On the connection between random waves and quantum fields: duality between nodal lines statistic and the Casimir energy

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Abstract

Using the statistical description common to random waves and quantum fields we show how the probability of having a nodal line close to a (translationally symmetric) reference curve γ is related to the Casimir energy of an appropriate configuration of conductors.

PACS number: 03.70.+k

1. Introduction

Random waves (RW) have been an object of interest for their statistical properties both in wave mechanics and statistical mechanics. In wave mechanics they turned out to be an incredibly interesting and rich scenario for studying the statistics of topological properties, like phase singularities [1]. In optics they made a good statistical model for speckle patterns [2] in laser beams. In quantum mechanics they have been studied [3, 4] in connection with semiclassical wave functions in chaotic billiards. In statistical mechanics their properties have been put in connection with the statistic of defects and vortices [5] and an interesting duality with a percolation problem has been put forward recently [6].

The purpose of this paper is to exploit in a new direction the description of RW in terms of quantum field theory (QFT).[1] I will make a quantitative connection between the ground state energy (or the Casimir energy) of a scalar field in a given configuration of semi-penetrable conductors in $d - 1$ space dimensions and the probability of having a certain configuration of nodal lines in d dimensions.

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[1] The statistical mechanics description of RW is related to this via the usual QFT-statistical mechanics duality (Wick rotation). In this sense this description is already contained in the work of B Halperin in [2] who employed it to study the statistic of vortices and defects.

This paper is far from being exhaustive or self-contained. I will briefly introduce the concept of random wave referring the reader to the existing literature for their statistical properties, in particular the properties of their nodal lines. I will then rephrase the concept of Casimir energy for a scalar field in a static background (a toy model for QED where the static background models the conductors) in a language closer to that of random waves statistics. I will then point out the connection between the Casimir energy in this background and the probability of having a certain nodal line configuration. Finally I will draw some consequences on the nodal line probability and comment on the possible extensions on which further work is needed.

2. Random waves

An isotropic random wave (RW) in d dimensions is the random function defined on a subset of $\mathbb{R}^d$ (we will not use any particular notation for vectors but there is little room for confusion) as

$$\phi(x) = \sum_{j=1}^{J} \sqrt{\frac{2}{J}} \epsilon(k_j) \cos(k_j x + \delta_j), \tag{1}$$

where the phases δ_j are uniformly distributed in $[0, 2\pi)$ and the vectors k_j are random variables as well. We will assume isotropy of ϵ, i.e. $\epsilon(k)$ is an even, analytic function of the length of the vector k.

For any finite J the moments $\langle \phi(x_1) \cdots \phi(x_n) \rangle$ are not factorizable, but in the limit $J \to \infty$ the Wick theorem holds [1] (among other things one requires the existence and finiteness of at least the second moment, i.e. $\langle \phi^2(x) \rangle < \infty$):

$$\langle \phi(x_1) \cdots \phi(x_{2n}) \rangle = \sum_{\text{Contractions}} \langle \phi(x_i)\phi(x_j) \rangle \cdots \langle \phi(x_k)\phi(x_l) \rangle. \tag{2}$$

In the following we will hence always assume that the limit $J \to \infty$ is taken.

Wick's theorem is equivalent to saying that the statistical properties of RW can be described by a Gaussian probability functional

$$P[\phi] = \frac{1}{Z} \exp\left(-\frac{1}{2} \int d^d x \, d^d x' \phi(x') h(x', x) \phi(x)\right), \tag{3}$$

where Z is a normalization constant and $h(x', x) = h(|x' - x|)$ for isotropic RW. From this probability functional the reader could already recognize the usual set-up of the statistical mechanics of a non-interacting real field ϕ. The function h is determined by the spectrum $\epsilon(k)$ (and vice versa). We will now determine their connection.

To this purpose is convenient to pass to the Fourier components of the field $\phi_k = \int d^d x \, e^{ikx} \phi(x)$ and define $h(k)$ through

$$\int d^d x \, d^d x' \, e^{ikx - ik'x'} h(x', x) = (2\pi)^d \delta^{(d)}(k' - k) h(k). \tag{4}$$

In terms of ϕ_k the probability functional is

$$P[\phi] = \frac{1}{Z} \exp\left(-\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} h(k) |\phi_k|^2\right). \tag{5}$$

The limiting Gaussian probability functional (3) or (5) can describe the statistical properties of (1) if we choose the spectrum $\epsilon(k)$ as

$$\lim_{J \to \infty} \frac{1}{J} \epsilon^2(k) = \frac{1}{h(k)} \frac{d^d k}{(2\pi)^d} \tag{6}$$

which means that in the limit $J \to \infty$ the sum over k_j must be substituted by the integral in $d^d k$ whose measure is given by the right-hand side of (6). This is the promised connection between $h(x)$ and $\epsilon(k)$. In this way when $J \to \infty$ the propagator G tends to

$$G(x,0) \equiv \langle \phi(x)\phi(0) \rangle = \lim_{J \to \infty} \sum_{j=1}^{J} \frac{1}{J} \epsilon^2(k_j) \cos(k_j x) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{h(k)} e^{ikx}, \qquad (7)$$

where we used the fact that ϵ is even in k to substitute e^{ikx} for $\cos(kx)$.

There are at least two 'natural' choices for the spectrum $h(k)$:

- The *scalar field* spectrum $1/h(k) = \theta(\Lambda - |k|)/(k^2 + m^2)$, where one has to introduce the cut-off Λ to ensure the finiteness of $G(x,x) = \langle \phi^2(x) \rangle$.
- The very singular *monochromatic* spectrum $h(k)$, such that $1/(2\pi h(k)) = \delta(|k| - K)$. This last choice gives $G(x,0) = J_0(Kx)$ which is a statistical model for the solutions of the Schrödinger equation $-\Delta \psi = K^2 \psi$ in chaotic billiards.

3. The Casimir energy and nodal lines

We now turn to the main point of this paper: the connection between nodal lines' properties and the Casimir energy. For simplicity at the moment we assume $d = 2$, the generalization to other d will be straightforward.

Following [4] we introduce the functional

$$X_\gamma[\phi] = \frac{1}{2} \int_\gamma ds \, \phi^2(x(s)), \qquad (8)$$

where the integral is defined over the reference line $\gamma = \{x(s) | s \epsilon [0, \ell]\}$ and parameterized with the length of the line itself, s. For any given reference curve γ, $X_\gamma[\phi]$ is a random variable whose generating function $S_\gamma(\lambda)$ is defined as

$$S_\gamma(\lambda) \equiv \langle \exp(-\lambda X_\gamma[\phi]) \rangle = \int \mathcal{D}\phi \, P[\phi] \exp\left(-\frac{1}{2}\lambda \int_\gamma ds \, \phi^2(x(s))\right). \qquad (9)$$

It has been shown in [4] that $S_\gamma(\lambda)$ can be interpreted approximately as the probability of having a nodal line in the tube of radius $r = (\langle(\nabla\phi)^2\rangle\lambda)^{-1/3}$ built around the reference curve γ (in d dimensions $1/3$ gets substituted by $1/(d+1)$). Note that the radius r of the tube goes to zero when $\lambda \to \infty$. The approximation allowing us to interpret S_γ as the probability of having a nodal line relies mainly on a mean-field approximation where $\phi^2/(\nabla\phi^2) \to \phi^2/\langle(\nabla\phi)^2\rangle$ as discussed in [4]. It is not easy to establish the limits of this approximation, so we will adopt it as a working hypothesis and we will see later a situation in which it possibly fails. From now on we will simply say that S_γ is 'the probability of having a nodal curve γ' without referring to the tube radius r or the approximation within which this interpretation has been derived.

Let us write in (9) $P[\phi]$ explicitly inside the probability functional

$$S_\gamma(\lambda) = \int \mathcal{D}\phi \frac{1}{Z} \exp\left(-\frac{1}{2} \int d^2x \, d^2x' \phi(x')(h(x',x) + \delta^{(2)}(x'-x)V(x))\phi(x)\right), \qquad (10)$$

where we have defined

$$V(x) = \lambda \int_\gamma ds \, \delta^{(2)}(x - x(s)). \qquad (11)$$

Let us now specialize the problem in two ways.

- Choose $h(x', x)$ to mimic a scalar field, with a cut-off Λ intended in all the momentum integrals

$$h(x', x) = \delta^{(2)}(x' - x)(-\Delta + m^2). \tag{12}$$

- Consider a random wave in the strip $[0, T] \times \mathbb{R}$. Denote the two Cartesian coordinates in the plane as x_0, x_1 so $0 \leqslant x_0 \leqslant T$ and $x_1 \in \mathbb{R}$. Choose the reference line γ as made of $n \geqslant 1$ disconnected lines parallel to the x_0 axis and intersecting the x_1 axis at the points $\{a_1, \ldots, a_n\}$

$$\gamma = \{a_1, \ldots, a_n\} \times [0, T]. \tag{13}$$

With these assumptions the final expression for the generating function $S_\gamma(\lambda)$ is then

$$S_\gamma(\lambda) = \frac{1}{Z} \int \mathcal{D}\phi \exp\left(-\frac{1}{2} \int_{[0,T]\times\mathbb{R}} \mathrm{d}^2x \, \phi(x)(-\Delta + m^2 + V(x))\phi(x)\right), \tag{14}$$

where $\Delta = \partial^2/\partial x_0^2 + \partial^2/\partial x_1^2$. This expression itself is reminiscent of two intertwined concepts in QFT and statistical field theory: the Casimir energy $\mathcal{E}$ in the first and the free energy F in the second. The connection with the latter is evident, without any need for formal manipulations, $F = -\log(S_\gamma(\lambda))$. The connection with the Casimir energy becomes evident as well if we perform a clockwise (inverse) Wick rotation in the x_0 coordinate, $x_0 \to it$. Then (14) becomes

$$\frac{1}{Z} \int \mathcal{D}\phi \exp\left(i\frac{1}{2} \int_{[0,T]\times\mathbb{R}} \mathrm{d}t \, \mathrm{d}x_1 \phi(t, x_1)(-\partial^2 - m^2 - V(x_1))\phi(t, x_1)\right) = \mathrm{e}^{-i\mathcal{E}_\gamma T}. \tag{15}$$

Here $\partial^2 = \partial^2/\partial t^2 - \partial^2/\partial x_1^2$ and $\mathcal{E}_\gamma$ is the Casimir energy in the background V and

$$V(y) = \sum_{i=1}^{n} \delta(a_i - y). \tag{16}$$

We can now establish the promised connection between the generating functional $S_\gamma(\lambda)$ and the Casimir energy $\mathcal{E}_\gamma$ of the corresponding background as

$$S_\gamma(\lambda) = \mathrm{e}^{-T\mathcal{E}_\gamma}. \tag{17}$$

In words: *the probability of having a (translationally symmetric) nodal line γ in a random wave ensemble is related to the Casimir energy of a configuration of conductors given by a constant-time section of γ.*

The generalization to d dimension is easily obtained (and is already understood in the previous paragraph). Since a nodal hypersurface has codimension 1 so does its constant-x_0 section. Hence the problem maps to the Casimir energy of codimension 1 surfaces. The dual Casimir problem then is the usual problem of penetrable, codimension 1 surfaces (see [7] for the case with arbitrary codimension). For example, the $d = 4$ case maps into the $\mathbb{R}^3$ Casimir problem with penetrable two-dimensional surfaces. The limit $\lambda \to \infty$ is the limit of perfect conductors (or the Dirichlet limit, because $\phi = 0$ on the conducting surfaces).

In the rest of the paper we will use this duality to make statements on the nodal line statistic from the knowledge of the properties of the Casimir energy.

4. Applications

Let us start with a well-known problem of a Casimir energy calculation: the presence of various divergences, when taking $\Lambda, \lambda \to \infty$. We will now discuss the interpretation of these divergences for the nodal lines probability.

A volume divergence $\propto V\Lambda^{d+1}$ (divergent when $\Lambda \to \infty$) is removed by the factor of $1/Z$ in our definition of $P[\phi]$. This term is however independent of the presence and shape of γ. In QFT it would represent a cosmological constant term. For what concerns the divergencies arising when $\lambda \to \infty$ (the so-called Dirichlet limit) let us recall that the tube radius r built around the reference line γ, goes to 0 when $\lambda \to \infty$. It is then natural that the probability $S_\gamma(\lambda)$ of having a nodal line within a distance r from γ goes to zero when $\lambda \to \infty$. This reflects the fact that $\mathcal{E} \to +\infty$ when $\lambda \to \infty$.

We also know that the interaction energy (the one that depends on the distance between the bodies) remains finite when $\Lambda, \lambda \to \infty$. This means that there are some properties of the nodal lines, connected with the interaction part of $\mathcal{E}$, which are well defined also when the tube radius $r \to 0$ and the cut-off goes to infinity. In order to identify them we must define a quantity which stays finite in this limit. Led by the intuition about the Casimir energy of rigid conductors we recognize that this problem is related to the removal of the self-energy for rigid, disconnected bodies. Suppose then our curve γ is composed of two disconnected pieces $\gamma = \gamma_1 \cup \gamma_2$ (we always require them to be straight and both parallel in the x_0 direction). Their Casimir energy can be written as

$$\mathcal{E}_\gamma = \mathcal{E}_{\gamma_1} + \mathcal{E}_{\gamma_2} + \mathcal{E}_{\text{int}}, \tag{18}$$

where the first two terms are independent on the distance between the nodal lines and the last term goes to zero when the distance between the two curves goes to infinity (this can be taken as a definition of $\mathcal{E}_{\text{int}}$). The terms $\mathcal{E}_{\gamma_{1,2}}$ are the energies of isolated plates.

Let us define the quantity $\mathcal{P}$ as the ratio of the probability of having $\gamma_1 \cup \gamma_2$ and the probability of having both γ_1 and γ_2 independently of each other:

$$\mathcal{P} = \frac{S_{\gamma_1+\gamma_2}(\lambda)}{S_{\gamma_1}(\lambda)S_{\gamma_2}(\lambda)} = \mathrm{e}^{-T\mathcal{E}_{\text{int}}}. \tag{19}$$

The interpretation of $\mathcal{P}$ is the following: if $\mathcal{P} > 1$ ($\mathcal{P} < 1$) then it is easier (more difficult) to find a nodal line γ_2 if another line γ_1 is present.

The interaction energy $\mathcal{E}_{\text{int}}$ is always finite (even when $m \to 0$ and/or $\Lambda, \lambda \to \infty$) and hence so is $\mathcal{P}$. Moreover, we know from quantum field theory that $\mathcal{E}_{\text{int}} < 0$ and that it increases when the nodal lines are pulled apart. Hence we can say that the presence of a nodal line γ_1 makes it easier for another nodal line to be born. Hence in this case nodal lines *induce* other nodal lines in their vicinity.

The choice of the scalar field spectrum allows us to use all the machinery of QFT (including the Hamiltonian formulation) to calculate the Casimir energy $\mathcal{E}_{\text{int}}$. Depending on the values of λ, Λ, m and the distance between the nodal lines a, we can use a weak or strong coupling approximation for $\mathcal{E}_{\text{int}}$. Since λ has dimension ℓ^{d-1} (ℓ is a length scale) the relevant dimensionless parameter is $\epsilon = a\lambda^{1/(d-1)}$. Moreover assuming $\Lambda \gg m$, $\langle(\nabla\phi)^2\rangle \sim \Lambda^2$, we have $\lambda \sim 1/r^{d+1}\Lambda^2$. By choosing a, r, Λ, we have $\epsilon \ll 1$ for weak coupling and $\epsilon \gg 1$ for the strong coupling regime.

We will now make some explicit sample calculation in these two regimes.

In the *weak coupling* regime one can use a Feynman diagram expansion [8] (one must use the Euclidean cut-off Λ on the k integrals) for the Casimir energy $\mathcal{E}$

$$\mathcal{E} = \lambda\mathcal{E}_1 + \lambda^2\mathcal{E}_2 + \cdots, \tag{20}$$

where $\mathcal{E}_1$ is given by the tadpole diagram, $\mathcal{E}_2$ is given by the two-legs diagram and so on. We will now calculate the first two terms of the series (20) showing that $\mathcal{E}_1$ drops between numerator and denominator in $\mathcal{P}$ and then calculating the first correction to $\mathcal{P}$, i.e. $\mathcal{E}_2$. We will also show that $\mathcal{E}_2 < 0$, which implies $\mathcal{P} > 1$, in a region of order $1/m$ around any nodal line.

For the tadpole diagram we have

$$\mathcal{E}_1 = \int d^{d-1}x\, V(x)\langle\phi^2(x)\rangle = \langle\phi^2(0)\rangle \int d^{d-1}x\, V(x) \tag{21}$$

or in Fourier space

$$\mathcal{E}_1 = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} V(k) \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2}, \tag{22}$$

where $V(k)$ is the Fourier transform with respect to the $d-1$ spatial dimensions

$$V(k) \equiv \int d^{d-1}x\, V(x)\, e^{ikx} \tag{23}$$

so that in $d = 2$ and with $V(x) = \delta(x) + \delta(x - a)$ we have

$$V(k) = 1 + e^{ika}. \tag{24}$$

Since $\int dx\, V(x)$ does not depend on a, the tadpole diagram does not contribute to the interaction energy $\mathcal{E}_{\text{int}}$ and hence does not contribute to $\mathcal{P}$. We must then go to the next diagram, the one with two legs to find the first non-zero correction to $\mathcal{E}_{\text{int}}$. The two-legs diagram contribution can be written as

$$\mathcal{E}_2 = -\int \frac{d^{d-1}k}{(2\pi)^{d-1}} V(k)V(-k) \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q+k)^2 + m^2} \frac{1}{q^2 + m^2}. \tag{25}$$

It contains an a-dependent interaction term. To calculate this a-dependent term we can send $\Lambda \to \infty$ (for $d = 2$ we can take this limit safely) and by means of the usual technology for handling Feynman diagrams we find

$$\mathcal{E}_2 = -\frac{1}{2\pi} \int_0^1 dx \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ika} \frac{1}{m^2 + x(1-x)k^2}. \tag{26}$$

Performing the integrals gives

$$\mathcal{E}_2 = -\frac{1}{2m}(1 - \Phi(2\sqrt{ma})), \tag{27}$$

where Φ is the error function. Then $\mathcal{P}$ can be written, to this order in λ, as

$$\mathcal{P} = \exp\left(T\frac{\lambda^2}{2m}(1 - \Phi(2\sqrt{ma}))\right). \tag{28}$$

As we said before, $\mathcal{P}$ decreases when a increases. Moreover, for $a \gg 1/m$ we can do an asymptotic expansion for Φ finding

$$\mathcal{P} \simeq \exp\left(T\frac{\lambda^2}{4m\sqrt{am}} e^{-4am}\right), \tag{29}$$

so $\mathcal{P} \simeq 1$ effectively for $a \gg 1/m$.

The strong coupling limit has to be tackled with different, non-perturbative techniques.

The $d = 2$ case can also be solved exactly for any number n of parallel nodal lines by using the techniques in [7]. The resulting exact expression for $n \geqslant 3$ is too cumbersome to be presented here and we refer the reader to [7] for details.

Two parallel nodal lines separated by a distance a in the limit $\Lambda \to \infty$ are dual to the problem of two points in 1 space dimension at a distance a. The Casimir energy for this configuration is

$$\mathcal{E}_{\text{int}} = \frac{1}{4\pi} \int_0^\infty \frac{dE}{\sqrt{E}} \ln\left(1 - \frac{e^{-2a\sqrt{E+m^2}}}{\left(1 + \frac{2}{\lambda}\sqrt{E+m^2}\right)S^2}\right). \tag{30}$$

We can use this formula to make some predictions about $\mathcal{P}$. To begin we know that $\mathcal{E}_{int} < 0$ and that it has a minimum at $a = 0$ as $\mathcal{E}_{int}(a = 0, \Lambda \gg \lambda) \simeq -\lambda \log(2)/2\pi$ (here we put for simplicity $m = 0$). It can be proved that this is also equal to $\mathcal{E}_{single}(2\lambda) - 2\mathcal{E}_{single}(\lambda)$ (where $\mathcal{E}_{single}$ is the energy of a single delta function when $\Lambda \rightarrow \infty$), which appeals to intuition since at $a = 0$ we are just superposing two delta functions to create a delta function with double strength. If $m > 0$ it can be proved that $\mathcal{E} \propto \exp(-2ma)$ and hence again $\mathcal{P} \simeq 1$ when $a \gg 1/m$. In any case we can say that $\mathcal{P}$ decreases when a increases.

A difficulty must be noticed here concerning how far one can push the interpretation of S_γ as the probability of having a nodal line γ. Reasoning like in [9], assuming Dirichlet boundary conditions on a line γ intersecting the x_1 axis at say $x_1 = 0$ we have then to expand our RW in series of $\sin(k_j x_1)$ (with random coefficients). Reasonably the subset of RW that has a nodal line on γ should be expandable in this basis as well. If moreover our spectrum is cut off at Λ then one expects that for $a \ll \pi/\Lambda$ one should find much fewer nodal lines (the first zero of $\sin z$ is at $z = \pi$). In fact a similar phenomenon is found in [9] for the monochromatic spectrum. The nodal line length density normalized to its asymptotic value goes to ~ 0.5 for $x_1 = 0$. However, increasing x_1 the nodal line length density suddenly increases to a value higher than the asymptotic value and then relaxes, oscillating, to 1. In analogy our quantity $\mathcal{P}$ should then start from a value < 1 at $a = 0$, increase in a region $1/\Lambda$ to a value $\mathcal{P} > 1$ and then relax to $\mathcal{P} = 1$. Evidently the first, $\mathcal{O}(1/\Lambda)$ region is not captured by our analysis, while the second one is. This, as we said in the discussion after equation (9), can possibly be traced back to the failure of the 'mean field' approximation that was used to link S_γ with the true probability of finding a nodal line [4]. We have hence learned that we must assume $a \gg 1/\Lambda$ for our results to hold. Equation (30) for $a \gg 1/\Lambda, 1/\lambda$, and $m = 0$ gives

$$\mathcal{E}_{int} = -\frac{\pi}{24a}, \tag{31}$$

yielding

$$\mathcal{P} = e^{\pi T/24a}. \tag{32}$$

The higher dimension $(d > 2)$ case cannot be solved in general, due to its strong geometry dependence. The constant time section of γ can be any hypersurface representing disconnected conductors in space. The Casimir problem is the most generic one and we do not posses an efficient way of solving it. We know however how to solve the case of parallel, large (actually, infinite) $d-2$ hyper-planes (lines in $d = 3$, planes in $d = 4$ etc). The result for $m = 0, \lambda \rightarrow \infty$ is

$$\mathcal{E}_{int} \propto -\frac{S}{a^{d-1}}, \tag{33}$$

where S is the $(d-2)$-dimensional area of the hyper-planes, a is their separation and the proportionality constant depends on d.

One of the main problems of Casimir physics is to find effective (analytical or numerical) ways of calculating the Casimir energy for arbitrary configurations of perfect conductors. Despite recent developments [10, 11], this problem escapes analytical solution for all but the simple parallel plates case. We then expect to gain some insights from the other side of the duality, namely the nodal lines' distributions.

5. Extensions and further developments

Extension to different spectra. It would be interesting to know how much of what we said in this paper, based on the scalar field spectrum, is valid for other spectra (like the monochromatic spectrum). The monochromatic, as well as other kinds of isotropic spectra cannot be modelled by an Hamiltonian field theory, even though their probability functional is Gaussian. The high

degree of non-locality of these spectra implies that the free energy F is not extensive. Hence we could not even define a T-independent quantity like the Casimir energy $\mathcal{E}$. It is hence of great interest for a field theorist to grasp some of the properties of these generalized free QFTs in terms of some, more intuitive perhaps, statistical properties of random waves.

Extension to codimension > 1. Generically nodal lines of real fields have codimension 1 (lines in the plane, etc) because they are defined by a single condition, namely $\phi(x) = 0$. Codimension 2 or higher nodal lines are non-generic and have extremely low probability of occurring. For example the probability of having $\phi(x) = 0$ at an isolated point requires both $\phi(x) = 0$ and $|\nabla\phi(x)| = 0$ at the same point. This is *extremely* unlikely in the sense that it has measure 0, and would never show up in a Monte Carlo simulation. We know in fact from [7] that conductors of codimension 2 and higher cannot be defined with $\lambda > 0$. They *must* be defined as a limit $\lambda \to 0^-$. However the generating functional $S_\gamma(\lambda)$ is not well defined for $\lambda < 0$. It diverges badly. Actually, since $S_\gamma(\lambda) = \langle e^{-\lambda X_\gamma} \rangle$, for $\lambda < 0$ it is finite and only if the probability distribution of X_γ decays at infinity faster than $e^{\lambda X_\gamma}$. It turns out that one can take the limits ($\lambda \to 0^-$ and shrinking γ to codimension > 1) in such a way that this divergence and the infinitesimal probability of a codimension > 1 nodal line occurring compensate, giving a finite value for S_γ.

Extension to complex fields and phase singularities. A nodal line of a complex field is a more interesting object than that of a real field [1]. Complex field nodal lines are phase singularities whose strength can be interpreted as a topological charge [5]. Various correlation functions of these charges have been calculated by means of the Gaussian field technology. It would be interesting to see what the Casimir energy analogy has to say on these objects.

Numerics. One of the main reasons this duality is interesting is that it could lead to a more efficient numerical algorithm for computing Casimir energies of conductors of arbitrary shape. However this issue is beyond the scope of this paper and we leave it for future work.

6. Conclusions

We have shown that there is a dual description of random waves in terms of quantum field theory. In particular we put forward and started the exploration of the duality between the probability of having a nodal line close to a given disconnected reference curve and the Casimir energy of a configuration of conductors.

We used this duality to infer some properties of the distribution of nodal lines and we proved that, for the scalar field spectrum, nodal lines induce other nodal lines in their proximity. This last statement just follows from the attractive nature of Casimir interactions.

This duality can be used in the other direction to gain information on the Casimir energy of an arbitrary configuration of conductors from the statistical properties of the nodal line.

Acknowledgments

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