SUPPLEMENTAL MATERIAL

We show how one can get the estimates (5) and (6) for the mobility edge by analyzing the perturbative wavefunctions in the forward approximation.

At finite energy density $\epsilon = E_a/N$ resonances, i.e. values of the energy denominators $\delta_n = E_a - E_i$ particularly small, are quite rare. One has to go a distance of $O(N)$ to find such a resonance. Let us suppose the first resonance occurs at distance n. Prior to the resonance, the paths sum coherently with each contributing a typical value, so that we estimate

$$\psi_{n-1} \simeq (n-1)! \left(\frac{\Gamma}{\epsilon N}\right)^{n-1} \tag{1}$$

for all amplitudes at distance $n-1$. The are n ways to reach a site at distance n from sites at distance $n-1$. Hence,

$$\psi_n \;=\; \frac{\Gamma}{\delta_n} n \psi_{n-1} \tag{2}$$

We have a resonance when $|\psi_n| > 1$, namely if

$$|\delta_n| < \Gamma n |\psi_{n-1}| \tag{3}$$

Therefore the (small) probability to have a resonance is

$$\begin{aligned} p \;&=\; \int_{\epsilon - \Gamma n|\psi_{n-1}|}^{\epsilon + \Gamma n|\psi_{n-1}|} d\epsilon \sqrt{\frac{N}{\pi}} e^{-N\epsilon^2} \\ &\simeq\; 2\Gamma n |\psi_{n-1}| \rho(\epsilon) \end{aligned} \tag{4}$$

with $\rho = \sqrt{\frac{N}{\pi}} e^{-\epsilon^2 N}$ comes from the distribution of levels. Define P_n as the probability that none of the $\binom{N}{n}$ points at level n gives a resonance. In order to proceed we need to assume that these events are uncorrelated. This is an approximation which gives a lower bound to the probability P_n and we will see how good this is compared to the numerical data. In this approximation,

$$P_n = (1-p)^{\binom{N}{n}} \tag{5}$$

which, inserting (4) gives

$$\begin{aligned} P_n \;&=\; \left(1 - 2\Gamma n! \rho \left(\frac{\Gamma}{\epsilon N}\right)^{n-1}\right)^{\binom{N}{n}} \\ &\simeq\; e^{-e^{N f(x,\epsilon)}} \end{aligned} \tag{6}$$

where $x \equiv n/N$ and,

$$f = N^{-1} \ln \left(2 \binom{N}{n} \Gamma n! \rho \left(\frac{\Gamma}{\epsilon N}\right)^{n-1} \right). \tag{7}$$

We obtain to leading order in $1/N$,

$$f(x,\epsilon) = -(1-x)\ln(1-x) - \epsilon^2 + x \ln\left(\frac{\Gamma}{e\epsilon}\right). \tag{8}$$

As $N \to \infty$, if $f < 0$ we have $P_n \to 1$, while for $f > 0$ we have $P_n \to 0$. In order to see where the first resonance occurs we need to find the smallest n such that $P_n = 0$, so we have to find f^*, the maximum of the function $f(x,\epsilon)$ over x for any given ϵ. After some algebra we find

$$f^* = \frac{\epsilon}{\Gamma} + \ln\left(\frac{\Gamma}{e\epsilon}\right) - \epsilon^2. \tag{9}$$

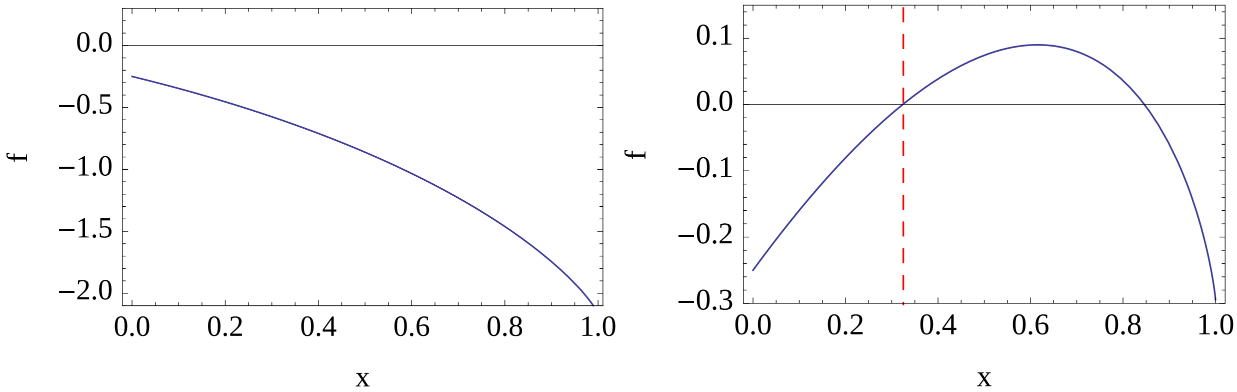

FIG. 1: (color online) The function f in (8) for $\epsilon = 0.5$ and $\Gamma < \Gamma_c$ (left) and $\Gamma > \Gamma_c$ (right). In the right panel the red, dashed line is the position of $x^* = n^*/N$.

Solving $f^* = 0$ for Γ we find an explicit form for $\Gamma_c(\epsilon)$ which can be expanded for small ϵ as

$$\Gamma_c = \epsilon + \sqrt{2}\epsilon^2 + \frac{4}{3}\epsilon^3 + \ldots \, . \tag{10}$$

At constant Γ, varying E_a therefore defines a many-body mobility edge. It is also instructive to see at which value of n the maximum occurs, which gives the most probable position of the first resonance:

$$n^* = N\left(1 - \frac{\epsilon}{\Gamma}\right) \simeq N(\sqrt{2}\epsilon - 2\epsilon^2/3 + \epsilon^3/(9\sqrt{2})\ldots). \tag{11}$$

We see from here that for any finite ϵ the position of the first resonance is at $O(N)$ away. As $\epsilon \to 0$ the first resonance comes quite close to the origin of the locator expansion. If we want to find the finite-N corrections to Γ_c at $\epsilon = 0$, we need to consider this possibility more carefully.

This leads to the discussion of the case at infinite temperature, i.e. where $E_a = 0$. Let us define the variable $y_i = -\ln(|E_i|/\sigma)$ for some σ which we will fix shortly. $y_i \to \infty$ at a resonance $E_i = E_a = 0$.

We find

$$P(y_i) = \frac{2\sigma}{\sqrt{\pi N}} e^{-y_i - \frac{\sigma^2}{N}\exp(-2y_i)}. \tag{12}$$

We choose now

$$\sigma = \frac{\sqrt{\pi N}}{2} \tag{13}$$

so, since we are interested in rare fluctuations where $y_i \gg 1$, we have that

$$P(y_i) \simeq e^{-y_i} \quad \text{for } y_i \gtrsim 1. \tag{14}$$

We need to study the distribution of the amplitudes

$$A_p = \prod_{i=1}^{n} \frac{\Gamma}{0 - E_i}, \tag{15}$$

over all the paths p which go out to distance n. Consider all the $\mathcal{N} \equiv \prod_{i=0}^{n-1}(N-i)$ paths that go out to one of the $\binom{N}{n}$ points. They appear clustered in sums but since the distribution of their contributions is very large this does not matter: $O(1)$ of the paths will dominate both the sum to get to the point b and the total probability of resonance at distance n. To control the latter, we will look for the probability that *none* of these paths gives resonance. We already know that the first path to break this condition will be similar to the greedy path but performing the calculation will give an extra $\ln N$ correction, typical of Anderson localization problems on large connectivity graphs [1].

Consider the log amplitude A_p of a given path,

$$\ln |A_p| = n \ln(\Gamma/\sigma) + \sum_{i=1}^{n} y_i. \tag{16}$$

We have a resonance if

$$|A_p| > 1, \tag{17}$$

namely if

$$\sum_{i=1}^{n} y_i > n \ln(\sigma/\Gamma) \equiv Y_c. \tag{18}$$

Introducing $Y = \sum_{i=1}^{n} y_i$ one finds that it is distributed as

$$P(Y) = \frac{Y^{n-1}}{(n-1)!} e^{-Y}, \tag{19}$$

and so we have now all the ingredients to find the probability to have a resonance $|A_p| > 1$ at distance n (see also [1–3]).

Since $P(|A_p| > 1) = P\left(Y > Y_c\right)$, where $Y_c = n \ln\left(\frac{\sigma}{\Gamma}\right) \gg 1$ we have

$$\begin{aligned} p \equiv P(Y > Y_c) &= \int_{Y_c}^{\infty} dY \frac{Y^{n-1}}{(n-1)!} e^{-Y} \\ &\simeq \frac{Y_c^{n-1}}{(n-1)!} e^{-Y_c} + O(Y_c^{(n-2)}) \end{aligned} \tag{20}$$

doing the integral by parts. Using Stirling's approximation:

$$\begin{aligned} p &\simeq \frac{Y_c^n e^n}{n^n} e^{-Y_c} \\ &= \exp\left[-n\phi\left(\frac{\sigma}{\Gamma}\right)\right], \end{aligned} \tag{21}$$

where $\phi(x) = \ln(x/(e \ln x)) \geq 0$. Again, assuming that all $\mathcal{N}$ paths resonate independently (an underestimate), the probability that we do not have any resonant paths is

$$(1-p)^{\mathcal{N}} \simeq e^{-\mathcal{N}p}. \tag{22}$$

If $\mathcal{N}p \gg 1$ then the probability that no resonating path exists goes to zero. Defining $f = \ln(\mathcal{N}p)/n$ we have the condition

$$\begin{aligned} f &= \frac{1}{n} \ln \mathcal{N} - \phi(\sigma/\Gamma) \\ &\simeq \ln N - \ln\left(\frac{\sigma}{e\Gamma \ln(\sigma/\Gamma)}\right) = 0, \end{aligned} \tag{23}$$

the condition for the transition gives

$$\frac{\sigma}{e\Gamma_c \ln(\sigma/\Gamma_c)} = N. \tag{24}$$

The numerical solution of this equation for $N = 8, ..., 14$ are reported in the text. We cannot solve this equation for Γ_c exactly but in the large N limit, the solution is

$$\Gamma_c \simeq \frac{\sigma}{eN \ln eN} = \frac{\sqrt{\pi}}{2eN^{1/2} \ln(eN)} + O\left(\frac{1}{N^{1/2} \ln^2 N}\right), \tag{25}$$

as quoted in the main text.

Replica treatment– The statistical properties of the wavefunctions can also be studied using the replica method, which provides complementary but non-rigorous understanding [4, 5]. In this approach, we view the amplitude ψ_b as the partition sum of a directed random polymer (the path) living on the *hypercube* with the long-tailed random weights $w_i = \Gamma/(E_a - E_i)$. Notice that these weights do not have any finite moments so we expect the directed random polymer to condense onto a small number of large weight paths [6]. We focus on the most interesting case of infinite temperature states, where the replica approach is most useful as it naturally regulates the divergence of the weights.

The typical value of the forward scattering wavefunction $f = \overline{\ln|\psi|}$ admits a straightforward replica treatment exploiting the usual relationship $\overline{\ln|\psi|} = \operatorname{Re} \lim_{m\to 0} \frac{\overline{\psi^m}-1}{m}$. In the 1RSB ansatz, the dominant configurations contributing to $\overline{\psi^m}$ consist of m/x tightly bound groups of x paths each. This gives rise to the 1RSB free energy:

$$f(x) = \frac{n}{x}\left(\log n - 1 + \log \overline{w_i^x}\right) \tag{26}$$

where $x \in [0,1]$ is the Parisi parameter and $w_i = \Gamma/(E_a - E_i)$ is the weight on site i. Minimizing over x, we find that the saddle point of the replicated free energy arises at $x^* = 1 - \frac{1}{\log\sqrt{2/\pi n}} + \cdots$ as $n \to \infty$, indicating condensation of the paths.

Solving for the resonance condition $\operatorname{Re} f = 0$ at $n = N$, we find the estimate

$$\Gamma_c = \frac{\sqrt{\pi}}{2\sqrt{N}\log\sqrt{2/\pi N}} + \cdots \tag{27}$$

for the critical value of the transverse field. We note that this estimate is larger by a factor of e than the estimate from the direct probabilistic analysis above, but it has the same scaling with N. This is natural as the resonance condition used here is that the amplitude at the far side of the hypercube should diverge as opposed to a small (but entropic) collection of atypical resonances appearing somewhere in the cube, as estimated above.

[1] R. Abou-Chacra, D. Thouless, and P. Anderson, J. Phys. C **6**, 1734 (1973).

[2] B. L. Altshuler, Y. Gefen, A. Kamenev, and L. S. Levitov, Phys. Rev. Lett. **78**, 2803 (1997).

[3] A. De Luca, A. Scardicchio, V. E. Kravtsov, and B. L. Altshuler, arXiv (2013), 1401.0019.

[4] M. Mézard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond*, Lecture Notes in Physics Series (World Scientific Publishing Company, Incorporated, 1987), ISBN 9789971501150.

[5] M. Mézard and A. Montanari, *Information, Physics, and Computation*, Oxford Graduate Texts (OUP Oxford, 2009), ISBN 9780198570837.

[6] M. Kardar, Les Houches Summer School on Fluctuating Geometries in Statistical Mechanics and Field Theory (1994).