The goal of this seminar is to study the connections between the space of stability condition and scattering diagrams given in Bridgeland [4]. As a complementary reference we will be reading [3].

**Introduction**

Scattering diagrams are collections of codimension one cones (walls) in a vector space together with attached functions constructed from combinatorial data. They were introduced by Kontsevich and Soibelman in [17] to study families of K3 surfaces and by Gross and Siebert in [10] to study mirror symmetry for toric varieties.

A quiver $Q$ is an oriented graph and a mutation is an operation on quivers. The scattering diagram associated to $Q$ (the cluster scattering diagram) encodes in a consistent way the quantum dilogarithm relations associated to $Q$ and the wall-crossing structure given by quiver mutations. In [9], they use the fact that the functions attached to walls have positive coefficients [9, Theorem 1.3] to prove several conjectures on cluster algebras.

For each quiver $Q$ with a potential $W$ (a linear combination of cyclic paths), we can assign a CY$_3$ triangulated category $D(Q,W)$ that “categorifies” quiver mutations as follows: if $(Q',W')$ is a quiver mutation of $(Q,W)$, then $D(Q,W)$ is equivalent to $D(Q',W')$. See [14, Theorem 3.2].

In [4], Bridgeland defines the stability scattering diagram, a scattering diagram associated to the space of stability conditions $Stab(D(Q,W))$. The primary purpose of [4] is to compare the cluster scattering diagram, a combinatorial construction and the stability scattering diagram, an algebro-geometric construction.

The main goal of our seminar is to prove the following theorem.

**Theorem 0.1.** [4, Theorem 1.5] If $Q$ is acyclic (and hence $W = 0$), then the stability scattering diagram for $(Q,W)$ is equivalent to the cluster scattering diagram.

The stability scattering diagram is defined on the motivic Hall algebra. A Hall algebra of an abelian category $\mathcal{A}$ encodes all extensions between objects in $\mathcal{A}$ and can be thought of as the algebra of finitely supported functions on the moduli space of objects of $\mathcal{A}$. They have been used to compute cohomology of moduli spaces and DT-invariants. See [22], [18].

Joyce in [12] introduced the motivic Hall algebra. In the case of CY$_3$ categories, it is possible to define a Poisson algebra morphism, the so-called integration map, from the motivic Hall algebra to a power series ring. Quoting Nagao in [21]: simple categorical statements (e.g. given by HN-filtration with respect to a stability condition) provide identities in the motivic Hall algebra. Pushing them out by the integration map we get power series identities. See [5] and [3]. Moreover by varying the stability condition in $Stab(D(Q,W))$ and comparing the identities above, we obtain wall-crossing formulas. We finally apply the integration map to stability scattering diagram to produce a scattering diagram in a much simpler Lie algebra.
Plan of the talks

1. Scattering diagrams and the reconstruction theorem of Kontsevich–Soibelman.
The aim of this talk is to introduce the definition of scattering diagram and to prove the reconstruction theorem of Kontsevich–Soibelman.
State the equivalence between finite dimensional nilpotent Lie algebras $g$ and unipotent algebraic groups $G$ as in [20, Theorem 14.27]. Define scattering diagram [4, Definition 2.3] and give their basic properties. See [4, Section 2]. Prove the reconstruction theorem of Kontsevich-Soibelman i.e. that the equivalence classes of consistent scattering diagrams are in bijection with the elements of $\hat{G}$ [4, Proposition 3.4]. See also [9, Theorem 1.17]. Give some examples of scattering diagrams.

References: [4, Section 2 and 3], [9, Section 1].

2. Moduli of representations of quivers and first examples of scattering diagrams.
The first goal of this talk is to introduce quivers, the abelian category of representation of quivers, to study King’s $\theta$-stability and to construct the corresponding moduli spaces of semistable objects via GIT. The second goal is to give the first examples of cluster scattering diagrams.

First introduce quivers and quiver representations. Study [15, Section 1-5] to prove [15, Theorem 4.3, Proposition 5.2 and 5.3]. Then define a potential and the Jacobi algebra a quiver $Q$ with a potential $W$. Introduce cluster scattering diagrams [4, Section 1.7]. State and explain the statement of [4, Theorem 1.1] and study in detail [4, Section 1.3].

References: [15, Section 1-5], [4, Section 1.2, 1.3 and 1.7]

3. Introduction to stacks.
The goal of this talk is to give an introduction to stacks.
Start by studying different moduli problems to motivate the definition of a stack. See for example [8, Section 2.1]. Then, study briefly Grothendieck topologies, sheaves and algebraic spaces. See [8, Appendix A] or [16, Section I.1-I.3]. Afterwards, define 2-categories and 2-functors as in [8, Appendix B]. Define a stack as a sheaf of groupoids [8, Definition 2.10] and as a category fibered in groupoids [8, Definition 2.15]. Explain some properties of stacks as being of finite type, separated and proper as in [8, Section 2.5] (Skip the valuative criteria).

Then, define an Artin stack [8, Definition 2.22] and explain with details [8, Example 2.24 and 2.25]. Finally explain fiber products of stacks as in [3, Appendix A.1].

References: [8], [16].

4. Representation of quivers
Let $(Q, I)$ be a quiver with relations. The goal of this talk is to prove that the fibered category $M$ over the category of schemes define in [4, Section 4] is an algebraic stack of finite type over $\mathbb{C}$ with affine stabilizers that parametrizes all objects of $\text{rep}(Q, I)$.

Recall the definition of the abelian category $\text{rep}(Q, I)$ of representation of a quiver with relations and prove that $M$ is a stack that parametrizes all objects of $\text{rep}(Q, I)$.

Define a stack (locally of finite type over $\mathbb{C}$) with affine stabilizer [3, Definition 3.4], give $\text{Rep}(Q, I)$ as an example and state [3, Proposition 3.5]. This condition will allow us to define the Poincaré invariant. See [5, Section 3.1]. Introduce the stack of short exact sequences and explain diagram (9) in [4, Section 4.5]. Finally prove [4, Lemma 4.2] and explain that the morphism $(a_1, a_2)$ is not representable.

References: [4, Section 4]
Additional references: [3, Section 4.1]

5. Motivic Hall algebra.
The goal of this talk is to define the motivic Hall algebra in order to define a suitable Lie algebra where the stability scattering diagram is going to be defined.
First define a Zariski fibration of algebraic stacks [3, Definition 3.3]. Then, give the definition of the relative Grothendieck group of a stack $K(\text{St}/\mathcal{M})$ [4, Definition 5.1]. Explain the structure of as a $\mathcal{K}(\text{St}/\mathbb{C})$-module and define $\mathcal{K}_\gamma(\text{St}/\mathcal{M})$. After defining the convolution product [3, Section 4.2], adapt the proof of [3, Theorem 4.3] to our case to show that $\mathcal{K}_\gamma(\text{St}/\mathcal{M})$ is an associative algebra over $\mathbb{C}(t)$.

We denote the resulting algebra as $H(Q,I)$. Introduce the completion of the motivic Hall algebra [4, Section 5.7] and define the subalgebra of regular elements, which roughly speaking correspond to varieties. After computing the semi-classical limit define the semi-classical Hall commutative algebra and the Poisson bracket induced on it. Sketch the proof of [4, Theorem 5.2]. Finally, pass to the completion and define the Lie algebra $\hat{g}_{\text{reg}}$ and its corresponding pro-unipotent group. State Joyce’s Theorem [4, Theorem 5.3], that asserts that in fact there is an element in $\hat{g}_{\text{reg}}$ corresponding to the open substack of semistable objects.

References: [4, Section 5] and [3, Section 4].

6. The Hall algebra scattering diagram. The aim of this talk is to define a scattering diagram associated to $\mathcal{M}$ and to show its relation with King’s stability conditions. Briefly review $\theta$-semi-stability for representations quivers with relations [4, Section 6.1 and 6.2]. Define $1_{\text{ss}}(\theta) \in \hat{G}_{\text{Hall}}$. The Harder-Narasimhan filtration allows to give an important identity given in $\hat{G}_{\text{Hall}}$ [4, Proposition 6.4]. Follow [4, Section 3] to prove [4, Theorem 6.4], which shows the existence of a scattering diagram $\mathcal{D}$ with values in $\hat{g}_{\text{Hall}}$ and whose walls are in correspondence to weights $\theta$ for which there are non-zero $\theta$-semistable representations. Explain in detail [6, Example 3.10]. If times permits prove [4, Lemma 6.6].

References: [4, Section 6]

7. Review on Bridgeland stability conditions and nearby stability conditions

The first part of the talk concerns Bridgeland stability conditions. Let $\mathcal{D}$ be a triangulated category. Define the heart of a bounded t-structure [19, Definition 5.1] and state without proof [19, Lemma 5.2 and Exercise 5.4]. As an example of how to construct a heart of a bounded t-structure from a given one state [19, Lemma 6.3]. Give the definition of a slicing [19, Definition 5.5] and explain the Harder-Narasimhan filtration for each object. Give the definition of a Bridgeland stability condition $\sigma = (Z,\mathcal{P})$ as in [19, Definition 5.8]. Define $\mathcal{P}(I)$ for an interval $I$. Explain that $\mathcal{P}(0,1)$ is the heart of a bounded t-structure. Define $\text{Stab}(\mathcal{D})$. State [19, Theorem 5.15] and explain in detail [19, Remark 5.14] and [19, Example 5.12].

Let $(Q,W)$ be a quiver with potential, potential, review the definition of the bounded derived category $\mathcal{D}(Q,W)$ of the Ginzburg dg-algebra of $(Q,W)$. Briefly define a Serre functor as in [11, Definition 1.28] and definite a $n$-CY category. The triangulated category $\mathcal{D}(Q,W)$ is particularly important for us because it is a CY 3 triangulated category and moreover $\text{rep}(Q,W)$ is the heart of a bounded t-structure in $\mathcal{D}(Q,W)$.

The second part of the talk concerns the walls of the Hall scattering diagram and its relation with the space of stability conditions i.e. to prove [4, Theorem 1.2]. Introduce walls of type II as in [4, Section 1.4]. Introduce nearby stability condition [4, Definition 7.2] and prove [4, Proposition 7.4].

References: [4, Section 7], [19, Section 5], [11].

Additional references: [2], [21, Section 4].

8. Framed representations. The aim of this talk is to study framed representations and its stacks. Framed representation of quivers has been used (as the analogue of stable pairs) to produce smooth models as in [7] and to give a factorization formulas related to wall-crossing formulas of DT type invariants, see for example [23, Theorem 2.1]. Our
goal is to describe the stability scattering diagram using Euler numbers of moduli space of framed quivers. First motivate the use of framed representations of quivers. See for example [13, Section 7.4], [23] and [7]. Cover [4, Section 8 and 9]. Focus on [4, Lemma 8.2], [4, Lemma 9.3 and Lemma 9.4].

References: [4, Section 8 and 9]

9. **Theta functions.** The aim of this talk is to define the theta functions given by a scattering diagram in terms of generating functions for Euler characteristic of framed quiver moduli. Motivate the introduction of theta functions and briefly explain their role in [9]. Prove in detail [4, Theorem 10.1] and [4, Theorem 10.2].

References: [4, Section 10]

10. **Stability scattering diagram of \((Q, W)\): the \(\text{CY}_3\) case.** The aim of this talk is to define the stability scattering diagram of the pair \((Q, W)\) and to prove the Theorem 0.1. Define the integration map and prove that it induces a homomorphism of graded Poisson algebras as in [4, Lemma 11.5] and [3, Theorem 5.2]. Define the stability scattering diagram and prove the main theorem as in [4, Lemma 11.5].

References: [4, Section 11]

11. **Scattering diagrams, stability conditions, and coherent sheaves on \(\mathbb{P}^2\).**

References: [1]

**References**


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