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A Memory Model for Seasonal Variations of Temperature in Mid-Latitudes

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To Larry Sirovich, on the occasion of his 70th birthday.

ABSTRACT The Earth receives, on the average, the largest amount of radiation from the Sun on summer solstice (June 21), which is the longest day of the year. However, the warmest day occurs usually later; the time lag is about a month for 50 deg latitude, and decreases with increasing latitude. There is comparable time lag between the shortest day of the year (winter solstice, December 21) and the coldest day of the year. We model these and related observations by a linear Maxwell-type viscoelastic model. By comparing predictions of the model with observations, we extract, as functions of the latitude, two free parameters representing the memory and the effective viscosity coefficient. Some interpretation of the results is provided.

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1 Introduction

The yearly seasons are related to Earth's position with respect to the Sun, so it is traditional and seemingly logical to assume that the longest day of the year, which is nominally when the Earth receives the largest amount of radiation from the Sun, is also the warmest. The following quote from Smart [1956] exemplifies this thinking: "...the days increase in warmth from March

21 to June 21 corresponding to the increasing [length of day]...; from the latter date, a decrease ensues." However, the reality is different. A survey of some fifty years of the temperature data for New Haven, CT, shows that the warmest day occurs on July 24 (with a standard deviation of about 10 days), which is 33 days after summer solstice (June 21). The time lag between the coldest day and winter solstice (December 21) is similar. The situation is akin to the commonly known fact that, often, the warmest temperature on a winter day is reached not at noon but a few hours later (see Figure 1.1). It is also of interest to be able to explain the fact that the temperature changes occur most rapidly around late April and October, a month or so after the occurrence of equinoxes (the two days of the year, March 21 and September 21, on which day and night have equal duration).

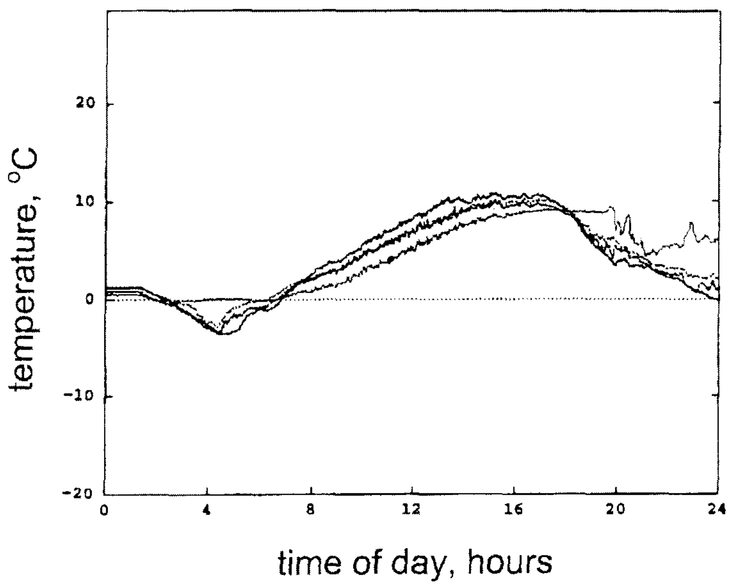

FIGURE 1.1. The variation of temperature as a function of time during a typical winter day. Different curves represent measurements at different places in the same neighbourhood.

A proper explanation of these observations should incorporate detailed modelling of solar radiation received by the Earth, the differential storage and rerelease of this radiation by oceans and the land, cloud coverage, albedo effect, ocean-land interactions, global circulation, and so forth. That would be a major task. Our limited goal here is to model these behaviors in a simple way by subsuming the details by two free parameters in a Maxwell memory model. One of the parameters is a relaxation time (or memory coefficient) and the other an (effective) viscosity coefficient; see, for example, Joseph [1990]. If the model has some value, one should be able to extract these parameters from comparison of its predictions with empirical observations, and to interpret them usefully. We show that it is possible to do so.

Section 2 describes the model briefly while Section 3 summarizes the data analysis. Section 4 is a disscussion of the model, with Section 5 providing a few concluding remarks.

2 The model

The simplest version of the model assumes that one can associate, with a given latitude, a mean temperature for the same day of the year. This is the value of the temperature averaged over many years, all of which correspond to a given day; further, the temperature is averaged for all positions on a given latitude so that the longitude-variations are ignored for this first look. Let us denote this mean temperature by $\theta(\ell, t)$, where ℓ is the latitude and t denotes, within a year, the number of days elapsed since the occurrence of one of two solstices. Let $\langle\theta\rangle$ be the time average of θ, taken over all days in a year, and T denote the difference $\theta - \langle\theta\rangle$. At any latitude, we are interested in the variation of T through the year, and wish to relate it to $L(t) = L^*(t) - \langle L^*\rangle$, where L^* is the length of day at the chosen latitude and $\langle L^*\rangle$, the yearly average of L^*, is approximately 12 hours for all latitudes. The proposed model is

$$\lambda \frac{\partial T}{\partial t} + T = \eta L(t)\,, \tag{2.1}$$

where λ is a relaxation (memory) parameter and η is a viscosity coefficient. The inevitable nonlinearities in the problem are hidden in the two parameters λ and η.

Models bearing some resemblance to equation (2.1) have indeed been proposed for seasonal variations of temperature, but they are overtly nonlinear and more complex; for example, Crowley and North [1991] incorporate ice-albedo feedback and turbulent eddy diffusivity—each of which is the subject of extensive and unfinished study in its own right. We know of no effort identical to ours, carrying out data analysis to the same degree.

3 Data analysis

Data on length of day are available in tabulated form in List [1951] and, in forms more suitable for computer manipulations, in Meeus [1988] and Montenbruck and Pflaeger [1994]. Figure 3.1 shows a plot of $L(t)$, taken from List [1951], for 50 deg latitude in the northern hemisphere over a

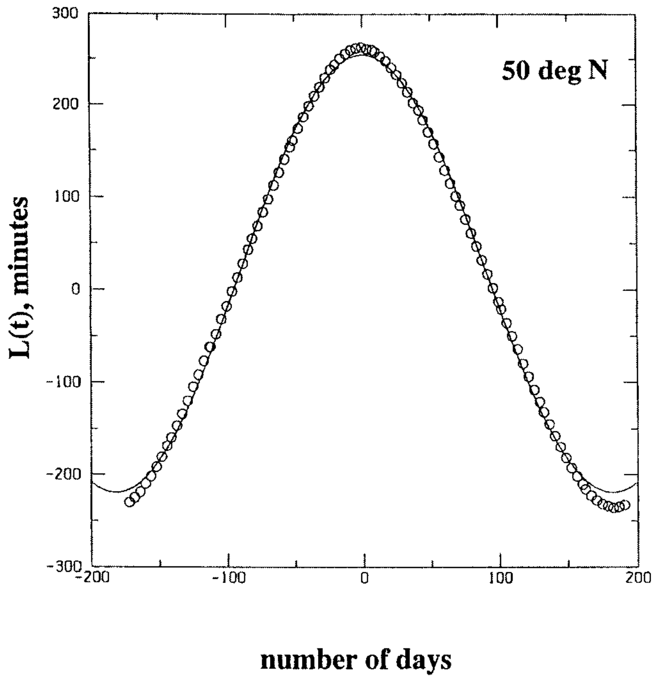

FIGURE 3.1. The variation of the length of day, $L(t)$, at 50 deg latitude, as a function of the number of days counted from the summer solstice. The data are from List [1951]. The cosine term shown by the full line fits the data well. The deviations from the fit can be accommodated by including higher harmonics, but the effort was deemed unnecessary for present purposes.

one-year period, as a function of the number of days from the winter solstice. For simplicity of further analysis, we fit the data in Figure 3.1 by $A\cos(\omega t)$, where A is the amplitude of the daylight variation and the circular frequency $\omega = 2\pi/365$ days^{-1}. This fit is adequate for our purposes; small discrepancies that exist between the fit and the data are not worth emphasizing, given the gross features we wish to understand. Figure 3.2 shows that the amplitude A, as determined from such fits, varies smoothly with the latitude and can be fitted by a simple polynomial.

If $L(t)$ can be fitted by a cosine term, the solution of equation (2.1) is simply

$$T(t) = a\cos\omega t + b\sin\omega t\,, \tag{3.1}$$

where

$$\lambda = \frac{b}{a\omega}\,, \quad \eta = (a + b\lambda\omega)\,. \tag{3.2}$$

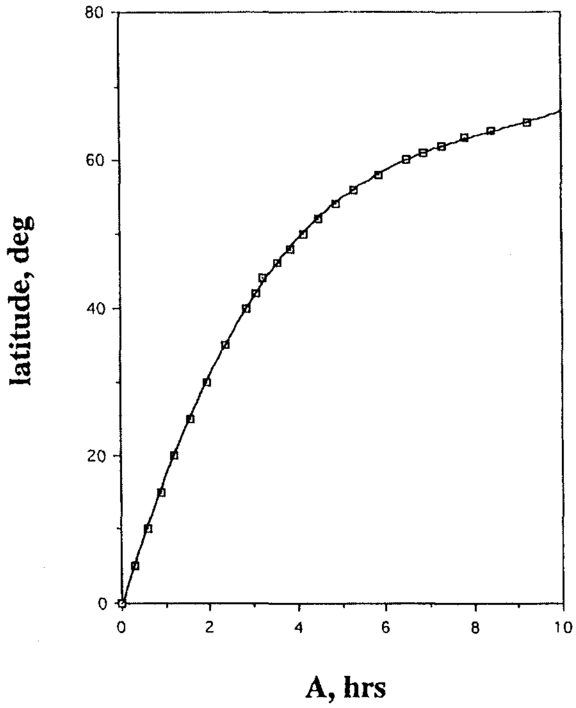

FIGURE 3.2. The variation of the coefficient A as a function of the latitude. The fit to the data (full line) is $l = 19.47A - 2.10A^2 + 0.082A^3$, where the latitude ℓ is expressed in degrees and the amplitude of daylight variations A in hours.

Equivalently, we have

$$T(t) = \frac{A\eta}{\sqrt{1+\lambda^2\omega^2}} \cos\, \omega(t-\phi)\,, \tag{3.3}$$

where

$$\phi = \omega^{-1} \tan^{-1}(\lambda\omega)\,. \tag{3.4}$$

We shall verify if this solution adequately expresses the mean temperature variation for a fixed latitude through the year, and if so, obtain a and b (and thus extract the parameters λ and η or ϕ). This will be done below.

The latitude-averaged temperature data have been compiled in Oort and Rasmuusen [1971] for a five-year period between 1955 and 1960. These data are compiled from observations from a number of weather stations (of the order of a few hundred) in the northern hemisphere. There are more weather stations over land than on water, so the data may be biased in some way. Figure 3.3 shows a comparison of these temperature data with the solution

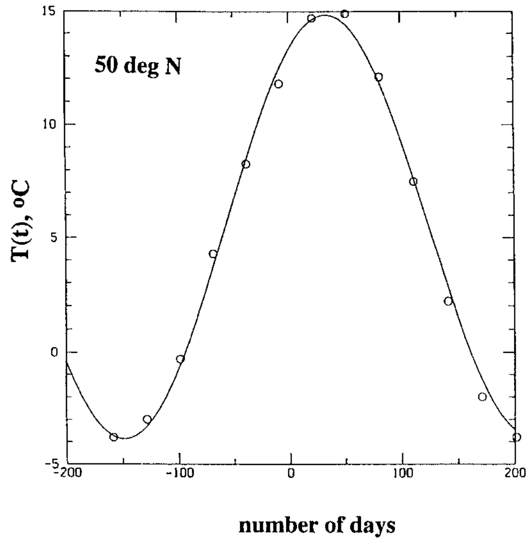

FIGURE 3.3. The variation of T at 50 deg latitude as a function of the number of days, counted with respect to the summer solstice, showing that it can be fitted quite well by equation (3.1). The observational data, taken from Oort and Rasmuusen [1971], were five-year averages. The authors remark that scanty evidence available over a period of some twenty years are substantially the same as the five-year averages, thus implying that reasonable stationarity has been reached. The weather stations are not distributed on a uniform grid on the globe (see, for example, figure 1a and 1b of Oort and Rasmuusen [1971]). This may introduce some unknown bias.

of the model equation. The agreement is good and one has

$$\lambda = 38 \text{ days}, \quad \text{and } \eta = 2.69°/\text{hr}.$$

This gives a phase lag ϕ of about 30 days, consistent with expectations raised earlier. Considering that η represents the increase in temperature per hour of heating by the Sun, a plausible interpretation is that, on the average, an hour of solar heating raises the temperature at 50 deg latitude by about 2.7 K.

In analyzing temperature data for other latitudes, two restrictions should be noted. First, data on length of day become increasingly uncertain for latitudes above 65 deg because small changes in atmospheric reflectivity can cause relatively large changes in daylight. This difficulty restricts useful consideration to lower latitudes (even though the model seems to work for latitudes at least as high as 75 deg). For latitudes below about 25 deg, the relatively small seasonal variations are influenced by a variety of minor effects, none of which—including the length of day—appears to have a particularly dominant influence. Possibilities for improving the model for low altitudes will be mentioned briefly later, but, for now, we restrict attention to the latitude range between 25 deg and 65 deg—which is what we mean by mid-latitudes.

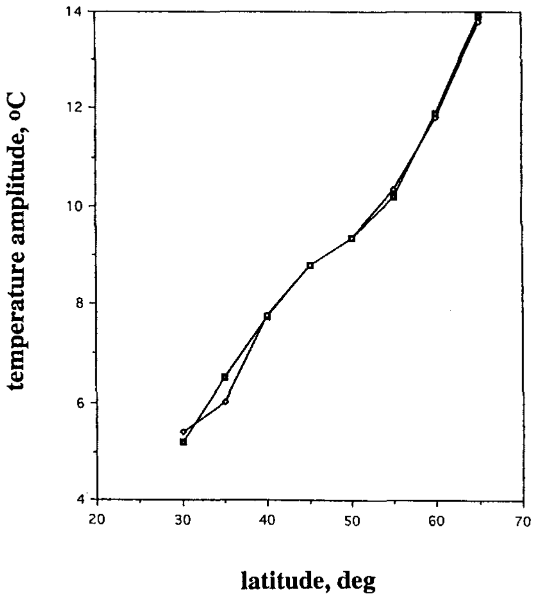

FIGURE 3.4. A comparison between the amplitude of the harmonic fit to the temperature data and the observed amplitude of temperature variation. Their good agreement is a measure of the goodness of the fit to the temperature variations in mid-latitudes.

Our experience is that temperature variations at all mid-latitudes can be fitted well by equation (3.1). A quick feel for the goodness of fit can be had from Figure 3.4, which compares half the difference amplitude of T at various mid-latitudes with the amplitude $(a^2 + b^2)^{\frac{1}{2}}$ obtained by fitting equation (3.1) to the data. The agreement is good, and the data can be approximated crudely by a straight line which intercepts the latitude axis at a finite value of 13 deg (to which no particular attention need be paid because of the limitation of the model for tropical latitudes). One can combine this approximate and empirical formula with equations (3.1) and (3.2) to obtain

$$\frac{\eta A}{\sqrt{1+\lambda^2\omega^2}} \text{ (in } {}^\circ C) \approx 0.23\ell \text{ (in deg)} - 3.06 \text{ (in } {}^\circ C)\,. \qquad (3.5)$$

The interpretation of the numbers on the right-hand-side of the equation is unclear. Equation (3.5) probably possesses no greater significance than the

city	latitude, deg.*	λ, days
Fairbanks, AL	64	20
Anchorage, AL	60	27
Minneapolis, MN	45	30
Burlington, VA	44	27
Boston, MA	42	38
Chicago, IL	42	34
New Haven, CT	41	39
Boulder, CO	40	36
Columbus, OH	40	34
Little Rock, AR	35	47
Jacksonville, FL	30	42
Miami, FL**	26	57

*Latitude rounded off to the nearest degree

** At this latitude, temperature variations are not fitted well by equation (3.1).

TABLE 3.1. The coefficient λ for a few cities in the United States. Despite the scatter, the trend towards larger λ for lower latitudes is unmistakable.

fact that it relates the three parameters λ, η and A. It can be simplified even further by noting that λ/η is approximately constant to within 15%.

A summary of the principal results is given in Figure 3.5 which plots the relaxation parameter λ (or, more usefully, the phase lag ϕ) and viscosity η as functions of the latitude. (Although the data come from the northern hemisphere, the model should hold equally for the southern hemisphere.) The increasing value of the time lag was not expected at the outset. To assuage the reader's skepticism, we plot in Figure 3.6 the temperature variations for latitudes of 30 and 75 deg. It is clear, without any help from the model, that the time lag is smaller for higher latitudes than for lower latitudes. The feature will be explained subsequently.

The data examined so far are averages for a given latitude, but the model works for local areas as well. Figure 3.7 shows the temperature variation through the year for the city of New Haven. Examination of these data for several other American cities shows that equations (3.1)–(3.3) adequately describe the observed temperature variations—although, not unexpectedly, λ varies somewhat between two cities that lie on the same latitude (see Table 3.1). The general trend, however, is quite similar to that for longitude-averaged data.

We now note three other aspects briefly:

a. *Temperature variations at different heights:* We have examined the temperature data at several altitudes from the ground. There are only minor variations with respect to height, at least until the tropopause is reached.

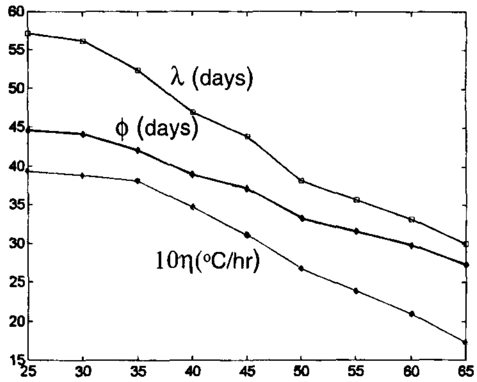

FIGURE 3.5. The parameters λ and η as functions of the latitude. Note the scale change between the two parameters. All data correspond to a pressure altitude of 1000 mb.

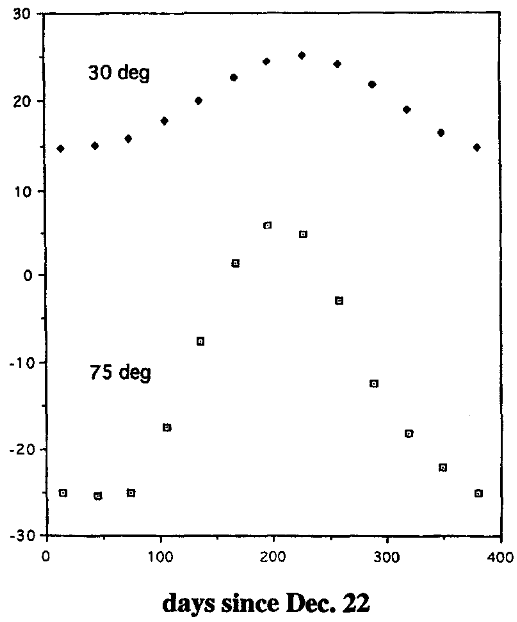

FIGURE 3.6. A comparison of the temperature variations at two substantially different latitudes. The data show without ambiguity that the warmest day occurs later in the year at lower latitudes than at higher latitudes.

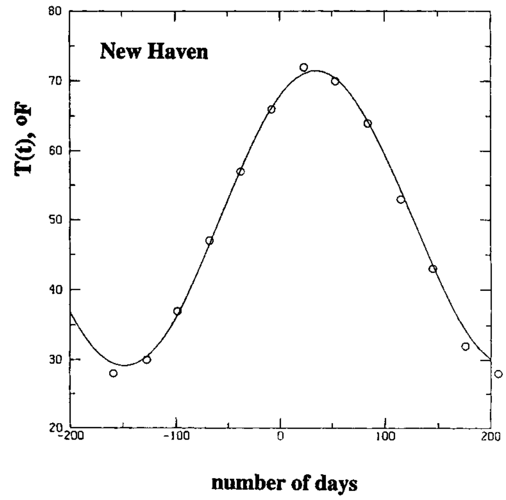

FIGURE 3.7. The variation of $T(t)$ for the city of New Haven as a function of the number of days, counted with respect to summer solstice, showing that it can be fitted well by equation (3.1). The observational data are from Miller and Thompson [1979] obtained over several (but unknown number of) years. A twelve-year average has a standard deviation of the order of 4°C.

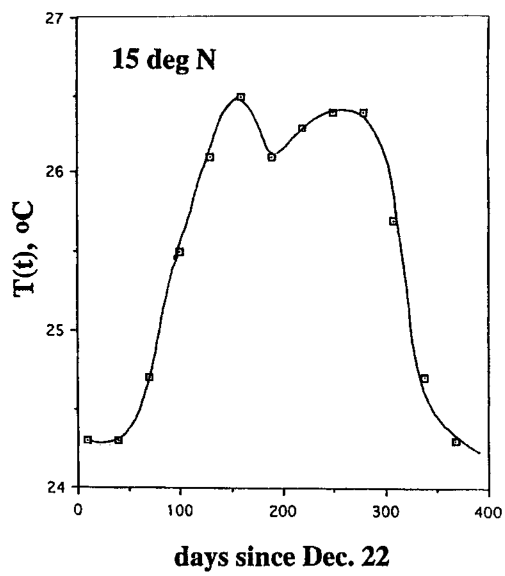

FIGURE 3.8. Temperature variations at 15 deg latitude, showing the limitations of the model for low latitudes.

Thereafter, temperature variations cannot be fitted by equation (3.1); and, if they can be fitted at all, the value of λ seems to be significantly smaller than at lower altitudes.

b. *Temperature variations at lower latitudes:* It has been remarked already that temperature variations at lower latitudes cannot be fitted well by equation (3.1). An example is shown for the latitude of 15 deg (Figure 3.8). Lower latitudes exhibit a more pronounced bimodality, or some other more complex behavior, depending on the altitude and latitude. Where bimodality is pronounced, a simple nonlinear version of the model could work, but this has not been tested extensively.

c. *Largest gradients in temperature variations:* For any given latitude, one can write from equation (3.1) that

$$\frac{dT}{dt} = \frac{\eta L - T}{\lambda}, \tag{3.6}$$

and compute dT/dt using the measured values of λ and η. The result is shown in Figure 3.9 for 50 deg latitude. The largest changes occur sometime in April and October, consistent with the known fact that the weather changes are most rapid in these months of Spring and Fall. If you are an outdoor jogger, you will need to change from long jumpers to shorts, and vice versa, sometime around these dates.

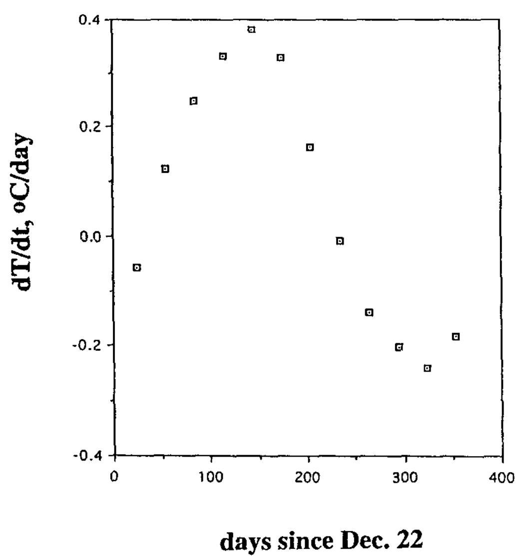

FIGURE 3.9. Temporal derivative of the temperature $T(t)$ through the year, as computed for 50 deg latitude using equation (3.6).

4 Discussion

A linear viscoelastic model describes several gross features of temperature variations in mid-latitudes. A prominent qualitative character of the model is the "memory" incorporated through the parameter λ. The best fit to the data shows that the parameter varies with latitude in a simple manner, as shown in Figure 3.5. (All these data correspond to a pressure altitude of 1000 mb. Slightly different definition of constant altitude could lead to slightly different numbers.) In the simple version, the present model averages temperature over all longitudes—and, at a given latitude, over many years—and is thus similar in spirit to general circulation models, though it is difficult to associate the two in detail.

To understand the parameter λ, let us make an energy balance for a thin slice of the globe centered around a given latitude ℓ. The radius of this slice, which is in the form of a disc, is $R\cos\ell$, R being the Earth's radius. The disc gets heated by the component of the solar radiation received normal to the surface, loses (gains) energy by convective heat loss to the neighboring discs as well as to the atmosphere, and stores (or releases) energy due to the imbalance between these two effects. This energy storage, associated with the cyclic heat-up and cool-down processes, occurs in the first few meters of the ground; as is well-known to divers and plumbers alike, one can identify a penetration depth below which the seasonal variations of temperatures are felt less conspicuously. The details of the energy loss (or gain) are not well understood, and the standard practice in the heat transfer literature is to represent this effect by means of a term $h(T - T^*)$, where one's ignorance is lumped into the heat transfer coefficient h; here, T^* is a reference temperature. The energy balance then takes the form

$$\frac{\rho c\Lambda}{h}\frac{d(T-T^*)}{dt} + (T - T^*) = \text{excess (deficit) energy supply}\,, \tag{4.1}$$

where ρ and c are the density and specific heat, respectively, of the soil in the upper few meters of the Earth's crust, and Λ is a characteristic length scale for the penetration of heat. It is clear that this characteristic length should be proportional to the radius of the disc under consideration. If we assume that the daily excess energy supply is proportional to L, then equation (4.1) coincides with equation (2.1) if

$$\lambda = \frac{\rho c\Lambda}{h}\,, \tag{4.2}$$

whence one may expect that

$$\frac{\lambda}{\cos\ell} = \text{constant}\,. \tag{4.3}$$

Table 4.1 shows the ratio is roughly constant with a value of about 64 days. With this information, and $A(\ell)$ given in the caption to Figure 3.2,

latitude, deg.	$\lambda / \cos \ell$
25	62.9
30	64.7
35	63.5
40	61.4
50	59.1
60	66
65	68.6

TABLE 4.1. The ratio $\lambda / \cos \ell$ at various latitudes. The ratio is a constant with a mean value of about 63.8 and a standard deviation of about 3.1.

equation (3.5) is an explicit formula for $\eta = \eta(\ell)$, whose graph resembles that shown in Figure 3.5.

Daily variations in temperature are caused mainly by ground absorption and release of heat. The thermal conductivity of the ground and its heat capacity are not very large, so (on a short term) it gives up at night what it receives during the day. Inserting reasonable values for ρ, c and h (or, equivalently, for the so-called Biot number, or for the soil diffusivity), one gets a plausible estimate for the penetration depth to be of the order of a meter. Clearly, more precise estimates of this depth will depend on the latitude, and on whether one is on land or in the ocean, and so forth.

Two further comments may be useful. First, our equation (2.1) is tantamount to assuming that the heating of the Earth by the Sun can be represented by the term $\eta L(t)$. In principle, this term can be computed exactly from the known information on the solar radiation arriving at the Earth. We have not attempted to do this. In effect, the "viscosity coefficient" η lumps these factors into an effective single number. Secondly, we have so far ignored temperature fluctuations about its mean value $\langle \theta \rangle$ at a given latitude; the model therefore has nothing to say about the latter. If one imagines that equation (2.1) without the derivative term holds for the mean temperature $\langle \theta \rangle$ as well, it is clear that one can define another constant η^* given by

$$\eta^* = \frac{\langle \theta \rangle}{\langle L^* \rangle}, \tag{4.4}$$

where the denominator $\langle L^* \rangle$, as already remarked, is approximately 12 hours at all latitudes. There is no reason to expect that this new coefficient η^* will be related to η. However, we find that the variation of η^* is roughly linear in ℓ and follows

$$\eta^* = 4.53\ell - 2.33. \tag{4.5}$$

5 Concluding remarks

A large number of non-equilibrium phenomena, involving dilute gases and a variety of rheological problems, have been represented successfully by relaxation (or memory) models. Various phenomena that occur in turbulent shear flows have also been modelled in this way. Two examples will suffice. In Narasimha and Prabhu [1972], a fully developed turbulent wake was distorted by a rapid pressure gradient, which was then released. The subsequent evolution of the flow, which is generally difficult to predict by conventional turbulence models, has been replicated well by a relaxation model. For a pipe flow with a periodically oscillating mass flow (Mao and Hanratty [1986]), the stress and rate of strain can be related quite well by a memory model. The present work has shown that the relaxation model performs similarly well for modelling seasonal variations of temperature. However, the ubiquitous success of memory models does not necessarily suggest that we understand the physics leading to relaxation effects. That understanding must come from the study of specific systems on hand.

Acknowledgments: Larry Sirovich is special for a number of reasons, one of which is that he takes delight in learning about new problems without being encumbered by their past foibles. We hope that the little problem discussed here will amuse him. KRS thanks the National Science Foundation, and DDJ thanks the US Army and DOE for supporting this work.

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