<sub>e</sub> th mass of the electron.



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## In these lectures we will.....

- Show how the basic symmetry principles determine all the properties of elementary particles
- Derive some predictions from the Standard Model and test them against experimental data
- This should serve as a good introduction to the rest of the school





## **Disclaimer!**

- This is NOT a complete course on the Standard Model!
- that would require all of the three weeks of the school and more
- This is just an introduction. You can consult my lecture notes for a more detailed introduction and can read the books suggested there for an even more detailed understanding.





## **Natural Units and Dimensional Analysis**

- Natural units are so-called because they intuitively express natural physical quantities in terms of mass scales or length scales.
- Natural units are defined by viewing the fundamaental constants  $\hbar$  and c as conversion factors
- $\hbar = 1$  converts 1 Joule into about  $10^{34}$  seconds<sup>-1</sup>. Equivalantly 1 eV =  $1.6 \times 10^{15}$ s<sup>-1</sup>
- c = 1 converts 1 second into approximately  $3 \times 10^8$  metres.
- In natural units  $[E] = [M] = [L]^{-1} = [T]^{-1}$ .
- With these units **any** quantity can be converted into (some power of) eV. This provides it with a mass scale.
- Excercise: Convert Newton's constant  $G_N$  into  $eV^{-2}$ .



## **Natural Units and Dimensional Analysis**

$$\hbar = 1 \sim 10^{-34} \text{J s} \sim 3 \times 10^{-26} \text{J m} \sim 3 \times 10^{-26} \frac{10^{19}}{1.6..} \text{eV m} \sim 2 \times 10^{-7} \text{eV m} (1)$$

$$1 \text{ m} \sim \frac{1}{2 \times 10^{-7} \text{ eV}}$$
(2)

which means that a distance of one metre is equivalent to an energy scale of  $2 \times 10^{-7}$  eV. Larger distances correspond to smaller energies and vice-versa. In natural units, the mass of the electron is about 0.5 MeV. This is the inverse of a distance of  $\frac{1}{0.5 \text{ MeV}} \sim 4 \times 10^{-13}$  m This is the Compton wavelength of the electron. The mass of proton is about 0.94 GeV, corresponding to a length scale of order  $2 \times 10^{-16}$  m. This is the characteristic size of a nucleus. The masses of the *W*-bosons, the *Z*-boson, the top quark and the Higgs boson are of order 100 GeV – corresponding to a distance of around  $2 \times 10^{-18}$  m. The LHC collisions in 2012 took place at energies of 8 TeV = 8 thousand GeV – a K distance scale of almost  $10^{-20}$  m. This makes the LHC the world's most

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## **Particle Interactions and Gauge Symmetries**

- The SM describes three forces: the strong nuclear force, the weak nuclear force and the electromagnetic force
- Associated with each of these is a symmetry called a gauge symmetry
- In electromagnetism, this symmetry leads to the interaction between photons and charged particles like electrons.
- In the strong nuclear force this leads to the interaction between gluons and particles "charged" under the strong gauge symmetry – the quarks.
- In the weak nuclear force this leads to the interactions between W and Z-bosons and particles charged under the weak gauge symmetry.



## Wave equations – a reminder

- Schrodingers equation derives from  $E = \frac{p_j^2}{2m}$  by replacing  $E \to i\partial_t$  and  $p_j \to -i\partial_j$
- The same exercise with  $E^2 = p_j^2 + m^2$  gives the Klein-Gordon equn:  $\partial_\mu \partial^\mu \Psi m^2 \Psi = 0$ .
- The K-G equation describes relativistic spin-0 particles.
- Dirac derived a relativistic equation which is first order in ∂<sub>t</sub> and ∂<sub>i</sub>.
- Dirac's ansatz:  $i\partial_t \Psi = (-i\alpha_j\partial_j + \beta m)\Psi$  with  $\alpha_i$  and  $\beta$  operators ie matrices
- $\alpha_i^2 = \beta^2 = 1$  and  $\{\alpha_i, \alpha_j\} = 0 = \{\alpha_i, \beta\}$  ensures  $H^2 \Psi = (p_i^2 + m^2) \Psi$



<sup>•</sup> Dirac equation in covariant form:  $(i\gamma^{\mu}\partial_{\mu}+m)\Psi=0$  with  $(\gamma_{0},\gamma_{i})=(\beta,\alpha_{i}\beta)$ 



In classical mechanics one considers generalised coordinates  $q_i(t)$  of a particle. Then the Lagrangian

$$L = T - V \tag{3}$$

which is the difference between Kinetic and Potential energy leads to the Euler-Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}_i}\right) - \frac{dL}{dq_i} = 0 \tag{4}$$

We can use this formalism to obtain the relativistic wave equations such as the Klein-Gordon equation, the Maxwell equations and the Dirac equation.

Instead of considering L to be a function of discrete coordinates  $q_i$ , we consider Lagrangians which are functions of the **fields** which are continuous functions of both  $x_i$  and t i.e. of  $x_{\mu}$ .



For example, for the Klein-Gordon equation L is a function of  $\phi(x_{\mu})$  as well as the derivatives  $\frac{\partial \phi}{\partial x_{\mu}} \equiv \partial_{\mu} \phi$ :

$$L(q_i, \dot{q}_i, t) \to L(\phi, \partial_\mu \phi, x_\mu)$$
 (5)

L is obtained from a Lagrangian density  $\mathcal L$  integrated over space

$$L = \int d^3x \mathcal{L}(\phi, \partial_\mu \phi) \tag{6}$$

Integrating over time gives the action, usually called S:

$$S = \int dt L = \int d^4 x \mathcal{L} \tag{7}$$

By varying S wrt  $\phi$  and  $\partial_{\mu}\phi$  and  $\partial_{\mu}\phi^*$  we obtain the Euler-Lagrange equations (this is derived at the end of the notes in the section on Noethers theorem):



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$$\partial_{\mu} \left( \frac{\delta \mathcal{L}}{\delta(\partial_{\mu} \phi)} \right) - \frac{\delta \mathcal{L}}{\delta \phi} = 0 \tag{8}$$

The Lagrangian density for the KG equation is

$$\mathcal{L} = \partial_{\mu}\phi^*\partial^{\mu}\phi - m^2\phi^*\phi \tag{9}$$

Substituting this into the Euler-Lagrange equations gives

$$\partial_{\mu}\partial^{\mu}\phi + m^{2}\phi = 0 \tag{10}$$

Note:

т

$$\left(\frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\phi)}\right) = \partial^{\mu}\phi^{*} \tag{11}$$

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$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$



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Here we consider  $\mathcal{L}$  as a function(al) of fields  $A_{\mu}$  and derivatives  $\partial_{\mu}A_{\nu}$ . ie since  $A_{\mu}$  has four components, we treat each component as a separate field.

In the presence of a current  $j_{\mu}$  there is an additional interaction term

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^{\mu} A_{\mu}$$
(13)



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Consider a small transformation in a field which leaves the Lagrangian invariant

$$\Psi \to \Psi + i\alpha\Psi \tag{14}$$

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$$0 = \delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \Psi} \delta \Psi + \left(\frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\Psi)}\right) \delta(\partial_{\mu}\Psi) + c.c.$$
(15)

so

$$0 = i\alpha\Psi\frac{\delta\mathcal{L}}{\delta\Psi} + i\alpha\left(\frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\Psi)}\right)(\partial_{\mu}\Psi) + ..$$
(16)  
$$= i\alpha\left[\frac{\delta\mathcal{L}}{\delta\Psi} - \partial_{\mu}\left(\frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\Psi)}\right)\right]\Psi + i\alpha\partial_{\mu}\left(\frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\Psi)}\Psi\right) + ...$$

where, to get to the last line from the previous one we use that:

$$\partial_{\mu} \left( \frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\Psi)} \Psi \right) = \left( \partial_{\mu} \frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\Psi)} \right) \Psi + \left( \frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\Psi)} \right) \partial_{\mu}\Psi$$
(17)  
Going back to the previous expression, the equation before the one above, there is are several key points:

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- The last term in (14) is proportional to a total derivative. Hence, it only contributes to the action at the boundary of space-time ie at infinity. Requiring this term to vanish at infinity implies that: the action is extremised ( $\delta S = 0$ ) exactly when the Euler-Lagrange equations are satisfied (the term in square brackets). We have thus derived the Euler-Lagrange equations.
- In the case that we require that  $\Psi \rightarrow \Psi + i\alpha\Psi$  is a **symmetry** of the action, then, because the terms in square brackets vanish due to the Euler-Lagrange equations, the total derivative term must vanish **everywhere** not just at infinity. Hence,
- When a variation of the fields is a SYMMETRY of the action, there exists a conserved quantity:  $j^{\mu} \equiv \left(\frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\Psi)}\Psi\right)$  which obeys:





Recall that to consider the motion of a particle of charge -e in an electromagnetic field generated by a vector potential  $A^{\mu}$  we replace the derivative  $\partial_{\mu}$  by

$$\partial_{\mu} \to \partial_{\mu} - ieA_{\mu}$$
 (19)

We call the rhs of this expression a covariant derivative. This is usually denoted by  $D_{\mu}$ :

$$D_{\mu} \equiv \partial_{\mu} - ieA_{\mu} \tag{20}$$

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By making the above replacement in the Dirac equation equation we obtain

$$i\gamma^{\mu}\partial_{\mu}\Psi + m\Psi = -e\gamma^{\mu}A_{\mu}\Psi \tag{21}$$

i.e. we get a "potential" for the field  $\Psi$ . Since the modified equation of motion was obtained by replacing  $\partial_{\mu}$  with  $D_{\mu}$ , the Lagrangian density for a charged fermion with mass m is

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}D_{\mu}\Psi + m\bar{\Psi}\Psi$$

$$= i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + m\bar{\Psi}\Psi + e\bar{\Psi}\gamma^{\mu}A_{\mu}\Psi$$
(22)

This is exactly:

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + m\bar{\Psi}\Psi + j^{\mu}A_{\mu}$$

The fact that  $j^{\mu}A_{\mu}$  appears in  $\mathcal{L}$  suggests that we can just "read off" the vertices allowed in Feynman diagrams from  $\mathcal{L}$ . This is a general rule for any Lagrangian! We just read off the Feynman rules from  $\mathcal{L}$ . In this example, the three point vertex between the photon and two charged particles is represented in  $\mathcal{L}$  by the presence of the  $j^{\mu}A_{\mu}$  term.





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The Lagrangian for Quantum Electrodynamics, QED has various symmetries.

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\gamma^{\mu}D_{\mu}\Psi + m\bar{\Psi}\Psi$$
(24)

Lorentz Invariance. Since all the Lorentz indices are contracted ( $\mathcal{L}$  is a scalar), the Lagrangian is invariant under Lorentz transformations.

Internal Symmetry. In addition to this "spacetime symmetry" it is invariant under an internal symmetry i.e. one which does not act on the coordinates, but just on the fields. This is intrinsic to electromagnetism and the other forces as we will see.





## **Gauge Symmetries**

We are going to consider a transformation of  $\phi$  by a **unitary**, **1-by-1** matrix, a U(1) transformation.

Any such matrix U is of the form  $U(\alpha)=e^{i\alpha}.$   $\alpha$  can take any continuous value between zero and  $2\pi.$ 

Clearly, under

$$\Psi \to U\Psi$$
 (25)

we have that

$$D_{\mu}\Psi \to UD_{\mu}\Psi$$
 (26)

L is clearly invariant under this transfomation since it is of the form  $D_{\mu}\Psi$  times the complex conjugate of  $\Psi$ , and  $UU^* = 1$ . Similarly  $\bar{\Psi}\Psi$  is invariant, so the mass term is invariant. Therefore  $\mathcal{L}$  is invariant under this transformation of  $\Psi$ . Noethers theorem proves that there is a conserved quantity when a Lagrangian is invariant under a symmetry transformation. In fact, when  $\alpha$  is small i.e. when  $U \approx 1 + i\alpha$  we see that the conserved current  $j^{\mu}$  is precisely that which we derived before.

## **Gauge Symmetry**

Now, we would like to consider when  $\alpha$  is different from point to point in spacetime. i.e. we make  $\alpha = \alpha(x_{\nu})$  – a function of the coordinates. Now, we will get terms proportional to derivatives of  $\alpha$ .

$$D_{\mu}\Psi \rightarrow U\partial_{\mu}\Psi + iU\partial_{\mu}\alpha\Psi - ieUA_{\mu}\Psi$$
 (27)

$$= UD_{\mu}\Psi + iU\partial_{\mu}\alpha\Psi \tag{28}$$

Because of this, the Lagrangian is no longer invariant. However, the unwanted term in the transformation of  $D_{\mu}\Psi$  can be removed if  $A_{\mu}$  also transforms:

$$A_{\mu} \to A_{\mu} + \frac{1}{e} \partial_{\mu} \alpha \tag{29}$$

which can be verified by replacing this transformed  $A_{\mu}$  in the "unwanted" term. Thus:

$$D_{\mu}\Psi \to e^{i\alpha(x)}D_{\mu}\Psi$$
 (30)

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and the Dirac terms Lagrangian are invariant under this **gauge transformation** (this is the name for transformations whose parameters depend on the coordinates. What about the Maxwell term in the Lagrangian?

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## Electromagnetism and U(1) Gauge Symmetry

- Maxwells equations can be written in terms of a gauge potential  $A_{\mu}$ .  $A_0$  = scalar potential and  $A_i$  = the vector potential
- The electric field has components  $E_i = \partial_0 A_i \partial_i A_0$
- The magnetic field has components  $B_1 = \partial_2 A_3 \partial_3 A_2$  plus  $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$
- They are unified into the EM field strength  $F_{\mu\nu} = \partial_{\mu}A_{\nu} \partial_{\nu}A_{\mu}$
- Maxwells equations in vacuum are then:
- $\partial_{\rho}F_{\mu\nu} + \partial_{\mu}F_{\nu\rho} + \partial_{\nu}F_{\rho\mu} = 0 \rightarrow \dot{\mathbf{B}} + \nabla \times \mathbf{E} = 0$

• 
$$\partial^{\mu}F_{\mu\nu} = 0 \rightarrow \dot{\mathbf{E}} - \nabla \times \mathbf{B} = 0$$

$$\partial^{\mu}F_{\mu\nu} = 0 \to \nabla \cdot \mathbf{E} = 0$$





## Electromagnetism and U(1) Gauge Symmetry

- Maxwells equations posess a large, infinite dimensional symmetry.
- Consider an **arbitrary** function  $\lambda(x, y, z, t)$ . (So  $\lambda = \frac{1}{e}\alpha$ )
- Then replace  $A_{\mu} \longrightarrow A'_{\mu} = A_{\mu} + \partial_{\mu}\lambda$

• 
$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \rightarrow \partial_{\mu}A_{\nu} + \partial_{\mu}\partial_{\nu}\lambda - \partial_{\nu}A_{\mu} - \partial_{\nu}\partial_{\mu}\lambda$$

- Thus, since  $\partial_{\mu}\partial_{\nu}\lambda = \partial_{\nu}\partial_{\mu}\lambda$ ,  $F'_{\mu\nu} = F_{\mu\nu} = \partial_{\mu}A_{\nu} \partial_{\nu}A_{\mu}$
- So, the gauge potentials  $A_{\mu}$  and  $A'_{\mu}$  describe **identical** E and B fields!
- These are the equations required by all electronic devices and technology and they have an infinite dimensional symmetry!
- So QED is invariant under U(1) gauge transformations



## Gauge Symmetry as a Principle

If we use gauge symmetry as a principle then it has far reaching consequences.

- The covariant derivative **must** be introduced otherwise the kinetic energy term would not be gauge invariant.
- This requires the introduction of a vector field A<sub>μ</sub> which couples to the matter current. A<sub>μ</sub> is usually called the gauge field.
- If we consider the kinetic energy of the gauge field, then gauge invariance requires it to be of the form  $F_{\mu\nu}F^{\mu\nu}$  (or more generally a function thereof). Thus, Maxwell's equations follow from gauge symmetry plus Lorentz symmetry
- The photon is massless





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## Gauge Symmetry as a Principle

This last point is crucial. It provides an **explanation** for why the photon essentially behaves as a massless particle. (Experimentally of course one cannot prove that the photon is exactly massless. Rather, one obtains an upper limit on its mass. The current upper limit is about  $10^{-18}$  eV.) To see why the photon is massless, we ask: what would a mass term look like?:

$$\partial_{\nu}\partial^{\nu}A_{\mu} + m^2 A_{\mu} = 0 \tag{31}$$

This mass term would arise from a term in the Lagrangian of the form

$$\Delta \mathcal{L} \sim -m^2 A_\mu A^\mu \tag{32}$$

Such a term is clearly not invariant under the gauge transformation of  $A_{\mu}$ , which is



$$A_{\mu} \to A_{\mu} + \frac{1}{e} \partial_{\mu} \alpha$$
 (33)

In fact, combining all of these points, the most general gauge and Lorentz invariant Lagrangian which is up to quadratic in the fields and their derivatives is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi} \gamma^{\mu} D_{\mu} \Psi + m \bar{\Psi} \Psi$$
(34)

Here we used the fact that  $[A_\mu]=[\partial_\mu]=[M]$  and  $[\Psi]=[M]^{3/2}$  so that  $[\mathcal{L}]=[M]^4$  (energy density)

**Thus: the symmetries determine the Lagrangian and, hence, the physics** This is a key point in particle physics. The Lagrangian for the Standard Model is essentially determined by its symmetries. In other words, symmetries determine the physics of all elementary particles!





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We would now like to consider generalising U(1) gauge theory (i.e. QED) to U(N) gauge theory. That is to say that U – the transformation matrix – will become a Unitary  $N \times N$  matrix  $U_j^i$  where i, j run from 1 to N each. An  $N \times N$  matrix acts naturally on N component vectors  $v_i$ . Hence we should introduce N complex fields  $\Psi_i$  on which these act. Under

$$\Psi_i \to U_i^j \Psi_j \tag{35}$$

we would like to impose a condition that a suitable covariant derivative transforms acting on  $\Psi_i$  transforms in the same way  $\rightarrow U^{\dagger}U = \mathbf{1}_N$  (Identity matrix) then would guarantee that

$$\bar{\Psi}^{j}\gamma^{\mu}D_{\mu}\Psi_{j}\longrightarrow \bar{\Psi}^{k}U^{\dagger}{}_{k}^{m}U^{n}_{m}\gamma^{\mu}D_{\mu}\Psi = \bar{\Psi}^{k}\gamma^{\mu}D_{\mu}\Psi_{k}$$
(36)

This would be the  $N \times N$  generalisation of the U(1) case.



#### But what is this covariant derivative?

If we try to introduce a gauge field, in general it is a matrix of gauge fields i.e. we can have up to  $N\times N$  gauge fields:

$$(D_{\mu}\phi)_{i} = \partial_{\mu}\phi_{i} - ig(G_{\mu})_{i}^{j}\phi_{j}$$
(37)

that is, for each of the values of i and j,  $(G_{\mu})_{j}^{i}$  is a different gauge field. So if U(N) gauge symmetry is a principle it requires N matter particles  $\Psi_{j}$  and  $N^{2}$  gauge bosons  $G_{\mu})_{i}^{j}$ 

How is this matrix of gauge fields defined?





## Unitary matrices, exponentials, group generators and all that

$$\exp i\theta = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots = \cos\theta + i\sin\theta$$
(38)

Now consider the rotation matrix

$$R(\theta) \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$
(39)

We want to Taylor expand this matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots & \theta - \frac{\theta^3}{3!} + \dots \\ -\theta + \frac{\theta^3}{3!} + \dots & 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots \end{pmatrix}$$
(40)

Just as  $e^{i\theta}$  is the exponential of a 1-by-1 matrix, the rotation matrix above is the exponential of a two-by-two constant matrix:

$$\mathbf{R}(\theta) = \exp(i\theta T) = \mathbf{1} + i\theta T - \frac{\theta^2}{2}T^2 - i\frac{\theta^3}{3!}T^3 + ... \rightarrow T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (41)$$
and  $T^2$  is the matrix product of  $T$  with itself.  
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So, since any rotation can be written as  $\exp i\theta T$  we say that T generates the rotations. Now consider some other examples of this, because we will need matrices like T to define the covariant derivative properly. Let U be a N-by-N unitary matrix ie

$$U^{\dagger}.U = 1 \tag{42}$$

Now assume that

$$U = \exp iM \tag{43}$$

What properties does M have? unitarity implies that

 $M = M^{\dagger}$  so M is Hermitian. (44)

If, additionally, we require that  $\det(U) = 1$  i.e. that U is special unitary, then one can show that the trace of M is zero



$$\det U = 1 \leftrightarrow \mathrm{Tr}M = 0 \tag{45}$$

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When N = 2, one can show that M is a linear combination of the Pauli matrices:

$$M = \alpha^a \sigma_a \tag{46}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
(47)

That is to say that the Pauli matrices generate SU(2) transformation matrices! An important fact about the Pauli matrices is that they obey an algebra:

$$[\sigma_a, \sigma_b] = 2i\epsilon_{ab}{}^c \sigma_c \tag{48}$$

 $\epsilon_{abc}$  is totally antisymmetric i.e. if we interchange neighbouring indices then we get a minus sign:

$$\epsilon_{abc} = -\epsilon_{bac} = \epsilon_{bca} = -\epsilon_{cba} \ etc \tag{49}$$



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and  $\epsilon_{123} = 1$ 



## SU(N) gauge symmetry

In general, for an SU(N) matrix

$$U = \exp iM \tag{50}$$

where M is traceless and Hermitian, there are  $N^2 - 1$  generators  $T_a$  such that

$$M = \alpha^a T_a \tag{51}$$

and the  $T_a$  's obey an algebra

$$[T_a, T_b] = i f_{ab}{}^c T_c \tag{52}$$

where the  $f_{ab}{}^c$  are constants called the structure constants. For SU(N) the algebra defined by the above equation is called the Lie Algebra of SU(N). If you choose a basis for this algebra, you can explicitly calculate the structure constants in that basis. They are totally antisymmetric, like  $\epsilon_{abc}$ . For SU(3) there is a basis for the eight 3-by-3  $T_a$  matrices which is used a lot in particle physics called the Gell-Mann basis. You can find these eight matrices in the textbooks.



## SU(N) gauge symmetry

Finally(!)

$$D_{\mu} = \partial_{\mu} + igT_a G^a_{\mu} \tag{53}$$

where the second term is an N-by-N matrix since  $T_a$  is a matrix. There are  $N^2 - 1$  gauge fields  $G^a_{\mu}$ .

The Standard Model is a gauge theory with SU(3) gauge symmetry, SU(2) gauge symmetry and U(1) gauge symmetry. There are 8 + 3 + 1 = 12 gauge bosons. These are the eight gluons, the two W-bosons ( $W^+$  and  $W^-$ ), the neutral Z boson and the photon.

The full covariant derivative is thus

$$D_{\mu} = \partial_{\mu} - i \frac{Y}{2} g_1 B_{\mu} - i g_2 \frac{\sigma_j}{2} W^j_{\mu} - i g_3 \frac{\lambda_a}{2} G^a_{\mu}$$
(54)

 $B_{\mu}$  is the gauge boson of the U(1). The photon is a linear combination of  $B_{\mu}$  and  $W^3_{\mu}$ . The Z-boson is the opposite linear combination. Y is called the hypercharge. The charge under SU(2) is called isospin I. The proton has isospin 1/2 and the neutron -1/2.



## SU(N) gauge symmetry

## Most of the physics of all non-dark matter comes from this covariant derivative!

Putting this covariant derivative into the Dirac Lagrangian gives

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}D_{\mu}\Psi = i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + j_{1}^{\mu}B_{\mu} + j_{i}^{\mu}W_{\mu}^{i} + j_{a}^{\mu}G_{\mu}^{a}$$
(55)

where we have three currents, one for each symmetry group.

So we have three interaction terms which lead to the Feynman diagrams for the quarks and leptons interacting with the gauge bosons





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## **Standard Model Interactions**



By putting these together we can describe **any physical process** involving Standard Model particles!





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## Calculating the decay properties of W-bosons

We need the formula

$$-ig_2\frac{\sigma_j}{2}W^j_{\mu} = -i\frac{g_2}{2} \begin{pmatrix} W^3_{\mu} & W^1_{\mu} - iW^2_{\mu} \\ W^1_{\mu} + iW^2_{\mu} & -W^3_{\mu} \end{pmatrix}$$
(56)

which we re-write as:

$$-i\frac{g_2}{2} \begin{pmatrix} W^3_{\mu} & W^1_{\mu} - iW^2_{\mu} \\ W^1_{\mu} + iW^2_{\mu} & -W^3_{\mu} \end{pmatrix} = -i\frac{g_2}{2} \begin{pmatrix} W^3_{\mu} & \sqrt{2}W^+_{\mu} \\ \sqrt{2}W^-_{\mu} & -W^3_{\mu} \end{pmatrix}$$
(57)

we have introduced  $W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(W^1_{\mu} \mp i W^2_{\mu})$  which are the actual positive and negative charge eigenstates. All the left-handed fermions pair up as doublets under the SU(2) gauge symmetry i.e.

$$e^{\nu_{e}}_{e} \right)_{L}, \left( \begin{array}{c} \nu_{\mu} \\ \mu^{-} \end{array} \right)_{L}, \left( \begin{array}{c} \nu_{\tau} \\ \tau^{-} \end{array} \right)_{L}, \left( \begin{array}{c} u \\ d \end{array} \right)_{L}, \left( \begin{array}{c} c \\ s \end{array} \right)_{L}, \left( \begin{array}{c} t \\ b \end{array} \right)_{L}$$
(58)



## **SM Interactions**

Remember that each of the entries of these doublets are 4-component Dirac spinors. We have suppressed the spinor labels in the last equation. Similarly the Lagrangian depends on the "bars" of all of these multiplets and hence the couplings to the W-bosons also involves the right-handed anti-particles to all of the above e.g. right handed-positrons also couple to W's, but left-handed positrons do not. Let's write the SU(2) gauge interactions for the muon doublet.





## Muon and Muon-neutrino Interactions with $\boldsymbol{W}$ bosons

It is

$$\left(\bar{\nu}_{\mu} \ \bar{\mu}^{-}\right)\gamma^{\nu} \times g_{2} \left(\begin{array}{cc} W_{\nu}^{3} & \sqrt{2}W_{\nu}^{+} \\ \sqrt{2}W_{\nu}^{-} & -W_{\nu}^{3} \end{array}\right) \frac{1}{2}(\mathbf{1}-\gamma^{5}) \times \left(\begin{array}{c} \nu_{\mu} \\ \mu^{-} \end{array}\right)$$
(59)

The "i"'s have multiplied to 1. The ×'s are there to remind us we have to multiply the matrices together. The  $\frac{1}{2}(1-\gamma^5)$  is there to project onto the left-handed component of the fermion doublet to the right of the matrix of W-bosons. The expression above is equivalent to

$$\sqrt{2}\frac{g_2}{2}\bar{\mu}_L^-\gamma^\nu\nu_{\mu L}W_\nu^- + \sqrt{2}\frac{g_2}{2}\bar{\nu}_{\mu L}\gamma^\nu\mu_L^-W_\nu^+ \qquad (60)$$

$$-\frac{g_2}{2}\bar{\mu}_L^-\gamma^\nu\mu_L^-W_\nu^3 + \frac{g_2}{2}\bar{\nu}_{\mu L}\gamma^\nu\nu_{\mu L}W_\nu^3$$

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The first line above is the charged current interaction. The second is a neutral current interaction. If we were to change any of the four coefficients, the Lagrangian would no longer be SU(2) gauge invariant. The relative strengths  $\underline{I}$  of these interactions  $\sqrt{2}:1$  is thus fixed by symmetry.



## **SM Interactions**

For all of the six fermion doublets, there is a similar expression for the SU(2) interaction terms in the Lagrangian. The interactions we have derived lead to some remarkable consequences:

- The couplings of  $W^+$  and  $W^-$  are universal for both quarks and leptons. e.g. the interaction between  $W^+$ , u and  $\bar{d}$  is the *same* as the interaction between  $W^+$ ,  $\mu$  and  $\bar{\nu}$ .
- $W^+$  can decay into  $(e^+, \bar{\nu}_e), (\mu^+, \bar{\nu}_\mu), (\tau^+, \bar{\nu}_\tau), (u, \bar{d}), (c, \bar{s})$
- Similarly for W<sup>-</sup>.
- W bosons cannot decay into  $(t, \bar{b})$  because  $m_t \sim 173 \text{GeV} \pm 1 \text{GeV}$  and  $m_W \sim 80.4 \text{GeV}$ .
- Since  $m_W$  is much larger than  $m_e, m_\mu, m_\tau, m_u, m_d, m_c, m_s$  the decay width of the  $W^{\pm}$  doesn't "care" about the fermion masses
- So the partial decay widths

 $\Gamma(W^+ \to e^+ \bar{\nu}) = \Gamma(W^+ \to \mu^+ \bar{\nu})$   $= \Gamma(W^+ \to \tau^+ \bar{\nu})$ (61)
(62)
(62)

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• The decays into quarks are not just two decay channels: (ud) and  $(c, \bar{s})$ ,

- The decays into quarks are not just two decay channels:  $(u\bar{d})$  and  $(c,\bar{s})$ , but three each, since there are three u quarks, three d quarks, three c quarks and three s quarks. This is because the quarks transform under the SU(3) gauge symmetry (the leptons do not). So the decays into quarks are actually six channels.
- This gives a total of three leptonic and six hadronic decay channels, nine in total.
- If  $\Gamma(W^+ \to all)$  is the total W decay width, the Standard Model predicts that

$$\frac{\Gamma(W^+ \to e^+ \bar{\nu})}{\Gamma(W^+ \to all)} = \frac{\Gamma(W^+ \to \mu^+ \bar{\nu})}{\Gamma(W^+ \to all)}$$
(67)

$$= \frac{\Gamma(W^+ \to \tau^+ \bar{\nu})}{\Gamma(W^+ \to all)}$$
(68)  
= 1/9 (69)

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$$\frac{\Gamma(W^+ \to hadrons)}{\Gamma(W^+ \to all)} = 6/9 = 2/3$$

## *W*-boson decays – what the data says

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#### W<sup>+</sup> DECAY MODES

 $W^-$  modes are charge conjugates of the modes below.

	Mode	Fraction $(\Gamma_i/\Gamma)$	Confide	ence level
Г1	$\ell^+ \nu$	[a] (10.80± 0.09)	%	
Γ2	$e^+\nu$	(10.75± 0.13)	%	
Гз	$\mu^+ \nu$	(10.57± 0.15)	%	
Γ4	$\tau^+ \nu$	(11.25± 0.20)	%	
Γ <sub>5</sub>	hadrons	(67.60± 0.27)	%	
Γ <sub>6</sub>	$\pi^+\gamma$	< 8	$\times 10^{-5}$	95%
Γ7	$D_s^+ \gamma$	< 1.3	$\times 10^{-3}$	95%
Г8	сX	$(33.4 \pm 2.6)$	%	
Γ9	cs	$(31  +13 \\ -11  )$	%	
Γ <sub>10</sub>	invisible	$[b]$ ( 1.4 $\pm$ 2.9 )	%	

- [a]  $\ell$  indicates each type of lepton (e,  $\mu$ , and  $\tau$ ), not sum over them.
- [b] This represents the width for the decay of the W boson into a charged particle with momentum below detectability, p < 200 MeV.



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- These branching fractions have been measured and agree with the Standard Model predictions to within 1%. Have a look at the PDG.
- This excellent agreement between theory and experiment represents a thorough check of the structure of the Standard Model. The interactions are derived entirely from symmetry principles. If we changed the number of leptons the result would change. If there were four u quarks instead of three (which would be the case if SU(3) were replaced with SU(4)) the result would change.
- Note: gauge symmetry, Lorentz invariance and charge conservation allows the possibility of **flavour-changing decays** that involve different quark families e.g.  $W^+ \rightarrow c\bar{b}$ . In fact these decays also occur, but are suppressed by so-called CKM-mixing (Cabibbo-Kobayashi-Maskawa). We will not discuss this in any detail.





## Notation

- We use natural units:  $\hbar = c = 1$
- $V^{\mu}$  is a 4-component vector with components  $V_0$  and  $V_x, V_y, V_z \equiv V_i$ .
- $V_{\mu}$  is a 4-component co-vector with components  $V_0$  and  $-V_x, -V_y, -V_z.$

• 
$$\partial_t \equiv \frac{\partial}{\partial t}$$
.  $\partial_i \equiv \nabla \equiv \frac{\partial}{\partial x_i}$ 

• 
$$V^{\nu}V_{\nu} \equiv V_0^2 - V_x^2 - V_y^2 - V_z^2 \equiv V_0^2 - V_i^2 \equiv V_0^2 - \mathbf{V}^2$$

• 
$$\partial_{\mu} \equiv (\partial_t, \nabla)$$
 and  $\partial^{\mu} = (\partial_t, -\nabla)$ 

• 4-momentum can be represented as  $p_{\mu} \to i \partial_{\mu}$  and  $p^{\mu} \to i (\partial_t, -\nabla)$ 



