

Structure of typical states of a disordered Richardson model and many-body localization

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We present a thorough numerical study of the Richardson model with quenched disorder (a fully connected XX model with longitudinal random fields). We find that for any value of the interaction the eigenstates occupy an exponential number of sites on the unperturbed Fock space but that single-spin observables do not thermalize, as tested by a microcanonical version of the Edwards-Anderson order parameter. We therefore do not observe many-body localization in this model. We find a relation between the inverse participation ratio and the average Hamming distance between spin configurations covered by a typical eigenstate for which we hypothesize a remarkably simple form for the thermodynamic limit. We also studied the random process defined by the spread of a typical eigenstate on configuration space, highlighting several similarities with hopping on percolated hypercube, a process used to mimic the slow relaxation of spin glasses. A nearby nonintegrable model is also considered where delocalization is instead observed, although the presence of a phase transition at infinite temperature is questionable.

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I. INTRODUCTION

Recently^{1,2} it has been pointed out that the phenomenon of Anderson localization (AL),³ usually associated with single-particle hopping in a random potential, can be present even in the many-body eigenstates of an interacting quantum system and manifest itself as a phase transition at finite and even infinite temperature. This phenomenon has been dubbed *many-body localization* (henceforth MBL) and it can be conceived as an example of AL in configuration space rather than real space. As the geometry of configuration space for a many-body system differs substantially from that of a regular lattice in few dimensions, MBL is thought to have properties distinct from those of the single-particle AL.

MBL should be responsible, among other things, of the exact vanishing of the dc conductivity of metals below a critical temperature¹ and of the failure^{4,5} of the simplest version (and possibly of all versions) of the quantum adiabatic algorithm⁶ for the solutions of NP-complete problems; it has also been studied in disordered Heisenberg spin chains^{7,8} where the phase transition has been linked to the infinite-randomness fixed point. The similarity of some features of MBL to the glass transition in spin and configurational glasses makes it the closest to a *quantum* analog of a glass transition, where the assumptions of equilibrium statistical mechanics fail.

As we said, in some problems MBL is found in typical many-body states,⁹ namely states sampled with uniform distribution from the spectrum (therefore corresponding to infinite temperature). These states are difficult to study directly, much more than the ground states for which many approximations (DMRG, MPS, etc.) can be devised: Indeed, the only strategy here seems to be exact diagonalization (as used in Ref. 8, for example), the exponential complexity of which limits the size of the systems to less than 20 spins. Alternatively, the study of correlation functions with time-dependent DMRG was used, whose failure to converge due to growing entanglement can signal the onset of delocalization.¹⁰ Analytic results have mainly been obtained by studying the behavior of the perturbation theory for increasing system sizes.⁵

In this work we report the results of our numerical study on the structure of typical states of a fully connected quantum spin model with quenched disorder (the Richardson model)^{11–13} that has been introduced as a model of nuclear matter and has been studied in connection with the finite-size scaling of the BCS theory of superconductivity. Integrability allows us to go to sensibly higher spin numbers ($N = 50$ spins for single states and we will collect extensive statistics up to $N = 40$) and therefore make some educated guesses on the thermodynamic limit of the system.

We will begin by discussing the method we devised for the solution of the Bethe ansatz equations (the Richardson equations), which is at variance with respect to those very refined ones, used in the study of ground states and low-lying excited states^{14,15} (our method will be close in spirit to that used in Ref. 16, which appeared while this work was being completed). We will then discuss the observables, since we will face the problem that the classical observables in localization studies, the inverse participation ratio (IPR), is computationally heavy (its complexity goes as 2^N , although still smaller than $\mathcal{O}(2^{3N})$ steps required by exact diagonalization). We devised a Montecarlo method for the measure of IPR and performed an extensive study of an Edwards-Anderson-like order parameter q ,¹⁷ which is related to the average $(1, N-1)$ entanglement (the Meyer-Wallach¹⁸ entanglement measure) and to the average Hamming distance L , between states in the computational basis whose superposition forms the eigenstate. As we show, q is related to the long-time spin relaxation and therefore thermalization is not achieved as long as $q > 0$.

We also observe that for the Richardson model, q is in one-to-one correspondence with the IPR \mathcal{I} (but the relation differs from what found for the disordered Heisenberg model).¹⁹ We find for the thermodynamic limit of q the deceptively simple expression as a function of the hopping g , $q = (1 + g)^{-1}$, which implies in the same limit for the average Hamming distance $L/N = \frac{g}{2(1+g)}$. We will define a local entropy density $s = \ln \mathcal{I} / 2L$, for which we find numerically a well-defined

thermodynamic limit, although the limiting form does not seem to have the simple character of the previous quantities.

We have also studied the clustering properties of the eigenstates, and we have not found any presence of clusters but rather as the hopping g is increased the eigenstates spread rather uniformly over the configuration space. Neighboring states in energy have very close values of q but their overlap (a measure common to spin-glass studies) is close to 0. This means that clusters can be formed if one takes a superposition of states in a small energy interval to make a microcanonical density matrix.

The picture that emerges from this analysis is that there is no many-body localization-delocalization phase transition in this model although the states appear delocalized on the computational basis for any finite g , the average single-spin observables are always localized.

Finally, we discuss the role of integrability in the previous predictions and the implications of our findings for more natural cases where integrability is broken.²⁰

II. THE MODEL AND ITS SOLUTION

The Richardson model^{11,12} is an XX model (i.e., with no $s^z s^z$ coupling) of pairwise interacting spins with arbitrary longitudinal fields

$$H = -\frac{g}{N} \sum_{\alpha,\beta=1}^N s_{\alpha}^{+} s_{\beta}^{-} + \sum_{\alpha=1}^N h_{\alpha} s_{\alpha}^{z}, \quad (1)$$

where $s^{x,y,z}$ are spin- $\frac{1}{2}$ representations of SU(2) algebra. This model can accommodate quenched randomness in the arbitrary choice of the fields h_{α} .²¹ We choose a Gaussian distribution for them, with $\bar{h} = 0, \bar{h}^2 = 1$. First, one notices that the total spin S^z is conserved and we focus on the subspace $S_z = 0$ that exists only for even N . The model belongs to the class of models that are integrable through the algebraic Bethe ansatz (see also Ref. 22 for a more general discussion), which, in particular, implies that all the states in the sector $S^z = (2M - N)/2$ can be found by applications of M generalized raising operators on the reference state with all spin down:

$$|E[w]\rangle = \prod_{j=1}^M B(w_j) |\downarrow \dots \downarrow\rangle, \quad (2)$$

where the roots w_j satisfy the system of M coupled Richardson equations:

$$\forall j=1, \dots, M : \quad \frac{N}{g} + \sum_{\alpha=1}^N \frac{1}{w_j - h_{\alpha}} - \sum_{k=1, k \neq j}^M \frac{2}{w_j - w_k} = 0 \quad (3)$$

in terms of which the raising operators are

$$B(w) = \sum_{\alpha=1}^N \frac{s_{\alpha}^{+}}{w - h_{\alpha}} \quad (4)$$

and the energy of the state is given by

$$E[w] = \sum_{j=1}^M w_j - \sum_{\alpha=1}^N \frac{h_{\alpha}}{2}. \quad (5)$$

We address the reader to Ref. 23 for an extensive review of the algebraic aspects of the model and its solution. We mention only that the integrability of the model is connected to the presence of N commuting integrals of motion:

$$\tau_{\alpha} = \frac{N}{g} s_{\alpha}^{z} - 2 \sum_{\beta \neq \alpha} \frac{1}{h_{\alpha} - h_{\beta}} \vec{s}_{\alpha} \cdot \vec{s}_{\beta} \quad (6)$$

that are constructed via the algebraic Bethe ansatz in a standard way. We will use later that:

$$\frac{\text{Tr} \tau_{\alpha} \tau_{\beta}}{2^N} = \delta_{\alpha\beta} \left[\frac{N^2}{4g} + \frac{3}{4} \sum_{\gamma \neq \alpha} \frac{1}{(h_{\alpha} - h_{\gamma})^2} \right] - \frac{3(1 - \delta_{\alpha\beta})}{4(h_{\alpha} - h_{\beta})^2}. \quad (7)$$

Since the first term on the right-hand side is of $O(N^2)$ and the second is $O(1)$, the conserved charges become orthogonal in the thermodynamic limit. As we said, we will focus on $S_z = 0$ so $M = N/2$, which means that we have to solve $N/2$ coupled nonlinear equations, which is numerically viable provided one has only a reasonably good initial condition for root-finding algorithms. A widely used technique is that of considering that when $g \rightarrow 0$ the roots tend to some of the fields h_{α} , and from Eq. (4) it is clear that such root configurations correspond to the different choices of sets of M spins that are flipped with respect to the ground state according to Eq. (2); the choice of the set can be used to label the state at any g .

When one adiabatically increases g , by moving it of some small amount and solving Eq. (3) at each step, the roots begin to depart from their initial h values toward the negative direction. The ensuing evolution depends on the initial configuration of roots, but, generally, two of them may collide and form a pair of complex conjugate solutions; they may also then recombine and return real. When $g \rightarrow \infty$, roots either diverge in the negative direction or stay trapped within a couple of levels. The number of roots that eventually diverge is equal to the total spin S of the state (which is a conserved quantum number at infinite g). An algorithm that can follow the evolution of the roots with g has to take into account these changes in the nature of the solution, where the roots become complex conjugates. These critical points, for random choices of the h 's, can occur at particularly close values of g and this can create troubles for the algorithm.²⁴ The reader may refer to Refs. 14 and 25 and references therein for further details on the solutions. Extensive study on critical points has been performed in Ref. 26.

When more than a pair of roots collide in a too-small interval of g this change of variables may not be sufficiently accurate and one should think of something else (if one does not want to reduce the step in the increment of g indefinitely). The most general change of variables that smooths out the evolution across critical points is that which goes from the roots w_j to the coefficients c_i of the characteristic polynomial $p(w)$, i.e., the polynomial whose all and only roots are the w_j 's.

This polynomial is quite interesting in itself as it satisfies a second-order differential equation whose polynomial solutions have been classified by Heines and Stieltjes.^{27,28} Following the evolution of the coefficients $c_i(g)$ is a viable alternative to following the roots but we found out that the best strategy is a combination of both evolutions. Therefore, we follow the evolution of the roots, extrapolating the coefficients and using

them to correct the position of the roots at the next step in the evolution. In this way we do not implement any change of variables explicitly and we do not have to track the position of critical points. This algorithm²⁹ can be used on a desktop computer to find the roots of typical states with about 50 spins, although in order to collect extensive statistics we have limited ourselves to $N = 40$.

III. ORDER PARAMETERS

A. IPR, entanglement, average Hamming radius of an eigenstate, and a local entropy

Once obtained the roots one is faced with the task of studying the state. The quantity characterizing the localization/delocalization properties of a state on the basis of the eigenstates of s_α^z , $\alpha = 1 \dots N$ (the computational basis or configuration space \mathcal{C}) is the inverse participation ratio of an eigenstate $|E\rangle$:

$$\mathcal{I} = \left(\sum_{s_1, \dots, s_N, \sum_\alpha s_\alpha = 0} |\langle s_1, \dots, s_N | E \rangle|^4 \right)^{-1}. \quad (8)$$

We will see that for all $g > 0$, $\ln \mathcal{I} \propto N$, i.e., an exponential number of sites of the hypercube \mathcal{C} of spin configurations is covered; nevertheless, we generally observe that

$$\lim_{N \rightarrow \infty} \frac{\mathcal{I}}{\binom{N}{N/2}} \rightarrow 0, \quad (9)$$

which flags instead a single-particle *localized* phase, according to the definition common in AL studies. In fact, the analysis of single-particle observables will confirm this scenario.

The amplitudes $\langle s_1, \dots, s_N | E \rangle$ can be calculated as ratio of determinants of $(N/2) \times (N/2)$ matrices (therefore in time $\sim N^3$) once the roots w_j are known. It is sufficient to consider the appropriate limit of the general formulas for the scalar products as given in Ref. 23. However, the number of terms in the sum is exponential in N so the calculation of \mathcal{I} requires an exponential number of terms³⁰ and we are limited again to 20 spins or so.

We found two ways around this difficulty: They are complementary and can be checked one against the other for consistency. First, we devised a Montecarlo algorithm for the evaluation of \mathcal{I} . Define the probabilities $p_a = |\langle a | E \rangle|^2$, where $a \in \mathcal{C}$ stands for one of the $\binom{N}{N/2}$ allowed classical configurations of spins which constitute the configuration space \mathcal{C} . We perform a random walk with the probabilities p_a 's, namely start from a random configuration a . The neighboring configurations are those living within the same subspace $S^z = 0$ and differing from a by the exchange of a pair of opposite spins. We move to a random one of the $(N/2)^2$ neighboring states, say b , by accepting the move with probability $\min(1, p_b/p_a)$. This involves only one computation of p_b , which takes time $\sim N^3$. The random walk proceeds in this way, generating a history of configurations a for which we can take the average over Montecarlo time of p_a . The inverse of this value gives \mathcal{I} .

The intensive quantity is $\ln \mathcal{I}/N$, which can then be averaged over different states and realizations. We observe that for all $g = \mathcal{O}(1)$ the value of $\ln \mathcal{I} \propto N$, testifying then

that each state occupies an exponential number of states in the configuration space.

The second method is to find another quantity that can be computed in polynomial time and to link it to \mathcal{I} . Since the average values $\langle E | s_\alpha^z | E \rangle$ can be expressed again in terms of determinants they can be calculated efficiently [in $\mathcal{O}(N^3)$ time]. Therefore, one is led to consider a *microcanonical* version the Edwards-Anderson (EA) order parameter associated to a single eigenstate

$$q(E) = \frac{4}{N} \sum_{\alpha=1}^N \langle E | s_\alpha^z | E \rangle^2, \quad (10)$$

with this normalization $q \in [0, 1]$. The average over eigenstates is

$$q = \frac{1}{2^N} \sum_E q(E). \quad (11)$$

To get the physical significance of this quantity, following Ref. 8 we start with a slightly magnetized spin α in an infinite temperature state:

$$\rho_0 = (\mathbb{I} + \epsilon s_\alpha^z) / 2^N \quad (12)$$

with magnetization $\langle s_\alpha^z \rangle_0 = \text{Tr}(\rho_0 s_\alpha^z) = \epsilon/4$ (as $s_z^2 = 1/4$). The same magnetization at large time t in the diagonal approximation reads

$$\langle s_\alpha^z \rangle_\infty = \lim_{t \rightarrow \infty} \text{Tr} e^{-iHt} \rho_0 e^{iHt} s_\alpha^z = \frac{\epsilon}{2^N} \sum_E \langle E | s_\alpha^z | E \rangle^2. \quad (13)$$

Therefore, averaging over α we obtain the equality with Eq. (11):

$$q = \frac{1}{N} \sum_\alpha \frac{\langle s_\alpha^z \rangle_\infty}{\langle s_\alpha^z \rangle_0}, \quad (14)$$

namely the previously defined EA order parameter is the average survival fraction of the initial magnetization after very long times.

We notice two more things¹⁹: (i) that $q(E)$ is related to the average purity of the state (here we use the total $S^z = 0$)

$$q(E) = \frac{2}{N} \sum_\alpha \text{Tr} \rho_\alpha^2 - 1 \quad (15)$$

and (ii) that $q(E)$ is related to the average Hamming distance of the points in configuration space when sampled with the probability distribution p_a relative to state $|E\rangle$:

$$\begin{aligned} d(a, b) &= \sum_{\alpha=1}^N (\langle a | s_\alpha^z | a \rangle - \langle b | s_\alpha^z | b \rangle)^2 \\ &= \sum_{\alpha=1}^N \frac{1 - 4 \langle a | s_\alpha^z | a \rangle \langle b | s_\alpha^z | b \rangle}{2}, \end{aligned} \quad (16)$$

and multiplying by p_a, p_b and summing over a, b we find:

$$L \equiv \langle d \rangle = \frac{N}{2} (1 - q). \quad (17)$$

So q is computationally easy and it captures both some geometric properties of the covering of the configuration space by an eigenstate and the long-time correlation function for s^z .

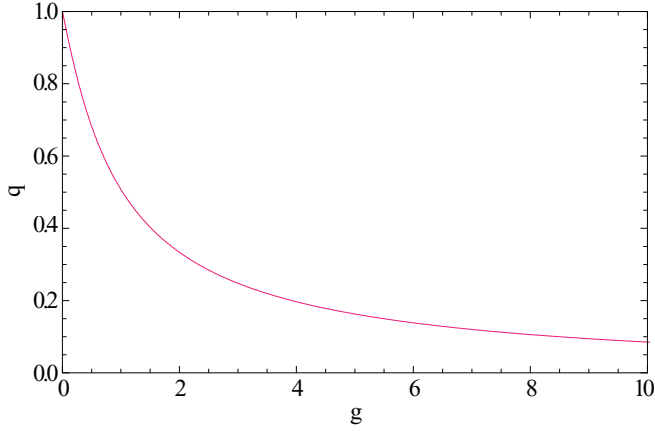


FIG. 1. (Color online) The pointwise extrapolation of the function q as a function of g . The fit $q = 1/(1 + g)$ is not distinguishable from the data.

We averaged q over the spectrum (sample over typical states) and then over realizations (the number of which depends on the size of the system but it will never be less than 100).

We found this average \bar{q} as a function of g for $g \in [0, 40]$ and $N = 16, \dots, 38$ and studied the pointwise finite-size scaling [in the form $q_N(g) = q(g) + c_1(g)/N + c_3(g)/N^3$] to obtain the thermodynamic limit of q (see Fig. 1). We fit the data using a ratio of polynomials with the condition that $q(0) = 1$ and we found that averaging over the state and the realization of disorder

$$\bar{q} = \frac{1 + 3 \times 10^{-8}g}{1 + 1.003g + 0.009g^2} \simeq \frac{1}{1 + g} \quad (18)$$

works in the whole range of data to an error of at most 0.5%. We therefore conjecture this to be the correct functional form of the EA order parameter at infinite temperature.

We can now go back to the relationship between the IPR and q , better expressed as a relation between $\ln \mathcal{I}$ and L . We notice a one-to-one correspondence between average values these two quantities already at the level of second-order perturbation theory in g starting from a given state with $N/2$ spins up S_\uparrow and $N/2$ spins down S_\downarrow :

$$\mathcal{I} = 1 + \frac{2g^2}{N^2} A + o(g^2), \quad (19)$$

where we defined a sum over pairs of up and down spins of the given state:

$$A = \sum_{\alpha \in S_\uparrow, \beta \in S_\downarrow} \frac{1}{(h_\alpha - h_\beta)^2}. \quad (20)$$

Since A is dominated by small denominators, it will be typically $A = O(N^4)$ and therefore from the expression for IPR we see the perturbative regime is valid for $g \ll 1/N$. With an analogous computation we get

$$L = \frac{4g^2}{N^2} A + o(g^2). \quad (21)$$

Eliminating g between the two relations and using Eq. (17) one obtains, independently of the state and of the quenched randomness (therefore the relation holds also on average),

$$\ln \mathcal{I} \simeq \frac{L}{2}. \quad (22)$$

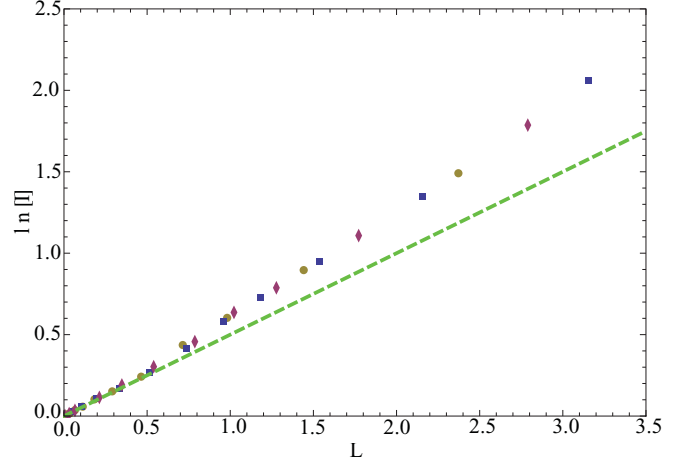


FIG. 2. (Color online) $\ln \mathcal{I}$ as a function of the average distance L . The points are $N = 28, 30, 32$ (square, diamond, and circle) averaged over 100 realizations: the dashed straight line is the second-order perturbation theory approximation Eq. (22).

So the relation is linear for small g . To see how this relation is modified at higher values of g we have again to resort to numerics. From the data it is clear that a strict relation exists between $\overline{\ln \mathcal{I}}$ and \bar{L} as one can see in Fig. 2.

By using the previous Montecarlo calculation for \mathcal{I} we can plot $\ln \mathcal{I}$ vs. L , showing that the relation is almost linear. The degree of nonlinearity is measured by the ratio

$$s = \frac{\ln \mathcal{I}}{2L}, \quad (23)$$

which can be interpreted as a local entropy.³¹ In fact, $2L = N(1 - q)$ can be interpreted as the number of free spins (whose value of s^z is close to 0) while \mathcal{I} is the number of configurations. If we want $2L$ spins to be responsible to \mathcal{I} states then each of these spins should account for a degeneracy of e^s , from which the interpretation as an entropy density.

The distribution of L over states and realizations becomes more and more peaked as N grows, since we observe the variance $\delta L^2 \propto N$. The same occurs to $\ln \mathcal{I}$, whose variance goes $\propto N$ in the region of g considered. Therefore, the average value of s becomes typical in the large- N limit.

In the curves of Fig. 3 the entropy s grows from the value of $1/4 = 0.250$ predicted by perturbation theory to an asymptotic value of $s = 0.383 \pm 0.003$.³² This value is not what one would expect from a uniform superposition over $\binom{N}{N/2}$ states, since in that case $L = N/2$, $\ln \mathcal{I} \simeq N \ln 2$, and the familiar value $s = \ln 2 = 0.693$ is roughly twice as much as we expect. This leads us to think that the most probable structure of the delocalized state at increasing g still retains a pair structure. We can build a toy model of delocalization in the typical eigenstates by assuming that Nq spins are localized on their $g = 0$ values and that the remaining $N(1 - q)$ spins are instead divided into couples, where couples are formed between almost resonating spins of opposite orientation. The couples are in one of the random valence bond states $|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle$

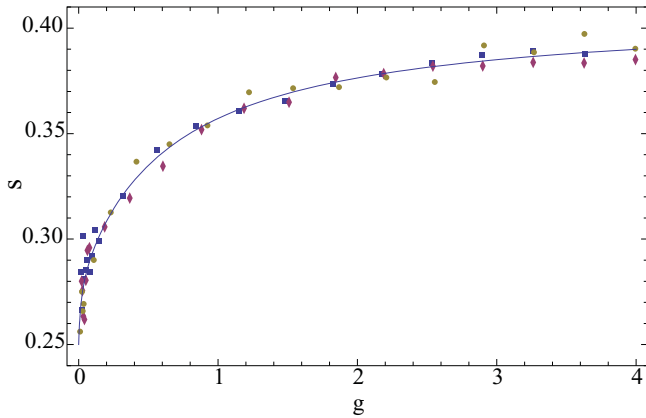


FIG. 3. (Color online) Local entropy as a function of g . The points are $N = 28, 32, 36$ (square, diamond, and circle), all averaged over 100 realizations. The fit is a (1,1) Padé' approximation conditioned to $s(0) = 1/4$.

which are indeed the two $S_z = 0$ eigenstates of the two-body Hamiltonian

$$H_2 = -\frac{g}{N}(s_1^+ s_2^- + s_2^+ s_1^-) + h(s_1^z + s_2^z), \quad (24)$$

where we have assumed that $h_1 \simeq h_2 = h$. This predicts that

$$\mathcal{I} \sim 2^{N(1-q)/2} = 2^L \quad (25)$$

and we should have a constant entropy $s = \ln(2)/2 = 0.347$, slightly smaller than the observed value at large g and off by 40% at small g . The pair structure of a given eigenstate can be observed in Fig. 4 where we plot the values of $m_n^2 \equiv \langle E | s_n^z | E \rangle^2$ for a given eigenstate $|E\rangle$, ordering the spins by increasing h_n (so that almost-resonant spins are nearest neighbors). We see a clear valence-bond like pair correlation in the values of the squared magnetization.

With the available data, we can discuss issues like the presence of multiple clusters in the same energy level $|E\rangle$. In fact, by randomly restarting the Montecarlo routine with the same p_a 's if multiple clusters exist, we would expect to sample them according to their basin of attraction. Moreover,

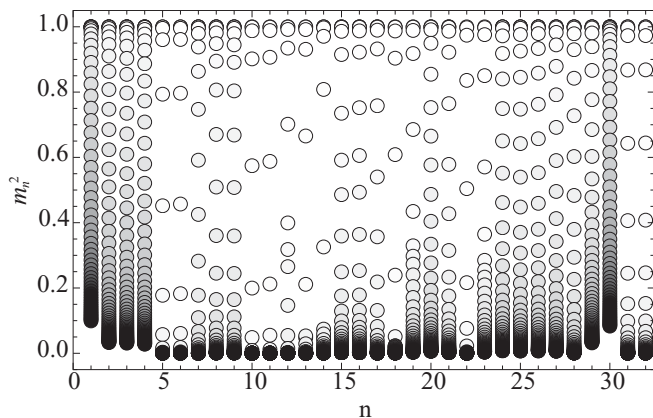


FIG. 4. Squared magnetizations for increasing g (from white to black), where the spins are ordered by increasing magnetic field h_n . A consistent number of valence bond pairing is observed as a good fraction of neighboring spins (e.g., $n = 8, 9$, $n = 15, 16$, and $n = 31, 32$) have the same speed.

we can rely on analytic results (such as those for q) to compare the Monte Carlo averages with clusterization and ergodicity breaking would translate in a difference between these two results (as the random walk would get stuck in a cluster and would not explore the whole configuration space). In the region where we can trust our numerics ($g \gtrsim 0.02$), Monte Carlo averages converge to the analytic results, though a slowdown of the dynamics is observed (see below).

B. Dynamics of Monte Carlo and other quantities

The Monte Carlo routine that allows for importance sampling of the distribution p_a allows other measures of the geometry of the state. We can now study the similarities between the dynamics of importance sampling on p_a and that of random percolation on the hypercube, which has been proposed as a model of relaxation in a glassy system.³¹ We will find that, in both cases, a stretched exponential is the best fit and that the exponent depends on the coupling constant g . This, we believe, is a remarkable similarity.

An important quantity in this sense is the time dependence of the average distance from the starting point. Consider the Hamming distance $H(t)$ from the starting point $H(t) \equiv d(a(t), a(0))$, where $a(0)$ represents a classical configuration of spins and $a(t)$ the one reached after t Monte Carlo steps. For $t \gg 1$, after averaging over many starting points $a(0)$, $H(t)$ is fit quite accurately by a stretched exponential ansatz of the form:

$$H(t) = L[1 - e^{-(t/\tau)^\beta}], \quad (26)$$

where L is the average distance introduced before and β is a new characteristic exponent. Let us consider the behavior of the exponent β with respect to g , as plotted in Fig. 5. Even if the results become quite noisy for small g , we can still see that for small values of g , β stays close to 1, while as g increases, β decreases, although quite slowly. Instead, for the time scale τ we find, apart from the monotonic decrease with g , which is to be expected on general grounds, that, for $g \gtrsim 1$, $\tau \propto N^{3/2}$, which we propose without explanation.

The small time behavior of $H(t)$ can be used to obtain some information about the local structure of the state. In particular we can set

$$k \equiv \frac{H(1)}{2} = \frac{4}{N^2} \sum_{\langle a, b \rangle} \min(p_a, p_b), \quad (27)$$

where the last equality follows from the Monte Carlo rate and the sum is over nearest-neighbor states. This quantity can be considered as a measure of the local connectivity, that is, the average fraction of active links.

From Fig. 6, we may deduce two things: one is that the connectivity stays well below 1 even for large g , confirming, as we claimed before, that the typical state is never uniformly spread over the hypercube; the second is that the connectivity scales with N as N^{-1} for small g and with $N^{-1/2}$ for large g (a fit $k = A/N^\alpha$ shows a continuously decreasing α from 1 to $1/2$). Since the number of diverging roots is proportional to the total spin of the eigenstate at infinite g , and since the more roots diverge, the more equally distributed the terms of each creation operator B are, we can argue the state will be better spread for larger total spin S . As for large N , the total spin of

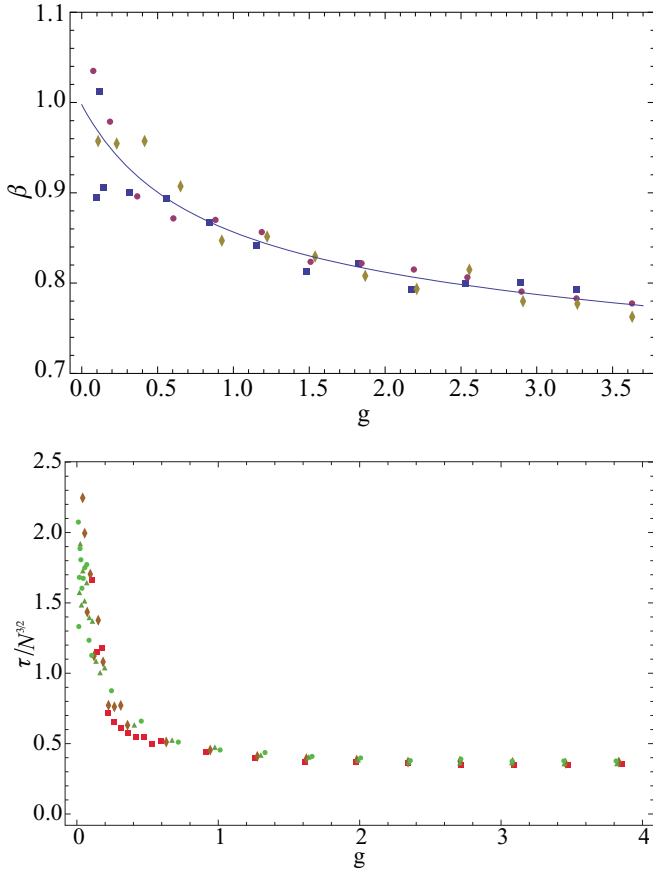


FIG. 5. (Color online) (Top) The stretched exponential exponent β data as a function of g for $N = 28, 32, 36$ (square, circle, and diamond) together with a fit of the form $(1 + a_1 g)/(b_0 + b_1 g + b_2 g^2)$. (Bottom) The time scale τ as a function of g for $N = 20, 24, 28, 32$ (square, diamond, triangle, and circle). The scaling $\tau \propto N^{3/2}$ is evidently good, in particular in the region $g > 1$.

a typical state will increase only as \sqrt{N} and this explains the depletion of the local connectivity.

C. Some perturbation theory

Let us see what the predictions of perturbation theory are for our system. Let us start at $g = 0$ from state a , with energy E_a . The states at distance 2 from a have energies

$$\Delta^{(2)} = E_b - E_a = h_\alpha - h_\beta \quad (28)$$

where the couple $(\alpha, \beta) \in \mathcal{S}_\uparrow \times \mathcal{S}_\downarrow$ defines the spins which have been flipped up and down in going from a to b . The typical value of $\Delta^{(2)}$ is $\sqrt{\langle (\Delta^{(2)})^2 \rangle} = \sqrt{2} = \mathcal{O}(1)$; however, the minimum value is $\mathcal{O}(N^{-2})$ which we write $x^{(2)}/N^2$, where $x^{(2)} = \mathcal{O}(1)$. So the corresponding term in perturbation theory for the wave function is

$$A_b = \frac{(g/N)}{x^{(2)}/N^2} = \frac{gN}{x^{(2)}}. \quad (29)$$

In this way, we can go on at arbitrary distance $2k$ to the state b^k , the amplitude thus having k denominators of $\mathcal{O}(1/N^2)$

$$A_{b^k} = \frac{(gN)^k}{x^{(2)}x^{(4)} \dots x^{(2k)}}, \quad (30)$$

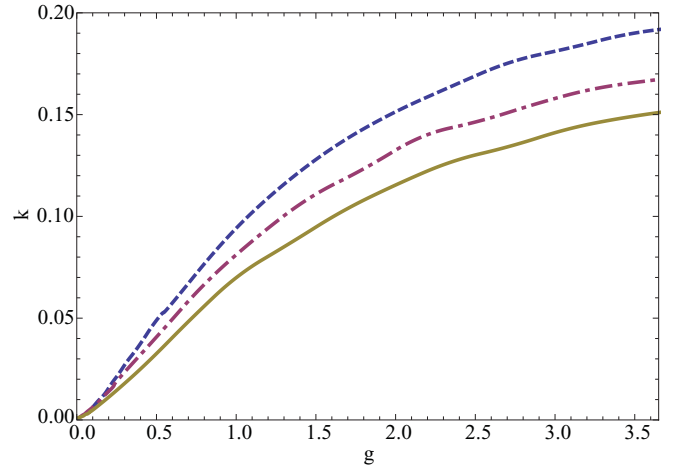


FIG. 6. (Color online) Local connectivity as a function of g . Different lines corresponds to $N = 18$ (dashed), 24 (dotted and dashed), and 30 (solid).

where x are random variables of $\mathcal{O}(1)$. For any given a there are only $\mathcal{O}(1)$ neighboring states with $\Delta \sim 1/N^2$, so the number of such b^k states at distance $2k$ from a is $\mathcal{O}(1)$ from N^{2k} (also the number of relevant paths does not grow as $k!$). This can be called a *direct* or *percolating* contribution. However, at distance 4 we already observe another type of contribution, which one is tempted to dub a *tunnelling* contribution, in which, although the final denominator $\Delta^{(4)} = h_\alpha - h_\beta + h_\gamma - h_\delta = z^{(4)}/N^4$, each of the two paths leading to the minimum $h_\alpha - h_\beta \simeq -(h_\gamma - h_\delta) = y^{(2)} = \mathcal{O}(1)$, where $\alpha, \gamma \in \mathcal{S}_\uparrow$ and $\beta, \delta \in \mathcal{S}_\downarrow$. Again, this contribution is of order:

$$A_b = \frac{(g/N)}{y^{(2)}} \frac{(g/N)}{z^{(4)}/N^4} = \frac{(gN)^2}{y^{(2)}z^{(4)}}, \quad (31)$$

while the amplitudes corresponding to the distance 2 intermediate steps are $\mathcal{O}(1/N)$. The distribution of x, z can be found by using the theory of extreme value statistics,³³ while y 's are typical values of field differences and none of these distributions depend on N . We will stop our analysis of perturbation theory here, as this would require a separate work by itself. It is sufficient for us to notice that only the combination gN appears in all terms of the series so scaling g to zero like $1/N^{1+\epsilon}$ for every $\epsilon > 0$ each term would go to 0 and the series would trivially converge unless the series is asymptotic in gN . Notice that an argument based on a Bethe-lattice approximation for the configuration space \mathcal{C} (see Ref. 34) would give $g_c \sim 1/N \ln N$ for the localization transition.

D. Setup of an exponential IPR

From perturbation theory (and from the Bethe-lattice approximation result)³⁴ one could argue that, if a phase transition occurs, it is at $g \sim 1/N$ (an extra factor $1/\ln N$ would not be noticed for our moderately large N). But does a phase transition in the geometric properties of the eigenstate occur?

We first analyze the quantity for which we have more extensive statistics (because of his polynomial complexity), L . A phase transition in L would mean that, set $\gamma = gN$,

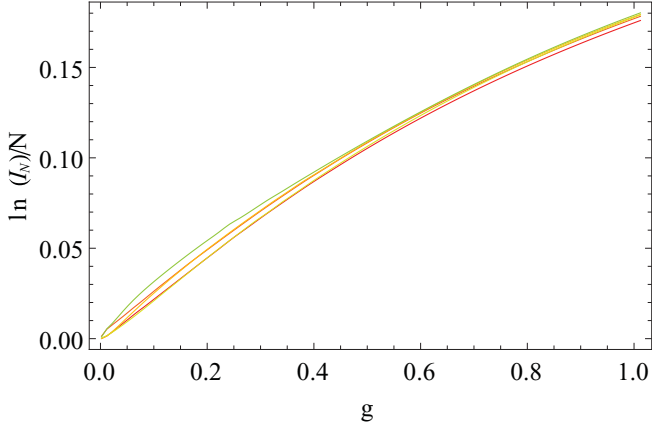


FIG. 7. (Color online) The logarithm of the inverse participation ratio divided by the size for N between 8 and 18, computed exactly and averaged over states and realizations. This graph shows no hint of a phase transition at $g \lesssim 1/N$.

there exists a $\gamma_c > 0$ such that for $\gamma < \gamma_c$, $L/N \rightarrow 0$ and for $\gamma > \gamma_c$, $L/N \sim (\gamma - \gamma_c)^\delta$ where δ is a critical exponent. We have analyzed our data for $g > 0.01$ and $g < 0.2$ and we can conclude that this is not the behavior observed. The behavior is more consistent with $\gamma_c = 0$, $\delta = 1$ or with a crossover, in which the limit L/N when $N \rightarrow \infty$ is a smooth function of g which vanishes at $g = 0$. The matching with the part of the curve at finite g is smooth and the limiting behavior is as described before.

There is the possibility, however, that although $L \sim N$ always, we have two phases: $\ln \mathcal{I} \sim 1$ and $\ln \mathcal{I} \sim N$ between which a transition occurs. This could happen if an eigenstate spread along one (or a few) directions without covering an exponential number of spin configurations. We have excluded this by both direct analysis of $\ln \mathcal{I}/N$ data and by the observation that the relation between $\ln \mathcal{I}$ and L remains valid all the way to small g (small here means $g \lesssim 1/N$). As $\ln \mathcal{I}/N$ becomes soon independent of N without any scaling of g needed (see Fig. 7) we are led to conclude that no phase transition occurs as the system occupies an exponential number of sites of the computational basis for any $g > 0$.

IV. BREAKING OF INTEGRABILITY

By considering the Richardson model essentially as a hopping process on the hypercube with random site energies given by the unperturbed energies we have found that the eigenstates are always covering an exponential number of spin configurations but, nonetheless, $q > 0$, meaning thermalization is not achieved. To understand better the role played by integrability, we make use of the *Mazur's inequality* for the long-time T average^{22,35,36} that in our case takes the form:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \langle s_\alpha^z(0) s_\alpha^z(t) \rangle \geq \sum_{\beta=1}^N \frac{|\langle \tau_\beta s_\alpha^z \rangle|^2}{\langle \tau_\beta^2 \rangle}, \quad (32)$$

where $\langle \cdot \rangle \equiv \frac{1}{2^N} \text{Tr}(\cdot)$.³⁷ Using that

$$\langle \tau_\beta s_\alpha^z \rangle = \frac{N}{4g} \delta_{\alpha\beta} \quad (33)$$

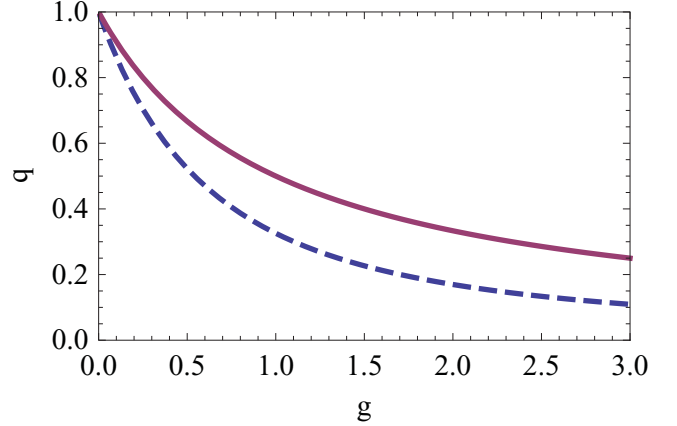


FIG. 8. (Color online) The comparison between our formula (18) for \bar{q} (solid line) and the lower bound coming from the conserved charges (dashed line). The Mazur inequality is satisfied.

together with Eq. (7), and then summing over α , we get the lower bound

$$q \geq \frac{1}{N} \sum_{\alpha} \frac{1}{1 + \frac{3g^2}{N^2} \sum_{\beta \neq \alpha} \frac{1}{(h_\alpha - h_\beta)^2}}. \quad (34)$$

By averaging both sides of the equation with respect to the fields we obtain, to leading order in large N , the exact expression:

$$\bar{q} \geq 1 - \sqrt{3\pi} g \int_{-\infty}^{\infty} dx \mathcal{P}(x)^2 e^{3\pi \mathcal{P}(x)^2 g^2} \overline{\Phi}(\sqrt{3\pi} \mathcal{P}(x) g),$$

where $\mathcal{P}(x)$ is the Gaussian distribution function of the fields h and $\overline{\Phi}$ is the complementary error function. A comparison between Eq. (18) and this bound is in Fig. 8. As this bound is > 0 for any g , this is sufficient to claim that thermalization is not achieved in our model. We, therefore, see explicitly that the integrability of the model plays an essential role beyond providing the methods used for its solution. One can then put on a more firm ground the claim that if the integrals of motion are too much local, in the way of Eq. (33), integrability can have the effect of freezing the expectation values of local quantities.

To further explore this issue, we investigated a very similar nonintegrable model, in which the hopping coefficients are not uniformly equal to g as in Eq. (1) but instead $N(N-1)/2$ random variables $g_{\alpha,\beta} = g(1 + \epsilon \eta_{\alpha,\beta})$, where $\eta_{\alpha,\beta} = \eta_{\beta,\alpha} = \pm 1$ with probability $1/2$. Randomness in the fields is retained. The Hamiltonian is

$$H = -\frac{g}{N} \sum_{\alpha,\beta} (1 + \epsilon \eta_{\alpha,\beta}) s_\alpha^+ s_\beta^- + \sum_{\alpha} h_\alpha s_\alpha^z, \quad (35)$$

is no longer integrable and must be solved by exact diagonalization.

We observe a decrease of the value of q (averaged both over E, η and h) as expected; in particular, for sufficiently large g , we are confident to say that $q \rightarrow 0$ for $N \rightarrow \infty$ and the system becomes ergodic.

For small g , however, the situation is not so clear. The limit $N \rightarrow \infty$ could actually be zero or not; what is clear is that the N dependence is not settled (compare the upper and

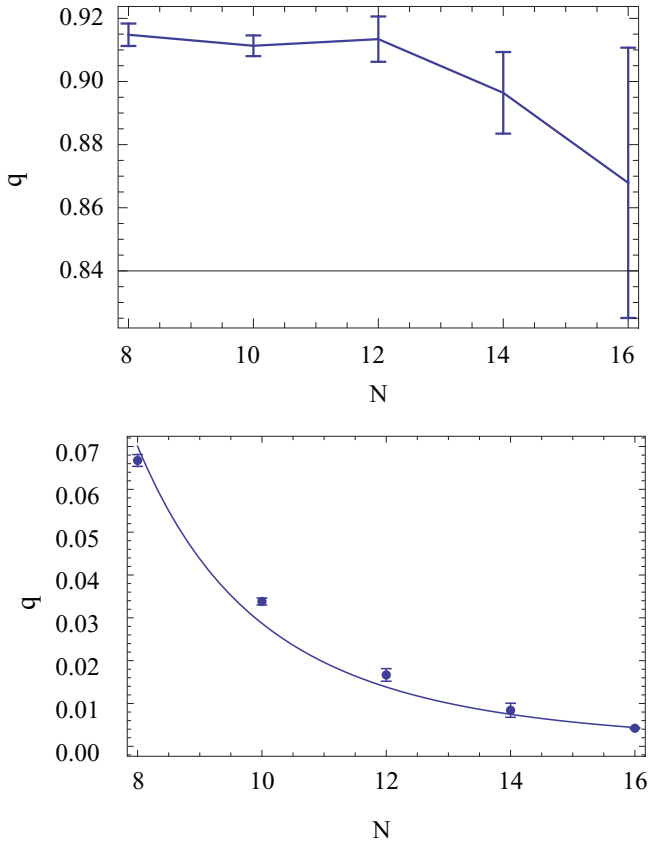


FIG. 9. (Color online) Averaged microcanonical q for the nonintegrable Hamiltonian Eq. (35) for (top) $g = 0.1, \epsilon = 0.4$ and (bottom) $g = 4.1, \epsilon = 0.4$. The data in the lower panel are fit by a power law $aN^{-\gamma}$ with $a = 290$ and $\gamma \simeq 4$. The exponent γ seems to be g dependent.

lower panel of Fig. 9) for $N = 16$, the largest system size that we can attain. This leads to two competing scenarios: In the first we have ergodicity as soon as $\epsilon > 0$; in the second, one could identify a finite $g_c(\epsilon)$ such that for $g < g_c$, $q > 0$ and for $g > g_c$ we have $q = 0$ in the thermodynamic limit. The latter would have an MBL transition at the said g_c . Much more extensive numerical work is needed to decide between these two scenarios. We leave the resolution of this issue for the future.

V. CONCLUSIONS AND SOME DIRECTIONS FOR FURTHER WORK

We have performed a numerical study of typical states of the Richardson model with quenched disorder (an example of Gaudin magnet and an integrable system). We have found no evidence of a delocalization phase transition, although typical eigenstates occupy an exponential number of states in the basis of s_α^z 's for any $g > 0$.

We have devised a method to calculate the IPR without summing over exponentially many states and studying its connections with a microcanonical version of the Edwards-Anderson order parameter, which measures the fraction of surviving magnetization at infinite temperature and for long times. Of this order parameter, we have hypothesized that the thermodynamic limit at infinite temperature as $q = 1/(1 + g)$. We were unable to obtain the temperature dependence of this quantity, as sampling from the Boltzmann distribution is not straightforward within our framework.

For what concerns the absence of a MBL phase transition we can point out two peculiarities of our system as responsible for its absence. One is integrability and the other is the infinite range of the Hamiltonian. We have therefore studied small-size systems (up to $N = 16$ spins) with an extra integrability breaking term of size ϵ . We observe a sharp reduction of q , which in some range of parameters could lead to think to a phase transition where $q = 0$ for $g > g_c(\epsilon)$. However, it is possible that in the complementary region [$g < g_c(\epsilon)$] the decrease with N starts from a value of N impossible to reach with our limited numerics so we are unable to see that $q = 0$ for all g as soon as $\epsilon > 0$. Unfortunately, this dichotomy is unlikely to be settled with the currently accessible values of N and in absence of an established scaling theory of MBL.

We point out that the Richardson model is one of a family of integrable spin systems (generalized Gaudin's magnets, see Ref. 38) which can be studied with minor modifications of the methods introduced in this paper. We leave this too for further work.

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