# Bridgeland stability conditions on the category of holomorphic triples

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# Contents

# $\mathrm{TCoh}(X)$

Let $X$ be a smooth complex projective variety.

**Definition (Holomorphic triples over $X$)**

$\mathrm{TCoh}(X) = \{(E_1, E_2, \phi) \mid E_1, E_2 \in \mathrm{Coh}(X), \phi \in \mathrm{Hom}(E_1, E_2)\}$.

**Goal**

To study $\mathcal{T}_X$ and $\mathrm{Stab}(\mathcal{T}_X)$, where $\mathcal{T}_X := D^b(\mathrm{TCoh}(X))$.

# Subcategories of $\mathcal{T}_X$

**$D_1 := i_*(D^b(X))$**

$$\begin{aligned} i_* : D^b(X) &\hookrightarrow \mathcal{T}_X \\ E &\mapsto (E \longrightarrow 0) \end{aligned}$$

**$D_2 := j_*(D^b(X))$**

$$\begin{aligned} j_* : D^b(X) &\hookrightarrow \mathcal{T}_X \\ E &\mapsto (0 \longrightarrow E) \end{aligned}$$

**$D_3 := l_*(D^b(X))$**

$$\begin{aligned} l_* : D^b(X) &\hookrightarrow \mathcal{T}_X \\ E &\mapsto (E \xrightarrow{\text{id}} E) \end{aligned}$$

- $D_i$ are admissible.
- $D_2^{\perp} = D_1$, ${}^{\perp}D_2 = D_3$.

# Semiorthogonal decomposition

**Definition (Semiorthogonal decomposition $\langle D_1, D_2 \rangle = \mathcal{T}$)**

*Given $D_1, D_2 \subseteq \mathcal{T}$ full triangulated admissible subcategories such that*

- $\mathrm{Hom}_{\mathcal{T}}(E_2, E_1) = 0$, *for all* $E_1 \in D_1$, $E_2 \in D_2$.
- *The smallest triangulated subcategory containing* $D_1, D_2$ *is* $\mathcal{T}$.

For every $X \in \mathcal{T}$

$$X_2 \longrightarrow X \longrightarrow X_1 \longrightarrow X_2[1],$$

where $X_1 \in D_1, X_2 \in D_2$.
If $\mathcal{A} \subset \mathcal{T}$ is admissible, then

- $\mathcal{T} = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$,
- $\mathcal{T} = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$.

# Semiorthogonal decompositions of $\mathcal{T}_X$

### $\langle D_3, D_1 \rangle$

$$\begin{array}{ccccccc} C(\phi)[-1] & \to & E_1 & \to & E_2 & \to & C(\phi) \\ \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_2 & \to & E_2 & \longrightarrow & 0 \end{array}$$

### $\langle D_1, D_2 \rangle$

$$\begin{array}{ccccccc} 0 & \to & E_1 & \to & E_1 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow \\ E_2 & \to & E_2 & \to & 0 & \to & E_2[1] \end{array}$$

### $\langle D_2, D_3 \rangle$

$$\begin{array}{ccccccc} E_1 & \to & E_1 & \longrightarrow & 0 & \longrightarrow & E_1[1] \\ \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow \\ E_1 & \to & E_2 & \to & C(\phi) & \to & E_1[1] \end{array}$$

# Numerical Grothendieck group of $\mathcal{T}_X$

$$\mathcal{N}(\mathcal{T}_X) = \mathcal{N}(\mathrm{Coh}(X)) \oplus \mathcal{N}(\mathrm{Coh}(X))$$

For a curve $C$,

$$\begin{array}{ccc} \mathcal{N}(\mathcal{T}_C) & \longrightarrow & \mathbb{Z}^4 \\ (E_1, E_2, \phi) & \mapsto & (r_1, d_1, r_2, d_2), \end{array}$$

where $r_i = \mathrm{rk}(E_i), d_i = \deg(E_i)$.

# Serre functor in $\mathcal{T}$

## Definition (Serre functor in $\mathcal{T}$)

*An exact autoequivalence $\mathcal{S}_{\mathcal{T}} : \mathcal{T} \longrightarrow \mathcal{T}$, such that for any $E, F \in \mathcal{T}$.*

$$\mathrm{Hom}_{\mathcal{T}}(E, F) \cong \mathrm{Hom}_{\mathcal{T}}(F, \mathcal{S}(E))^*,$$

*as $\mathbb{C}$-vector spaces and it is functorial in $E$ and $F$.*
$\mathcal{S}_X(E) = E \otimes \omega_X[\dim(X)]$.

## Definition (Fractional CY-category)

*If $\mathcal{S}_{\mathcal{T}}$ exists and there are $p, q \in \mathbb{Z}, q \neq 0$ s.t*

$$\mathcal{S}_{\mathcal{T}}^q = [p].$$

## Proposition

*The category $\mathcal{T}_X$ admits a Serre functor*

$$\mathcal{S}_{\mathcal{T}_X}(E_1, E_2, \phi) = (E_2 \otimes \omega_X[n], \mathrm{C}(\phi) \otimes \omega_X[n], \psi),$$

*with $n = \dim(X)$. If $X$ is a $n$-CY, then $\mathcal{T}_X$ is a fractional CY-category with $p = 3$ and $q = 3n + 1$.*

**Proof:** [Bondal-Kapranov'90]

## Example

$\mathcal{S}_C^3 = [4]$, *where $C$ is an elliptic curve.*
$\mathcal{S}_S^3 = [7]$, *where $S$ is a $K3$ surface.*

# Serre functor and semiorthogonal decompositions

## Remark

$$\mathcal{S}(^{\perp}\mathcal{A}) = \mathcal{A}^{\perp}$$

$$\mathcal{S}_{\mathcal{T}_X}(D_2) = D_3 \quad \text{and} \quad \mathcal{S}_{\mathcal{T}_X}(D_1) = D_2.$$

For $X \in \mathcal{T}_X$,

$$X_2 \longrightarrow X \longrightarrow X_1 \longrightarrow X_2[1],$$

after applying the Serre functor:

$$Y_3 \longrightarrow \mathcal{S}_{\mathcal{T}_X}(X) \longrightarrow Y_2 \longrightarrow Y_3[1].$$

# Stab($\mathcal{Q}_1$)

Let us consider the quiver $Q = \bullet \longrightarrow \bullet$ and $\mathcal{Q}_1 = \text{Rep}(\mathbb{C}Q)$.
**Indecomposable representations**:

| $S_1$ | $S_2$ | $S_3$ |
|---|---|---|
| $\mathbb{C} \to 0$ | $0 \to \mathbb{C}$ | $\mathbb{C} \to \mathbb{C}$ |

**Exceptional collections:**

$$(S_1, S_2) \mid (S_2, S_3) \mid (S_3, S_1)$$

**Remark**

*If $(E_1, E_2)$ is a complete Ext-exceptional collection, i.e. $\text{Hom}^{\leq 0}(E_1, E_2) = 0$, then $\langle E_1, E_2 \rangle$ is a heart of a bounded t-structure.*

$$\mathcal{N}(\mathcal{Q}_1) \cong \mathbb{Z}^2$$

**Serre functor:**

$$\mathcal{S}_{\mathcal{Q}}(E_1 \xrightarrow{\phi} E_2) = (E_2 \xrightarrow{\psi} C(\phi))$$

**Definition**

$\Theta_{ij} := \{\sigma \in \mathrm{Stab}(\mathcal{Q}_1) \colon S_i, S_j \textit{ are } \sigma\textit{-stable}\}$

$$\Theta_{12} \xrightarrow{\mathcal{S}_{\mathcal{Q}}} \Theta_{23}$$

**Theorem (Macrì'07)**

$\mathrm{Stab}(\mathcal{Q}_1) = \Theta_{12} \cup \Theta_{23} \cup \Theta_{13}$ *is a connected and simply connected 2-dimensional complex manifold.*

**Idea:** To use semiorthogonal decompositions instead of exceptional collections.

$$\langle D_1, D_2 \rangle \mid \langle D_2, D_3 \rangle \mid \langle D_3, D_1 \rangle$$

# CP-Gluing

**Lemma (Collins-Polischuck'10)**

*If* $\mathcal{T} = \langle D_1, D_2 \rangle$ *and* $\mathcal{A}_i \subseteq D_i$ *hearts of bounded t-structures on* $D_i$*, such that*

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}}(\mathcal{A}_1, \mathcal{A}_2) = 0.$$

*Then, there is a t-structure on* $\mathcal{T}$ *with heart*

$$\mathrm{gl}(\mathcal{A}_1, \mathcal{A}_2) = \{T \in \mathcal{T} \mid T_1 \in \mathcal{A}_1, T_2 \in \mathcal{A}_2\},$$

*where*

$$T_2 \longrightarrow T \longrightarrow T_1 \longrightarrow T_2[1].$$

# Stability conditions with heart $\mathrm{gl}(\mathcal{A}_1, \mathcal{A}_2)$

Given $\sigma_1 = (Z_1, \mathcal{A}_1), \sigma_2 = (Z_2, \mathcal{A}_2) \in \mathrm{Stab}(X)$, with $\mathrm{Hom}^{\leq 0}_{\mathcal{T}_X}(\mathcal{A}_1, \mathcal{A}_2) = 0$ and $\mathcal{A}_i \subseteq \mathcal{D}_i$ as above.

$$Z(T) = Z_1(T_1) + Z_2(T_2).$$

**Theorem (R-Martínez-Rüffer)**

$$\sigma = (\mathrm{gl}(\mathcal{A}_1, \mathcal{A}_2), Z) \in \mathrm{Stab}(\mathcal{T}_X).$$

- Harder-Narasimhan property.
- Support property.

## Definition

$$\Theta_i = \mathsf{gl}_{ij}\, \widetilde{\mathrm{GL}}^+(2, \mathbb{R}).$$

# Example

If $\sigma_1 = (Z_1, \mathrm{Coh}(C))$ and $\sigma_2 = (Z_2, \mathrm{Coh}(C))$ in $\mathrm{Stab}(C)$,

$$Z_1(r_1, d_1) = -d_1 - \alpha r_1 + r_1 i \quad \text{and} \quad Z_2(r_2, d_2) = -d_2 + r_2 i$$

for $\alpha \in \mathbb{R}$.

**Example ($\alpha$-stability)**

$$(Z, \mathrm{TCoh}(C)) \in \mathrm{Stab}(\mathcal{T}_C),$$

*where*

$$\mathrm{gl}(\mathrm{Coh}(C), \mathrm{Coh}(C)) = \mathrm{TCoh}(C)$$

*and*

$$Z_\alpha(r_1, d_1, r_2, d_2) = -d_1 - d_2 - \alpha r_1 + (r_1 + r_2)i$$

# GKR for $D^b(C)$

**Lemma (Gorodentsev-Kuleshov-Rudakov '04)**

*Given*

$$E \longrightarrow X \longrightarrow A \longrightarrow E[1]$$

*in* $D^b(C)$, $X \in \mathrm{Coh}(C)$ *and* $\mathrm{Hom}^{\leq 0}_{D^b(C)}(E, A) = 0$, *then*

$$E, A \in \mathrm{Coh}(C).$$

# GKR for holomorphic triples

## Lemma

*Given*

$$\begin{array}{ccccccc}
E_1 & \longrightarrow & X & \longrightarrow & A_1 & \longrightarrow & E_1[1] \\
\downarrow{\scriptstyle \varphi_E} & & \downarrow & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \varphi_E[1]} \\
E_2 & \longrightarrow & 0 & \longrightarrow & A_2 & \longrightarrow & E_2[1]
\end{array}$$

*in* $\mathcal{T}_C$ *with* $X \in \mathrm{Coh}(C)$ *and*

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}}(E, A) = 0,$$

*then,* $E_1, A_1 \in \mathrm{Coh}(C)$.

# HN of $\mathcal{L} \longrightarrow 0$

If we assume that $\mathcal{L} \longrightarrow 0$ is not $\sigma$-semistable, then

$$\begin{array}{ccccccc} \mathcal{L} & \to & \mathcal{L} & \longrightarrow & 0 & \longrightarrow & \mathcal{L}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{L} & \to & 0 & \to & \mathcal{L}[1] & \to & \mathcal{L}[1] \end{array}$$

is its HN-filtration.

# Main Theorem

| | $D_1$ | $D_2$ | $D_3$ |
|---|---|---|---|
| $\mathcal{L}_i$ | $\mathcal{L} \to 0$ | $0 \to \mathcal{L}$ | $\mathcal{L} \to \mathcal{L}$ |
| $\mathbb{C}(x)_i$ | $\mathbb{C}(x) \to 0$ | $0 \to \mathbb{C}(x)$ | $\mathbb{C}(x) \to \mathbb{C}(x)$ |

$\Theta_{ij} := \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \colon \mathcal{L}_i, \mathcal{L}_j, \mathbb{C}(x)_i$ and $\mathbb{C}(x)_j$ $\sigma$-stable$\}$, for $i, j \in \{12, 23, 31\}$.

**Theorem (R-Martínez-Rüffer.)**

$$\mathrm{Stab}(\mathcal{T}_C) = \Theta_{12} \cup \Theta_{23} \cup \Theta_{13}$$

*is a connected, simply connected 4-dimensional complex manifold.*

## Lemma

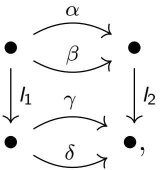
$$\mathcal{E} = \{i_*(\mathcal{O}), i_*(\mathcal{O}(1)), j_*(\mathcal{O})[1], j_*(\mathcal{O}(1))[1]\}$$

*is a strong, full exceptional collection.*

## Lemma

$$\mathcal{T}_{\mathbb{P}^1} \cong D^b(\mathrm{mod}(A)).$$

$A$ is the path algebra of




under some the relations.

# Exceptional collections on $\mathcal{T}_{\mathbb{P}^1}$

$$\mathcal{E}_{k,j} = \{i_*(\mathcal{O}(k)), i_*(\mathcal{O}(k+1)), j_*(\mathcal{O}(j)), j_*(\mathcal{O}(j+1))\}.$$

$$\Theta_{\mathcal{E}_{k,j}} = \bigcup_{\{p \in \mathbb{Z}^4 | \mathcal{E}_{k,j}[p] \text{ is Ext}\}} \Theta'_{\mathcal{E}_{k,j}[p]}.$$

**Lemma**

$$\Theta_1 = \cup_{k,j \in \mathbb{Z}} \Theta_{\mathcal{E}_{k,j}}.$$

$\Theta_{\mathcal{E}_{k,j}} \subseteq \mathrm{Stab}(\mathcal{T}_{\mathbb{P}^1})$ is an open, connected and simply connected 4-dimensional complex submanifold.

# Further remarks

- If $\mathcal{A}_i \subseteq D_i$ hearts without gluing condition, we use

  **Recollement** + [BBD]

  to construct "small" hearts that do not admit a stability function.
- We can generalize the construction for a n-Kronecker quiver over a nice abelian category $\mathcal{A}$.
- $\mathcal{T}_C$ is a "good" triangulated category.

# Thank you!