Casimir dynamics: Interactions of surfaces with codimension >1 due to quantum fluctuations

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We study the Casimir force between defects (branes) of codimension larger than 1 due to quantum fluctuations of a scalar field ϕ living in the bulk. We show that the Casimir force is attractive and that it diverges as the distance between the branes approaches a critical value L_c. Below this critical distance L_c the vacuum state $\phi = 0$ of the theory is unstable, due to the birth of a tachyon, and the field condenses.

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I. INTRODUCTION

Pointlike interactions have provided a remarkably useful idealization for many situations in physics. In the context of scattering theory the concept of a pointlike scatterer was introduced in 1934 by Bethe and Peierls [1]. Fermi [2] used and refined their results to describe the motion of neutrons in hydrogenated substances (such as paraffin) by introducing what is now known as the "Fermi pseudopotential." The idea is that when the scattering potential is concentrated on a very small scale r_0 (in the case studied by Fermi the range was that of nuclear interactions compared to the distances between the atoms), but its influence on the motion cannot be neglected, one can characterize the scattering in a simple and efficient way by means of a few quantities like the scattering length, finite in the $r_0 \to 0$ limit. The problem of "how to separate the scales" in the Schrödinger equation triggered by those 1930s papers was addressed and elegantly solved over the years at different levels of formalism [3–7]. The key to the solution relies in a proper definition of a "delta function interaction" in dimensions greater than 1.

Of course this paper will not be dealing with quantum mechanical scattering within matter, which is from many points of view a solved problem. The problem of "separation of scales," however, arises urgently in modern quantum field theory if stated as: "What is the quantum field theory response on length-scales L to a disturbance concentrated on a length scale $r_0 \ll L$?" How does quantum field theory respond to topological defects and singularities, in particular, of the metric? Once formalized in proper mathematical terms the two problems look much closer than one would think.

The Casimir effect falls in this class of problems. The penetration length (r_0) of the electromagnetic field inside conductors is much smaller than the distance (L) between the conductors, which sets the scale of the experimentally measured force. We are interested in studying the dynamics of the conductors on the larger scale L by integrating out the electromagnetic field. This motivates the nomenclature "Casimir effect" for a much wider set of problems than Casimir's original one.

Another example: in any candidate theory of the quantum geometry of space-time the problem of dealing with pointlike singularities will inevitably arise. Remember, for example, that the Ricci scalar for a pointlike particle (like Schwartzschild's solution) is a delta function centered on the position of the particle. Quadratic fluctuations of a nonminimally-coupled scalar (or of the metric) have hence a delta function term in their Lagrangian. The effect of such a term must be considered together with the other known effects of the black hole metric. It is then of paramount importance to analyze the problem of how one or more concentrated singularities influence the spectrum and low-energy behavior of the fluctuations of the field.

Analogous problems arise in condensed matter, quantum field theory, and string theory since localized disturbances appear in all these theories, essentially only their names are different (defects, domain walls, concentrated Aharonov-Bohm fluxes and branes to name some). With this in mind we will set up the problem in very general terms and, even though not all the details map one-to-one on specific examples, the main results will apply to a wide class of examples.

First, I will show that quantum fluctuations of a scalar field ϕ generate attractive forces between localized defects in the very same way Casimir forces act between metallic bodies. I calculate this force for an arbitrary number of defects with codimension[1] 1, 2, and 3 (see Table I). Note that previously the Casimir effect has been analyzed only for codimension 1. The main result of this paper is Eq. (22), which gives the interaction energy as a function of the scattering lengths of the defects and their relative separations. For codimension $d \geq 4$ the force disappears, as required by the properties of the self-adjoint extensions

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[1]Codimension is the number of dimensions transverse to a manifold. For example a point in 2 dimensions, a line in 3 dimensions and a surface in 4 dimensions all have codimension 2.

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TABLE I. Flat manifolds divided according to dimension and codimension. The first line is the well-known Casimir problem, from the fourth line down the perturbation is ''invisible'' to fluctuations. In this paper we will be dealing with manifolds in lines 2 and 3.

Codim/Dim	↶1	↶2	↶3	↶4
Casimir:1	point	line	plane	hyperplane
2		point	line	plane
3			point	line
↓ Trivial: 4				point

of the Laplace operator on the punctured $\mathbb{R}^d$ (see [8], Chap. X).

Second, in the presence of two or more of these defects (of codimension >1) the vacuum $\phi = 0$ is hopelessly unstable and a localized tachyon mode is formed when the defects approach closer than a critical distance. At this critical distance the attractive force diverges. I calculate the wave function of the tachyon and show that it leads to condensation of the bulk field ϕ to a vacuum expectation value (vev) $\overline{\phi}(x) \neq 0$—but only in a limited region of space. The consequences of these observations for some models will be discussed in Sec. VI.

II. THE INTERACTION ENERGY

In this section we will calculate the effective action [9] of a scalar field coupled quadratically to a static configuration of defects. The effective action $S_{\rm eff}$ and Casimir energy $\mathcal{E}$ are proportional to each other

$$S_{\rm eff} = -T\mathcal{E}, \tag{1}$$

where T is the interaction time. In the following we will be interested in the Casimir energy of the problem. We will see that the part of the Casimir energy responsible for the interaction between the defects is a cutoff-independent quantity, meaning that the separation of scales can be performed effectively in this quantum field theory.

We will consider the following action for the scalar field in $d+1$ dimensions ($\hbar = c = 1$):

$$S_\phi = \int d^dx dt \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\Big(m^2 + \sum_i^N \mu_i \delta(x - a_i)\Big)\phi^2. \tag{2}$$

Here δ is the d-dimensional Dirac's delta function, mimicking the concentrated disturbance on the field, μ_i are constants, meaning that they do not depend on the field ϕ, but in general they can depend functionally on other fields living on the defect.[2] The methods of [4] will be used in order to define these δ's correctly. In this section, in order to keep things simple we restrict our attention to points in 1, 2, and 3 dimensions. We will add an arbitrary number of flat directions in Sec. V, hence fulfilling our promise of studying codimensions 1, 2, and 3.

[2]The μ_i's are also called ''brane tensions.''

Actions like Eq. (2) arise in different contexts. For example consider the case of a scalar bulk field ϕ coupled with N branes in curved or flat space [curvature can be easily included in Eq. (2)]; or the case of a cosmic string (again the curvature outside the string must be considered); or the case of electrons coupled with Aharonov-Bohm fluxes (in this case one has fermions rather than bosons but, after squaring the Dirac equation, the analysis is analogous [7]). All these examples can be studied with the formalism introduced in this paper, so in full generality we will study the quantum fluctuations of the action Eq. (2).

Some features of Eq. (2) with only one delta function and constant m have been studied before, for example, in connection with cosmic strings scenarios [3,10,11]. The action (2) in one dimension with a single delta and a space-dependent mass term $m^2 = m^2(x)$ has been studied in [12].[3] In this paper we will consider the situation where a generic number of defects are present in $d \geq 1$ and m^2 is a constant. We will see that the situation will be different from that depicted in Refs. [3,10,12] and unexpected physics is found. Moreover, the generalization to x-dependent m^2 can be easily achieved by means of the techniques of Ref. [12] and we will not comment on it in this paper. The inclusion of any other term in the action (2) describing the dynamics of the surface itself would not affect our calculations. We consider the positions of the defects a_i fixed and obtain an effective action. In the usual way this action can be used to describe adiabatically moving $a_i(t)$ (i.e. if the velocities $|\dot{a}_{ij}| \ll c$).

The forces between the defects can be calculated by taking the derivatives of the Casimir energy $\mathcal{E}$ with respect to its arguments $\{a_i\}$. It will turn out that in general the forces are not additive, i.e. $\mathcal{E}$ is not a superposition of terms depending only on the relative distances $a_{ij} \equiv |a_i - a_j|$.

We use the following integral representation of the zero-point energy of the scalar field ϕ,

$$\mathcal{E} = \frac{1}{2}\int_0^\Lambda dE\rho(E)\sqrt{E}, \tag{3}$$

where Λ is a cutoff and ρ is the spectral density of the Hamiltonian operator H

$$H = -\nabla^2 + m^2 + \sum_{i=1}^N \mu_i\delta(x - a_i). \tag{4}$$

We define also the *unperturbed* Hamiltonian H_0 as

[3]The scale of variation of m^2 should be of the order of the ''long scale'' L. The purpose of this paper is to integrate out the physics at momenta $\gtrsim 1/r_0$, which is symbolized by the delta functions in Eq. (2).

$$H_0 = -\nabla^2 + m^2. \tag{5}$$

Equation (3) will look more familiar if the replacement $E \to \omega^2$ is done, and $\hbar$'s are restored. One can obtain the spectral density $\rho(E)$ as a functional of the propagator $G(E) = 1/(H - E)$ as

$$\rho(E) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \mathrm{Im}\, \mathrm{Tr} G(E + i\epsilon). \tag{6}$$

In the following we will often write $E + i0^+$ for $E + i\epsilon$ when $\epsilon \to 0^+$.

The propagator, $\mathcal{G}(x', x; E) \equiv \langle x'|G(E)|x\rangle$ satisfies the Schrödinger equation

$$\left(-\nabla'^2 + m^2 + \sum_{i=1}^{N} \mu_i \delta(x' - a_i) - E\right)\mathcal{G}(x', x; E) = \delta(x' - x), \tag{7}$$

and $\mathcal{G}_0 \equiv \langle x'|G_0(E)|x\rangle$ satisfies the analogous equation without δ's on the left-hand side. For m constant (which we will assume unless explicitly stated) and $\mathrm{Im}E > 0$ we have[4]

$$\mathcal{G}_0(x', x; E) = \begin{cases} \frac{i}{2\sqrt{E-m^2}} e^{i\sqrt{E-m^2}|x'-x|} & \text{if } d = 1 \\ \frac{i}{4} H_0^{(1)}(\sqrt{E - m^2}|x' - x|) & \text{if } d = 2 \\ \frac{e^{i\sqrt{E-m^2}|x'-x|}}{4\pi|x'-x|} & \text{if } d = 3, \end{cases} \tag{8}$$

where $H_0^{(1)}$ is Hankel's function of first kind of order 0.

For $d = 1$ the problem is that of a scalar field on the line $\mathbb{R}$ in the background of a stack of δ-functions centered on $x = \{a_i\}$ [13]. If we assume $\mu_i > 0$ they will attract each other, like metallic plates do via the Casimir effect. These forces are not confining and no new physics is obtained with the generalization obtained by adding n transverse directions. This is the usual Casimir problem. We will see how the situation changes dramatically when $d > 1$.

To see how the solution for $\mathcal{G}$ is obtained, consider first the case with a single delta function with strength $\mu_1 = \mu$, placed at $x = a$. By solving the Lippman-Schwinger equation [4,5,12] one finds

$$\mathcal{G}(x', x; E) = \mathcal{G}_0(x', x; E) + \frac{1}{\alpha - \mathcal{G}_0(a, a; E)} \mathcal{G}_0(x', a; E)\mathcal{G}_0(a, x; E), \tag{9}$$

where $\alpha = -1/\mu$. This solution is perfectly good in $d = 1$ and was the basis of the analysis in [12] for nonconstant m^2. For $d > 1$ it, however, suffers from a serious problem since $\mathcal{G}(a, a + r; E) \to \infty$ when the point splitting regulator $r \equiv |r| \to 0$. One can reabsorb this divergence [5] in a redefinition of α to obtain a finite result

$$\mathcal{G}(x', x; E) = \mathcal{G}_0(x', x; E) + \frac{1}{\alpha_r - B^{(d)}(E)} \mathcal{G}_0(x', a; E)\mathcal{G}_0(a, x; E), \tag{10}$$

where

$$B^{(d)} = \begin{cases} \frac{i}{2\sqrt{E-m^2}} & \text{if } d = 1 \\ -\frac{1}{2\pi} \ln\left(\frac{\sqrt{E-m^2}}{iM}\right) & \text{if } d = 2 \\ \frac{i\sqrt{E-m^2}}{4\pi} & \text{if } d = 3, \end{cases} \tag{11}$$

where for $d = 2$ it has been necessary to introduce an arbitrary mass scale M which stays finite when the point splitting regulator $r \to 0$. So α must be redefined such that when $r \to 0$

$$\alpha_r = \begin{cases} \alpha & \text{if } d = 1 \\ \alpha + \frac{1}{2\pi} \ln Mr & \text{if } d = 2 \\ \alpha - \frac{1}{4\pi r} & \text{if } d = 3, \end{cases} \tag{12}$$

so the "renormalized" α_r stays finite. It is clear from this equation that one needs a *positive* divergent α to reabsorb the negative divergences when $r \to 0$. Large positive α means *negative* very small μ (since $\alpha = -1/\mu$). A small negative μ corresponds to a weakly attractive potential. Hence the pointlike scatterer limit can be thought of as the limit of a concentrated attractive potential, zero outside a sphere of radius r_0, with at most one bound state whose energy stays finite when $r_0 \to 0$ [5,6]. For $d = 2$ any attractive potential has at least a bound state and so we always find a bound state also for $r_0 \to 0$ (for $d = 2$ the dependence of α on M is reminiscent of a renormalization group flow [7,10,14]); for $d = 3$ the bound state can be real or "virtual" (i.e. a pole of the propagator $G(E)$ located on the second Riemann sheet) its energy being finite in the limit $r_0 \to 0$. The scattering length is (both for $d = 2$ and 3) a function of α and is hence finite in the $r_0 \to 0$ limit.

Another interpretation of these results comes from the theory of self-adjoint extensions of symmetric operators [4,6]. Here the renormalized α_r corresponds to a choice of self-adjoint extension for the Laplacian operator $-\Delta$ on the punctured $\mathbb{R}^d$ [6]. In $\mathbb{R}^2$ the self-adjoint extensions are not positive definite, meaning that they all have at least one (but it turns out there is only one) negative eigenvalue. This corresponds to the bound state described in the paragraph above. In the punctured $\mathbb{R}^3$ the self-adjoint extensions of $-\Delta$ can be either positive semidefinite (with a virtual state on the second Riemann sheet) or not (due to the existence of a single real and negative eigenvalue).

The propagator with N deltas at positions $\{a_i\}$, $i = 1, ..., N$, and $1 \leq d \leq 3$ can be found [4]:

[4]In the following we will not use any special notation for vectors and we will indicate with $|x|$ the norm of a vector in 1, 2, and 3 dimensions.

$$\mathcal{G}(x', x; E) = \mathcal{G}_0(x', x; E) + \sum_{i,j=1}^{N} (\Gamma^{-1})_{ij} \mathcal{G}_0(x', a_i; E) \mathcal{G}_0(a_j, x; E). \tag{13}$$

The matrix Γ is defined as (from now on we drop the subscript r on the α_r's)

$$\Gamma_{ij} = (\alpha_i - B^{(d)}(E))\delta_{ij} - \tilde{\mathcal{G}}_0(a_i, a_j; E), \tag{14}$$

where

$$\tilde{\mathcal{G}}_0(a_i, a_j; E) = \begin{cases} 0 & \text{if } i = j \\ \mathcal{G}_0(a_i, a_j; E) & \text{if } i \neq j \end{cases} \tag{15}$$

It is now possible to explain why we limited our discussion to $d \leq 3$. The reason is that looking at the Laplacian Δ on the punctured $\mathbb{R}^4$ one realizes that this operator is essentially self-adjoint [4,8], meaning that it has a unique self-adjoint extension: the trivial one. The 4-dimensional delta function is "too small" a perturbation to be seen by the Laplacian. What does go wrong in the renormalization procedure? The propagator in $d = 4$ is ($|x' - x| \equiv r$)

$$\mathcal{G}_0(x', x; E) = \frac{i\sqrt{E - m^2}}{8\pi r} H_1^{(1)}(\sqrt{E - m^2} r) \tag{16}$$

$$\sim \frac{i}{8\pi}\left(\frac{1}{r^2} + \sqrt{E - m^2}\ln r + \mathcal{O}(1)\right) \quad \text{if } r \sim 0 \tag{17}$$

so we cannot choose α in an E-independent way (because of the $\sqrt{E - m^2}\ln r$ term) to remove completely the divergences as we did before. The low-energy limit of a 4-dimensional concentrated potential is hence trivial and we will not discuss this problem anymore.

Having solved for the propagator we can find the density of states ρ simply by taking the trace and the imaginary part. The result[5] is [12]

$$\rho(E) = \rho_0(E) - \frac{1}{\pi}\mathrm{Im}\frac{\partial}{\partial E} \ln \det\Gamma(E + i0^+, \{a\}), \tag{19}$$

where $\rho_0 = \pi^{-1}\mathrm{Im}\,\mathrm{Tr}G_0$ and the determinant of Γ is simply the determinant over the matrix indices ij.

The term ρ_0 in (19) is independent of the presence, strengths α_i and relative positions of the delta functions and we will neglect it in the following. The second term on the right-hand side of Eq. (19), can be used to calculate the Casimir energy as a function of the positions and strengths α of the scatterers:

$$\mathcal{E} = -\frac{1}{2\pi}\mathrm{Im}\int_0^{\Lambda} dE\sqrt{E}\frac{\partial}{\partial E} \ln \det\Gamma(E + i0^+, \{a\}). \tag{20}$$

The interaction part of this energy is obtained by subtracting from Eq. (20) the same quantity calculated with all the $L_{ij} = |a_i - a_j| \to \infty$. In this limit Γ becomes diagonal (considering that $\mathrm{Im}E > 0$) and the energy (20) becomes a sum of self-energies of isolated objects. The self-energy of an isolated brane contains all the usual ultraviolet divergences of the Casimir energy and must be treated with care [15]. The sharp δ-function limit $r \to 0$ and the strong potential limit (sometimes called *the Dirichlet limit*) $V_0 \to \infty$ are problematic for the Casimir energy already in $d = 1$ and the results depend on the order in which the $r \to 0$, $V_0 \to \infty$, and $\Lambda \to \infty$ limits are performed.[6] To avoid this problem we will consider the situation in which the potential $V(x)$ is finite, sufficiently smooth and localized over a finite distance $0 < r \ll L$, where L is the separation between the branes. Physically, this means that we assumed the cutoff Λ of the field theory to be much larger than any other momentum (square) scale, like V_0 or $1/r^2$. The divergences that then arise in the Casimir energy are entirely local: they will not affect the interaction energy. The procedure of subtracting the self-energies as described above then leaves a well-defined, finite interaction energy between the branes. After this subtraction is performed, one can take the appropriate limits for the potential $r \to 0$ and $V_0 \to \infty$ and, provided the interaction energy remains finite as we prove below, the resulting interaction energy is unique.

After performing the subtraction of the self-energies, the interaction energy can be written as

$$\mathcal{E} = -\frac{1}{2\pi}\mathrm{Im}\int_0^{\Lambda} dE\sqrt{E}\frac{\partial}{\partial E} \ln\frac{\det\Gamma(E + i0^+, \{a\})}{\det\Gamma(E + i0^+, \infty)}. \tag{21}$$

We keep using $\mathcal{E}$ to indicate the interaction energy, confident that this will not generate any confusion, since we will no longer be interested in the total energy. The integrand in Eq. (21) falls exponentially fast on the semicircle $|E| \to \infty$ of the complex E plane[7] which allows us to integrate by parts, Wick-rotate to the negative E axis[8] and send the cutoff $\Lambda \to \infty$. We can moreover remove the Im because all the quantities are real and positive on the negative real E axis (since the propagator $\mathcal{G}_0$ is real and

[5] The only algebraic identity worthy of notice is the fact that

$$\sum_{ij}(\Gamma)^{-1}_{ij}\frac{\partial}{\partial E}\mathcal{G}_0(a_i, a_j; E) = -\frac{\partial}{\partial E}\mathrm{Tr}\ln\Gamma. \tag{18}$$

[6] In this paper we are adopting a cutoff regularization of the quantum field theory. Different regularization schemes (like the widely used zeta-function, for example) give different results for the divergences. The finite part of the energy is, however, the same.

[7] In particular it goes to zero like $e^{-2\sin(\frac{\theta}{2})\sqrt{|E|}L}$ on the ray $E = |E|e^{i\theta}$, $\pi > \theta > 0$, where $L = \min|a_i - a_j|$ for any d.

[8] During the Wick rotation we do not pick any pole contribution on the positive imaginary semiplane of the first Riemann sheet because the total Hamiltonian Eq. (4) is self-adjoint.

positive for E real and below the spectrum) except for $\sqrt{-E + i0^+} = i\sqrt{E}$.

This leads us to a final, compact expression for $\mathcal{E}$

$$\mathcal{E} = \frac{1}{4\pi}\int_0^\infty \frac{dE}{\sqrt{E}} \ln\frac{\det\Gamma(-E,\{a\})}{\det\Gamma(-E,\infty)}. \tag{22}$$

This is the main result of this paper and together with the definition of Γ, Eq. (14), can be used to calculate the interaction energy of pointlike scatterers due to fluctuations of the field ϕ. In the rest of this paper we present several examples of the applications of this formula.

III. EXAMPLES

As a first example and a check for our result, Eq. (22), let us calculate the well-known interaction energy between two delta functions at distance L, in 1 dimension (we assume $\alpha_1 = \alpha_2 \equiv \alpha < 0$).

$$\mathcal{E} = \frac{1}{4\pi}\int_0^\infty \frac{dE}{\sqrt{E}} \ln\left(1 - \frac{e^{-2L\sqrt{E+m^2}}}{(1 - 2\alpha\sqrt{E+m^2})^2}\right). \tag{23}$$

This formula reproduces the usual results for the Casimir energy of two penetrable plates in 1 dimension [16].

As another example in $d = 1$ consider the case of three repulsive delta functions ($\alpha_i = -1$ for $i = 1, 2, 3$). The interaction energy can be calculated with the ease with which one can take a determinant of a 3 by 3 matrix. The result is plotted in Fig. 1 as a function of the position x of one of the three deltas while the other two are held fixed at $x = 0$ and $x = 5$. The interaction energy is not additive: the interaction energy of N semipenetrable plates does not split into a sum of $N(N-1)/2$ terms due to pairwise interactions. Rather, by expanding the logarithm a *reflection expansion* is obtained in the spirit of Ref. [17].

Before calculating the interaction energy for 2 or more deltas in $d > 1$ it is instructive to look at the case of a single delta function centered in $x = 0$ to introduce some properties of the bound state of a single delta. Consider the case $d = 3$. The propagator is (for $\mathrm{Im}E > 0$)

$$\mathcal{G}(x', x; E) = \frac{e^{i|x'-x|\sqrt{E-m^2}}}{4\pi|x'-x|} + \frac{1}{\alpha - \frac{i\sqrt{E-m^2}}{4\pi}} \frac{e^{i(|x'|+|x|)\sqrt{E-m^2}}}{16\pi^2|x'||x|}. \tag{24}$$

There is evidently a pole at $E = E_0$ such that $\sqrt{E_0 - m^2} = -i4\pi\alpha$. For $\alpha < 0$ this is a real bound state at $E_0 = m^2 - 16\pi^2\alpha^2$ and the wave function ψ_0 of this bound state is obtained by noticing that

$$\mathcal{G} \sim \frac{1}{E_0 - E}\psi_0^*(x')\psi_0(x), \tag{25}$$

for E near the pole E_0. We hence expand (24) about E_0 to find

$$\psi_0(x) = \frac{\sqrt{2(-\alpha)}}{|x|} e^{-4\pi(-\alpha)|x|}. \tag{26}$$

For $\alpha > 0$, on the contrary, the pole is on the 2nd Riemann sheet and hence is a virtual state and does not belong to the spectrum of H. Whether this pole is real or virtual, physically it represents the s-wave scattering over a concentrated attractive potential. $1/\alpha$ is indeed proportional to the scattering length in the s-wave channel [5]. The s-wave is the only contribution surviving in the limit when the scatterer is small compared to the wavelength $1/\sqrt{E - m^2}$.

We have to require the spectrum of H to be contained in the positive real axis for the vacuum $\phi = 0$ of our field theory to be stable. So if $\alpha < 0$ we have to choose $m > 4\pi(-\alpha)$. If $\alpha > 0$ any choice of m, in particular $m = 0$, is enough to ensure the stability of the $\phi = 0$ vacuum.[9]

Considering the case $d = 3$ further, let us now calculate the interaction energy between two identical delta functions with $\alpha_1 = \alpha_2 = \alpha > 0$ (so, according to the previous paragraph, no bound state exists for isolated scatterers) at a distance $|a_1 - a_2| = L$. After the Wick rotation and defining $k \equiv \sqrt{E}$ we obtain [see Fig. 2(a)]

$$\mathcal{E} = \frac{1}{2\pi}\int_0^\infty dk \ln\left(1 - \frac{e^{-2L\sqrt{k^2+m^2}}}{L^2(4\pi\alpha + \sqrt{k^2+m^2})^2}\right). \tag{27}$$

It is not difficult to see that there exists a critical distance L_c, being the positive solution of the equation

$$L_c e^{mL_c} = \frac{1}{4\pi\alpha + m}, \tag{28}$$

such that if $L < L_c$, the argument of the logarithm in Eq. (27) becomes negative for sufficiently small k and we get a negative imaginary part in the Casimir energy. A negative imaginary part of $\mathcal{E}$ means, as usual, an insta-

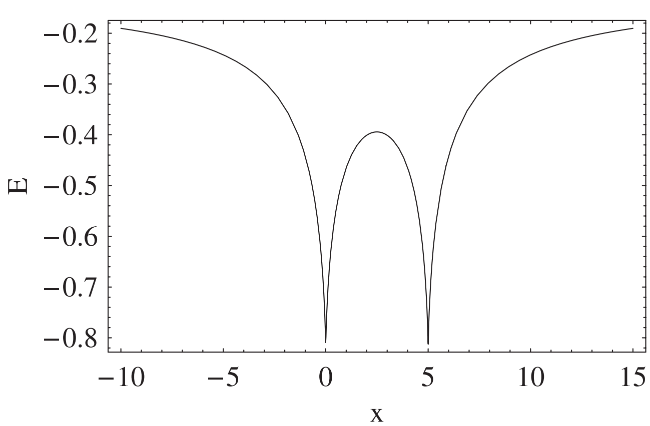
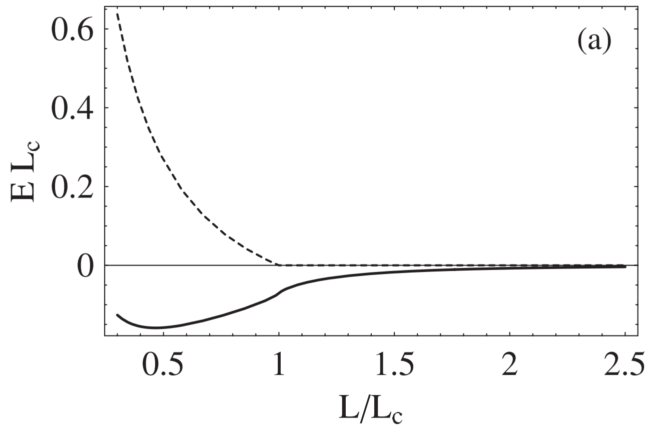


FIG. 1. The interaction energy, in arbitrary units, for three delta functions on the line as a function of the position x of one of them and $m = 0$. $\alpha = -1$ for all three deltas, one delta is held fixed at $x = 0$, another at $x = 5$.

[9] It is worth noting that the opposite choice for the sign of α is needed to avoid a bound state in the $d = 1$ case.



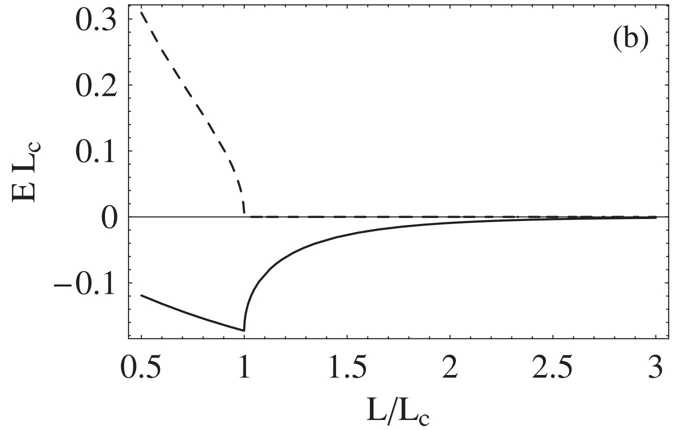


FIG. 2. Interaction energy $\mathcal{E}$ (the continuous line is $\mathrm{Re}\mathcal{E}$ and the dashed line is $-\mathrm{Im}\mathcal{E}$) in units of $1/L_c$, for two delta functions as a function of their distance L. (a) The $\mathbb{R}^3$ case with $m = 0$. (b) The $\mathbb{R}^2$ case with $m/M = 2$.

bility of the $\phi = 0$ vacuum in the presence of the two δ's. In fact, by studying the eigenvalues of the matrix Γ one can see that for $L < L_c$ the spectrum of H has a bound state with negative E, and since $E = \omega^2$ this is a clear indication for the existence of a tachyon. We will return to the implications of this instability for the low-energy physics.

The force $\mathcal{F} \equiv -\partial\mathcal{E}/\partial L$, always attractive and central, diverges logarithmically at the critical length L_c. For $m = 0$ and $(L - L_c)/L_c \ll 1$ one finds

$$\mathcal{F} \simeq -\frac{1}{4\pi L^2} \ln\left(\frac{L_c}{L - L_c}\right). \tag{29}$$

The long-distance behavior, $L \gg L_c$, of the force depends on the mass of the boson ϕ. For $m > 0$ the potential between the two δ's decreases exponentially. For $m = 0$, instead, a power-law tail is obtained:

$$\mathcal{E} \simeq -\frac{L_c^4}{4\pi L^5}. \tag{30}$$

This $1/L^5$ law is stronger than the Casimir-Polder law (induced polarization interaction [18]) which falls like $1/L^7$. This means that we should not think of these delta functions as mimicking polarizable molecules or metallic particles. Indeed, to correctly describe a metallic sphere of radius R, surface Σ, and penetration depth r_0 one should rather assume that $r_0 \ll R$ adding hence to H_0 in Eq. (5) a potential $V(x) = \int_\Sigma d^2y \mu \delta^{(3)}(x - y) = \mu\delta(r - R)$ and send $\mu \to \infty$ *before* sending $R \to 0$. This is clearly a different limit than the one we are describing here.

Now that we have discussed the divergences associated with Γ, its renormalization, and we know about the existence of vacuum instabilities related to negative E bound states of the Hamiltonian H, we are ready to tackle the two dimensional case where all these complications arise at the same time.

The free propagator is $\mathcal{G}_0(x', x; E) = \frac{i}{4} H_0^{(1)}(\sqrt{E - m^2}|x' - x|)$ if $\mathrm{Im}E > 0$. Notice that even for a single delta no choice of α eliminates the bound state. There will always be at least one bound state with energy $E_0 = m^2 - M^2 e^{-4\pi\alpha}$. This is due to the fact that any attractive potential in 2 dimensions has a bound state. We must choose our mass such that $E_0 > 0$ and the instability is not present (it suffices that $m > Me^{-2\pi\alpha}$). However, we will see that in $d = 2$, exactly as in the $d = 3$ case discussed above, in the presence of two or more δ's there exists a critical distance such that for closer approach a bound state has $E < 0$, generating a tachyon again.

Take N pointlike scatterers, each with renormalized strength α_i and a renormalization mass M. It is convenient to define $M_i \equiv Me^{-2\pi\alpha_i}$ so Γ in (13) is

$$\Gamma_{ij} = \left(\frac{1}{2\pi} \ln\frac{\sqrt{E - m^2}}{iM_i}\right)\delta_{ij} - \tilde{\mathcal{G}}_0(a_i, a_j; E). \tag{31}$$

The interaction energy for two identical deltas ($M_1 = M_2 = \mathcal{M}$, and $m > \mathcal{M}$ as required for the stability of isolated scatterers) separated by a distance L is ($k \equiv \sqrt{E}$)

$$\mathcal{E} = \frac{1}{2\pi} \int_0^\infty dk \ln\left(1 - \frac{K_0^2(L\sqrt{k^2 + m^2})}{\ln^2(\sqrt{k^2 + m^2}/\mathcal{M})}\right), \tag{32}$$

where K_0 is a Bessel function K of order 0, and the critical length is the solution of the equation $K_0(mL_c) = \ln m/\mathcal{M}$. The force diverges as $L \to L_c$ in $d = 2$ as well [see Fig. 2(b)], but the explicit expression is more difficult to recover. For $L \gg L_c$ the force is exponentially small, since we had to assume a mass $m > 0$ for the field ϕ.

IV. LOCALIZED VACUUM INSTABILITY

Let us calculate the shape of the tachyon in 3 dimension found in the discussion after Eq. (27) (take $m = 0$). Let us first notice that (for any number of scatterers) the Wick-rotated matrix $\Gamma(-E)$ is real and symmetric and can hence be put in diagonal form. In the case at hand we have only two delta functions with equal strength α and one can show that the spectral decomposition of Γ^{-1} is

$$(\Gamma^{-1})_{ij}(-E,\{a_1,a_2\}) = \frac{1}{\gamma_1} v_i^{(1)} v_j^{(1)} + \frac{1}{\gamma_2} v_i^{(2)} v_j^{(2)} \quad (33)$$

where

$$\gamma_1 = -\frac{e^{-L\sqrt{E}}}{4\pi L} + \frac{\sqrt{E}}{4\pi} + \alpha, \quad (34)$$

$$\gamma_2 = \frac{e^{-L\sqrt{E}}}{4\pi L} + \frac{\sqrt{E}}{4\pi} + \alpha, \quad (35)$$

and

$$v^{(1)} = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \quad (36)$$

$$v^{(2)} = \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}. \quad (37)$$

The bound state pole is generated by a zero E^* in the γ_1 eigenvalue. For $L < L_c = 1/4\pi\alpha$ we have E^* as the real positive solution of the equation $\gamma_1(E) = 0$ (remember: the integration variable E appears as $-E$ in Γ so positive E here are real, negative eigenvalues of H)

$$\sqrt{E} + \frac{1}{L_c} = \frac{e^{-L\sqrt{E}}}{L}. \quad (38)$$

Comparing the behavior of the propagator for E close to a pole E^*

$$\mathcal{G}(x', x; E) \simeq \frac{\psi_0^*(x')\psi_0(x)}{E^* - E} \quad (39)$$

with

$$\mathcal{G}(x', x; E) \simeq \frac{1}{\gamma_1} \sum_{i,j} v_i^{(1)} v_j^{(1)} \mathcal{G}_0(x', a_i; E) \mathcal{G}_0(a_j, x; E) \quad (40)$$

we find the wave function of the (not normalized) bound state as

$$\psi_0(x) = \frac{e^{-\sqrt{E^*}|x-a_1|}}{|x-a_1|} + \frac{e^{-\sqrt{E^*}|x-a_2|}}{|x-a_2|}. \quad (41)$$

Something can be said also in the case in which we have many identical defects (and assume all the a_{ij}'s are of the same order of magnitude), without necessarily having to solve the equations explicitly. Instructed by the previous analysis, we can state that the ground state will be a highly symmetric state $v \sim \{1/\sqrt{N}, ..., 1/\sqrt{N}\}$ which will then give a symmetric wave function $\psi_0(x) \propto \sum_i v_i \mathcal{G}(x, a_i; E^*)$ delocalized over the entire array of defects. The positivity of the v_i's coincides with the constraint that the ground state must not have any node.

Hence, for $L < L_c$, a free field theory coupled to these defects does not make any sense. Its vacuum state $\phi = 0$ is unstable. The imaginary part of the energy (as an analytic continuation to $L < L_c$) is related to the "decay time" of the vacuum state, due to particle creation.

Let us for a moment speculate on the consequences of this instability. Adding higher order terms in ϕ to the Lagrangian—one can, for example, think of adding a $\lambda\phi^4$ term—should eventually stabilize the field with a vacuum expectation value $\overline{\phi}(x) \neq 0$ in a somewhat large region around the two scatterers. However, the actual value of the vev $\overline{\phi}(x)$ and the size and shape of the condensation region cannot be easily constructed and will be the subject of future work.

This scenario of a local condensation and creation of localized vacuum instabilities due to defects could be interesting in inflation cosmology as well (the field ϕ being the inflaton). It must be also remarked that a similar scenario occurs in brane cosmology when an open string has its ends attached to two D-branes [19,20]. When the branes are pushed closer than a critical length one of the modes of the string becomes a tachyon.

V. EXTENSION TO $n - 1$ TRANSVERSE DIMENSIONS

Now that we know the density of states $\rho(E)$ for the "basic" problem of points in 1, 2, and 3 dimensions, we can move along the lines of Table I to generate solutions for manifolds with codimensions 1, 2, and 3. We shall then add $n - 1$ transverse, flat dimensions. The total dimension of the space is now $d + n - 1$. The calculations in the preceding part of this paper can be recovered by putting $n = 1$ in all the formulas. We will use the methods of [12,21] where one solves for the density of states $\rho(E)$ of the basic problem on a d-dimensional section and inserts the result in the equation for the energy per unit $n-1$-dimensional "area" S [21].

$$\mathcal{E}^{(n)} = \int_{\mathbb{R}^{n-1}} \frac{d^{n-1}p}{(2\pi)^{n-1}} \int_0^\infty dE \frac{1}{2}\left(\sqrt{p^2+E} - \sqrt{p^2}\right)\rho(E). \quad (42)$$

The subtraction $-\sqrt{p^2}$ removes a divergent but a-independent term, since the integral $\int dE \rho(E)$ is a-independent. We will also remove the a-independent "self-energy" terms by subtracting from $\rho(E)$ the density $\rho(E, \infty)$ with all $a_{ij} \to \infty$. We can then perform the (dimensionally regularized) integral over p, Wick-rotate, and perform an integration by parts on E to obtain

$$\mathcal{E}^{(n)} = \frac{1}{2\pi} \frac{\Gamma(1-\frac{n}{2}) \sin\frac{n\pi}{2}}{(4\pi)^{n/2}} \int_0^\infty dE E^{n/2-1} \ln\frac{\det\Gamma(-E;\{a\})}{\det\Gamma(-E;\infty)}. \quad (43)$$

For example, the interaction energy (per unit length) of two straight, infinite strings in 3 dimensions ($d = 2, n = 2$ so $d + n - 1 = 3$) put at a distance L is

$$\mathcal{E}^{(2)} = \frac{1}{8\pi}\int_0^\infty dE \ln\left(1 - \frac{K_0^2(L\sqrt{E+m^2})}{\ln^2(\sqrt{E+m^2}/M)}\right). \quad (44)$$

One can hence calculate the interaction energy of any two flat manifolds due to the quantum fluctuations of a bulk field ϕ. As an example in codimension 1 consider the Randall-Sundrum scenario [22] with two branes at a distance r_c from each other. The fluctuations of a given component of the metric G or of a bulk field ϕ (see [23] and references therein), have a space-dependent mass with two delta functions singularities on the two branes. The attractive force due to the quantum fluctuations of this field has a Casimir-like behavior. The curvature in the 5th direction does not change the physics.[10] If, however, the branes have codimension 2 or 3 (and are defined as the limit of an attractive potential) is in the class of problems that we have studied in this paper and a perturbation would eventually condense, if $r_c < L_c$. Following the same arguments above we can also say that if the branes have codimension >4 the fluctuations in the bulk will not see the brane. The cosmological implications of such a scenario will be subject of future work.

VI. OMISSIONS AND APPLICATIONS

The propagator, Eq. (13), comes directly from scattering theory. In that context it was natural to assume that the interaction between the particle and the scatterer (consider the Fermi [2] and Zel'dovich [5] examples) is attractive. One considers an attractive center whose attraction grows when $r_0 \to 0$ such that at most one bound state is present and its energy remains finite [i.e. of $\mathcal{O}(1)$]. Even though we assumed that only one bound state is present at energies of $\mathcal{O}(1)$ this is the most generic situation that can occur in scattering theory. In fact if a second bound state is present, it will be an energy $\mathcal{O}(1/r_0^2)$ below our bound state. In the limit $r_0 \to 0$ its influence on low-energy scattering disappears. It goes out of the spectrum. In scattering theory however, such a negative energy state is harmless. This is not the case for a bosonic field, for which it represents a tachyon.

One may wonder what happens if the potential is *repulsive* and concentrated. The answer is that for $d > 1$ its influence on the scattering matrix (and hence on the spectrum) disappears when $r_0 \to 0$. Obviously this is not true in 1 dimension because we cannot "go around" the scatterer. For a repulsive potential in $d > 1$ the renormalization procedure leading to (13) cannot be performed since μ has the wrong sign and sending $r_0 \to 0$ just kills the correction to $\mathcal{G}_0$ in (13).

More precisely, if in the Lagrangian we include a term $V_0\theta(r_0 - |x|)\phi^2(x)$ with $V_0 > 0$ and then we take the limit $V_0 \to \infty$ and $r_0 \to 0$ with $V_0 r_0^2$ finite, the spectrum we obtain is just the free one: the scatterer disappears. In a sense, the only smile the Cheshire cat can leave behind is the lightly [i.e. $\mathcal{O}(1)$ instead of $\mathcal{O}(1/r_0^2)$] bound (or virtual) state. If this is not present then the scatterer is invisible to the fluctuations.[11] If the purpose of calculating the effective action was to calculate quantum corrections to a classical solution (as often occurs), then we deduce that for a repulsive potential or for $d \geq 4$ the classical solutions are unchanged by quantum fluctuations.

Let us now comment on two possible applications of the formalism we have developed: cosmic strings and concentrated Aharonov-Bohm fluxes. We anticipate that further work is required in both cases. In the literature on cosmic strings the difficulty generated by a bound state tied to a *single* cosmic string has been recognized a long time ago [10,11]. In that context the bound state arising from Eq. (10) is rightly considered fictitious, because the smoothed potential is always positive ($\mu > 0$). Nonetheless, in [10] after projecting out this bound state at $E_0 < 0$, the propagator (13) is trusted and shown to be in good agreement with the numerical solution of the smoothed problem. It is not clear if projecting out a state from the propagator by hand has nontrivial (wrong) consequences on the density of states and the Casimir energy so we preferred not to follow this path even if it gives correct results for other quantities. We hence required the field to have a nonzero mass so that this bound state is stable. In the end it is not clear if the Casimir attraction and the birth of the tachyon could arise in cosmic strings coupled with bulk fields.

Another example to which the above techniques and results should be relevant is the case of a fermion around a concentrated tube of flux (Aharonov-Bohm case).[12] The spectrum of Dirac's equation can be inferred from that of a Klein-Gordon equation after squaring the former. The fact that we are dealing with fermions rather than bosons is not a difficulty. However, another difficulty arises: for Aharonov-Bohm fluxes and the more general case of cosmic strings charged under some U(1) symmetry, it has been shown [25] that the contribution to the scattering cross section given by the nonzero external vector potential is asymptotically larger in the low-energy regime than the contribution of the singularity in the core. Since we believe that cross sections and Casimir forces are tightly bounded

[10] However, If the brane is inside a horizon for the 5d metric, this assertion is most probably not true. But this is not the case for the Randall-Sundrum model.

[11] We have already remarked on the impossibility of considering the Casimir-Polder interaction between small metallic spheres as the interaction of pointlike scatterers the way we construct them here. Small metallic particles of radius r still have a penetration depth $r_0 \ll r$, and so effectively are codimension 1 surfaces.

[12] The one loop energy of QED flux tubes has been calculated in [24] using a combination of analytical and numerical methods. Our method could be used to calculate the interaction energy of two such tubes in the limit where their radius is small compared to their relative distance.

quantities, we would not apply any of the above arguments without treating the propagation in the external space properly. This will be done elsewhere.

Renormalization of branes coupling for a single brane (or δ-function in our case) with codimension 2 (and dimension 5) in a conical space has been studied in [14]. Arising from local divergences, the renormalization flow is not affected by the presence of other branes and the results in [14] apply also to our situation. Their renormalization of the brane coupling μ (λ_2 in their notation) corresponds to our renormalization $\alpha \rightarrow \alpha_r$. Their renormalization of the effective action corresponds to our subtraction of the a-independent terms in the Casimir energy. These two are the only subtractions needed (if ϕ^4 terms are not present) and it is heartening to see that our results coincide with those of [14]. Moreover, one can make an amusing observation if one compares the two approaches to the delta function, the one in terms of scattering (that we used here) and the one in [14] in terms of renormalization group. Notice that the renormalization group flow for μ is IR free and has a Landau pole: the location of the Landau pole coincides with the location of the bound state in our approach.

VII. CONCLUSIONS

We have calculated the force between an arbitrary number of surfaces (branes) with codimension >1 due to the quadratic fluctuations of a boson ϕ living in the bulk. The force turns out to be attractive and it diverges when the distance between the branes approaches a critical value L_c. This phenomenon has no analogues in the widely studied codimension 1 case.

The divergence of the force is accompanied by the birth of a vacuum instability, a mode with negative mass squared localized around the scatterer. In 3 dimensions, the long-range properties of this force (decreasing like $1/L^6$) are shown to be different from the Casimir-Polder $1/L^8$ law, the explanation relying in the proper mathematical definition of the pointlike limit.

Some implications of these effects have been pointed out.

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