Complexity and hydrodynamic turbulence

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Physics is a seamless dialogue with Nature that transcends the divisions made by practising physicists—often for convenience, often imposed by their own limitations. It is difficult to sustain the notion that one branch of physics can satisfactorily answer all the questions of Nature. I expand on this theme briefly, and also technically, using the example of hydrodynamic turbulence.

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1. Introduction

At the Symposium I reported the properties of cryogenic turbulence, both above and below the lambda-point. Much of that work has been published[1], so it is hard to justify another account of the same work. Since it is not especially useful to write for these proceedings a paper on specialized developments that have occurred since then on the subject of my talk, I have chosen to set down a few general remarks on the physics that is close to my interests, and add a few technical details to justify them.

Physics is about intelligent and rigorous dialogue with Nature; the aim is to discover exact relationships between measurable quantities, and the preferred language is mathematical. Agreement with experiments is essential but not in the way a lay person might imagine. Newton's first law, for example, is evidently not satisfied in our everyday experience (bodies set in uniform motion do not stay in uniform motion forever), but one can seek the confirmation of the law in controlled circumstances. By successively decreasing the friction between a moving body and its surroundings, one can presumably observe the body to remain in motion for longer and longer periods of time, leading us to infer that Newton's first law is correct in the absence of friction.

Within the broad definition of physics mentioned above, its different branches may have different characteristic approaches. One common element, however, is the goal of discovering general laws that underlie Nature's myriad manifestations. Questions about these general laws have led 20th century physics to inquire into the structure of increasingly smaller constituents of matter[2], culminating in modern particle physics and its more recent developments such as superstrings.

The reductionist approach assumes that the world around us can be understood in terms of the properties of a few small building blocks (particles or strings), and that everything can be understood from them, or the "first principles". The approach has worked well in some instances—for example in characterizing the structure of the hydrogen atom from quantum mechanics—but this "first principles" understanding is not quantitative for anything but the simplest of problems. No one has yet deduced the structure of macromolecules using the principles that were successful with respect to the hydrogen atom. In general, one cannot connect hierarchies of physical description without some empirical input; no effort has been made, for instance, to compute the structure of organic molecules from particle physics, and to go from these molecules to cells and living organisms. The question is not simply one of logistics but one of principle: it is not clear, even in principle, if one can understand the collective motion of a large number of nonlinearly interacting scales from this bottom-up approach. Perhaps more provocatively, it is not even clear if the knowledge organized at one level is essential for a deep understanding of the knowledge at a distantly coarser level.

Are there different fundamental laws that are applicable at different levels of description, with broad generality of their own? For instance, is self-organization, which is characteristic of living systems (among others), subject to some fundamental laws that one cannot deduce even *in principle* from particle physics?

A large body of scientists today thinks that reductionism, the cornerstone of much of the 20th century physics, has limitations of principle despite its enormous successes; that a deductive link does not exist between the finest constituents of matter and phenomena that occur on the human scale; that one needs an equally deep understanding of the so-called emergent phenomena regulated by different organizing principles ("the psychology of the lynch mob vanishes when you interview individual participants"[3]); that these organizing principles are equally deep in content and structure; that a knowledge of the constituent parts, however complete, cannot describe the whole. Perhaps it is too much to say that physics at the turn of the 21st century is undergoing a crisis similar to that at the turn of the last, but physics is changing its landscape.

As mentioned already, physics looks for generality of laws. The question is whether there are any such laws at all that describe a broad class of systems with many nonlinearly interacting degrees of freedom. Here, I pose the question at the quantitative level of rigor, not simply at a qualitative one; for instance, it is not enough that daily variations in financial markets *look* closely similar to gradients of the velocity field in a turbulent flow. If quantitative laws do cut across seemingly unrelated problems, in what form will they appear? Since it is known that nonlinear systems cannot be predicted except for short periods of time, what form should comparisons with experiment take? These questions are not new; equilibrium statistical mechanics faced them in a certain way more than 100 years ago. The domain of our consideration now is much broader and the urgency for the answer far greater, because a multitude of problems in Nature that spill out of the reductionist box are known not to obey the laws of equilibrium statistical mechanics.

2. The case for hydrodynamic turbulence

If we are interested in discovering laws underlying systems with many strongly interacting degrees of freedom and are far from equilibrium, it is important to begin with a study a few of them with the same rigor and control for which particle physics, say, is well known. Here, I make the case that hydrodynamic turbulence, which arises in flowing fluids, is an ideal paradigm. My first point is that the dynamical equations for the motion of fluids are known to great accuracy, which means that understanding their analytic structure can greatly supplement experimental queries; in just the same way, computer simulations—even if they require much investment of time and money—can be far more useful here than for many other problems of the condensed phase, in which the interaction potential among microscopic parts is often simply an educated guess. When driven hard, fluids develop irregular motion in time and space—this being the state of turbulence. The stochasticity of turbulence (and of all systems that are driven hard) means that one may discern only laws that concern statistical behavior. If we are fortunate, these laws are universal in some well-understood sense. This is the aspect I wish to address briefly.

To be specific, consider a jet of fluid emerging from an orifice. It is immaterial if the fluid is air (emerging into air medium), water (emerging into water medium) or mercury (emerging into mercury medium)—already a great generalization. Let the jet velocity be V, the orifice diameter D and fluid viscosity ν. The Reynolds numbers $Re = VD/\nu$ should be "high enough" for our considerations to hold. The requirement of high Reynolds number is no more constraining than one on the energy levels needed to study elementary particles. Some forcing is essential to maintain the flow steady, else it will grind to a halt eventually because of dissipation. The forcing in this instance is provided by the momentum at the jet orifice; typically the scale of forcing is of the order D, but not exactly so. Let us call it L.

Further from the orifice there appear a range of motions much smaller that L juxtaposed randomly in space and time. These scales are produced by the nonlinearity of the equations of motion. It is believed that all important properties of turbulence are contained in the Navier-Stokes equations for the fluid motion which can be written for unit density as

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mathbf{f} + \nu \Delta \mathbf{v}\,, \qquad (1)$$

supplemented by the incompressibility condition $\nabla \cdot \mathbf{v} = 0$. Here $\mathbf{v}$ is the velocity vector, p is the local pressure, f is the forcing and the suffix t stands for time-derivative. In (1), f is a stochastic forcing term that suitably abstracts the mechanism for sustaining turbulence.

The production of smaller and smaller scales is supposed to proceed in a cascade-like process[4]. The process is operational until viscous dissipation truncates it at a characteristic scale η (which is proportional to the 3/4-power of ν) that is much smaller than L. The scales between L and η, which remain unaffected directly by forcing as well as viscosity, or by L as well as η, form the so-called inertial range. The kinetic energy injected at scales of the order L is transferred through the inertial range and dissipated eventually at scales of the order η. Indeed, nonlinear energy transfer process is a dominant feature in the inertial range.

Because neither L nor η is relevant in the inertial range, it is natural to ask if some attributes of the inertial range are universal. Another question of the universality problem concerns features that are common to different non-equilibrium systems. The quest for universality is motivated by the hope of identifying some general principles that govern far-from-equilibrium systems, just as variational principles govern thermal equilibrium. As already mentioned, the dynamical features of turbulence are irregular, and so the questions can be posed and answered only in terms of statistical averages—computed for appropriate regions of space, intervals of time, or suitably defined ensembles.

I wish to emphasize that there are many other interesting and important problems in turbulence, especially when linked to practical applications.

3. The nature of the inertial range

The universality of the inertial-range is studied by constructing the so-called structure functions, which are moments of velocity differences δv_r across a scale of size r. Structure functions of order n are defined as $S_n \equiv \left\langle [(\mathbf{v}(\mathbf{r},t) - \mathbf{v}(\mathbf{0},t)) \cdot \mathbf{r}/r]^n \right\rangle \equiv \left\langle (\delta v_r)^n \right\rangle$, with $r \equiv |\mathbf{r}|$. The angular brackets indicate a suitable average. Generalization of structure functions, which we will not consider here, are statistical correlations of velocity differences at the vertices of more complex configurations such as triangles, quadrilaterals, and so forth. The study of correlators of different types is thus a study of different geometrical configurations.

The third-order structure function in the inertial range of scales can be related to the energy flux by an exact flux-constancy relation derived by Kolmogorov[5] from (1):

$$S_3 \equiv -\frac{4}{5}\,\langle\epsilon\rangle\, r\ . \tag{2}$$

This is known as the 4/5-ths law. Here $\langle\epsilon\rangle = \nu\langle|\nabla\mathbf{v}|^2\rangle$ is the average rate of energy dissipation per unit mass that can be identified with the energy flux over scales.

A few comments are in order. The above result for the inertial range is a direct consequence of the dynamical equations without any hypothesis. The only requirement is that the Reynolds number must be "sufficiently high"; further increase of the Reynolds number merely extends the inertial range without altering its scaling. Lest this result be thought of as merely mathematical, it should be stressed that measurements in real flows do indeed obey the 4/5-ths law; see, e.g.,[6]. The interesting point is that the law is valid for turbulence in all fluids, no matter what their molecular structure: Water, mercury, nitrogen and liquid helium all follow the same law. The nuclear structure of the molecules or atoms making up the fluid makes no difference. Indeed, one can say that the atomic properties and the behavior of turbulence are disconnected, in so far as we are concerned with this law. It is thus impossible to say that one can work up towards this law from the bottom, except in some generic fashion.

No one has been able to deduce theoretically moments of other orders. Kolmogorov[5] assumed scale invariance and obtained a tidy result. Scale invariance means that the symmetry broken by large-scale forcing is restored when turbulence scales become small compared to the forcing scale, L. Thus, the structure functions are thought to be universal power laws containing neither L nor η. This picture readily yields exponents ζ_n that depend linearly on the moment order, or $S_n \propto r^{\zeta_n}$ with $\zeta_n = n/3$.

4. Dissipation anomaly and anomalous scaling

The modern experimental evidence[7] is that the exponents depart from the scale-invariant result, and the forcing scale L appears explicitly in characterizing the inertial range. This non-restoration of scale invariance even at small scales, is arguably an important feature of turbulence, and sets it apart from the usual critical phenomena[8]: One needs to work out the behavior of each order moment independently without succumbing to dimensional analysis. Anomalous scaling in turbulence is such that $\zeta_{2n} < n\zeta_2$ so that S_{2n}/S_2^n for $n > 2$ increases as $r \to 0$. Relative growth of high moments means that strong fluctuations become more probable as the scales become smaller. One practical consequence is that it limits our ability to produce realistic models for small-scale turbulence.

We have already mentioned that scale invariance yields the explicit expression $\zeta_n = n/3$ for the scaling exponents. However, if scale invariance is violated in fluid turbulence, what else can be used to determine the scaling exponents? Conservation laws impose constraints on the dynamics, and so conserved quantities (or their fluxes) play an essential role in addressing this question. Conservation laws are broken in fluid turbulence by the large-scale forcing (usually through boundary conditions) as well as the dissipation at small scales (usually through fluid viscosity). Indeed, the basic dynamics of the inertial range is not the conservation of energy (whose application gives a drastically unrealistic result of equipartition) but of the energy flux across scales from the large to the small. In the steady state, this flux of energy equals the dissipation of the turbulent kinetic energy at small scales of the order η. For the first sight, it might appear from the definition of $\langle\epsilon\rangle$ below equation (2) that the dissipation rate of turbulent energy would vanish as $\nu \to 0$ (or as $Re \to \infty$), but an important feature of turbulence is that $\langle\epsilon\rangle$ remains finite in this limit: no matter how wide is the scale-range participating in the energy cascade, it carries the same flux.

The finiteness of energy dissipation even in the limit of vanishing viscosity is probably the first example of what is called "anomaly" in modern field-theoretical language: A symmetry of the inviscid equation (here, time-reversal invariance) remains broken even as the symmetry-breaking factor (viscosity) vanishes[9]. Recall from equation (2) that the third-order statistics of turbulent velocity increments are determined completely by $\langle\epsilon\rangle$, whose finiteness is a consequence of the breakdown of time-reversal symmetry in the inertial range. Indeed, velocity changes sign under time reversal so nonzero S_3 means time-irreversibility. Just as conservation of the kinetic energy is broken by viscosity (no matter how small), and this broken symmetry governs the third-order structure function, are there other candidate integrals of motion whose broken symmetries yield other structure functions? This is a fundamental question of turbulence—indeed, in a suitable analogous fashion, of modern statistical physics as a whole.

5. Statistical conservation laws

I shall now briefly discuss laws which are conserved only on the average, yet determine the statistical properties of strongly fluctuating systems. Since it is always possible to find some averages that do not change in random systems, our question should be posed better: Is it possible to find quantities that are expected to change on dimensional grounds yet remain constant because they are governed by some statistical conservation laws? How to build specific functions that are conserved on the average, and how to determine the scaling exponents from them? As fluid particles move in a random flow, an n-particle cloud disperses and grows in size while fluctuations in the shape of the cloud decrease in magnitude. Therefore, one may look for suitable functions of size and shape that are conserved because the growth of distances is compensated by the decrease of shape fluctuations.

For the simplest case of Brownian diffusion, the time derivative of the mean of any function of distances between particles is the Laplacian of this function. Harmonic polynomials turn Laplacian into zero (and can thus be called zero modes of Laplacian), and so are conserved on the average. For Laplacian diffusion, the zero modes are polynomials in R^2 or t so that the scaling dimension of the n-particle mode is $\zeta_n = 2n$. The dependence on n is linear because the particles move independently.

The zero modes governing the decrease of shape fluctuations exist for turbulent diffusion as well. The question of how the statistical conservation laws lead to anomalous scaling has been analyzed quantitatively for a model problem of advection of a passive scalar[10]. A scalar is a quantity such as temperature or concentration of an admixture that is advected by the turbulent velocity field. The scalar is passive if it does not affect the velocity field advecting it. For instance, if the scalar is the temperature, the heating must be small enough so as not to affect the dynamics. The model of Ref. [10] considers the case of a passive scalar field that is advected by a Gaussian velocity field that oscillates infinitely rapidly (the auto-correlation is a delta function); however, its spatial correlation follows the power-law characteristic of the real velocity field. For this model, it is shown that the anomaly arises from the statistical conservation laws of geometric origin, not by the dynamical conservation laws. The statistical conservation laws break the scale invariance in the inertial range and scalar turbulence "knows more" about forcing than just the value of the flux. For a summary of the important developments on this problem, see Ref. [11].

6. Closing remarks

My main point here is that a number of systems of great practical interest operate under conditions that are far from equilibrium and develop seemingly common properties. It is unlikely that these properties can be explained from the finest ingredients of matter. It would be most interesting if these properties possess quantitative universality but this quest has not yet been addressed satisfactorily. The first step in this direction requires that we study at least a few such systems precisely, and understand their structure well. Hydrodynamic turbulence is one of the best examples in this category.

To make the case further, I discussed the anomalous scaling of the inertial range in turbulence. The anomaly arises because the scale invariance is broken. Under such circumstances, the scaling exponents are determined by new types of statistical conservation laws. The solutions of the Kraichnan model[10] have elucidated this point quantitatively. The conserved quantities involve the geometry of multi-point configurations carried by the flow. It is possible that this lesson is of universal validity, i.e. for other nonlinear multiscale systems which possess statistical conservation laws and anomalous scaling exponents.

A postscript may be appropriate. In part through direct and indirect influence of Professor Abdus Salam, the founding director of ICTP, Pakistan has had a few excellent physicists in particle physics and related fields.

However, it has not developed a similarly strong tradition in the science of non-equilibrium and nonlinear physics. It is indeed high time that this development occurred. My hope is that the National Centre for Physics in Islamabad, which hosted this Symposium, will be able to develop and foster such an activity.

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