Kolmogorov's Third Hypothesis and Turbulent Sign Statistics

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The breakdown of turbulent eddies can be characterized by sets of "multipliers," defined as ratios of velocity increments at successively smaller scales. These quantities were introduced by Kolmogorov, who hypothesized their self-similar statistics and independence at distant scales. Here we report experimental and numerical results on the statistics of these multipliers, for both their magnitude and sign. We show that the multipliers at adjacent scales are not independent but that their correlations decay rapidly in scale separation. New scaling laws are thereby predicted and verified for both roughness and sign of turbulent velocity increments. The sign oscillations per cascade step are found to decrease at points of increasing roughness or singularity of the velocity.

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In 1962, Kolmogorov presented a set of hypotheses on the local structure of turbulence in order to account for the intermittency phenomena [1]. This work is now perhaps best known for its hypotheses on the fluctuations of locally volume-averaged energy dissipation [2]. However, in the same work, Kolmogorov proposed another formulation of the theory of inertial-range intermittency which, in his own words, is "freed from the special selection of the values of [energy dissipation]." In this formulation, he considered "multipliers" defined by ratios of velocity increments

$$w_{ij}(\mathbf{x};\ell,\ell') = \delta_i v_j(\mathbf{x},\ell)/\delta_i v_j(\mathbf{x},\ell'). \tag{1}$$

Here $\delta_i v_j(\mathbf{x},\ell) = v_j(\mathbf{x}+\ell\mathbf{e}_i) - v_j(\mathbf{x})$ is the increment of the jth component of the velocity vector $\mathbf{v}$ along the unit vector $\mathbf{e}_i$. Kolmogorov hypothesized that, at very high Reynolds number, these multipliers should have distributions which are universal functions only of the scale ratio ℓ/ℓ' and not of the absolute scale. He postulated further that multipliers corresponding to widely separated scales should be statistically independent. This is what Kolmogorov referred to as his *third hypothesis*. Surprisingly, the alternative formulation based upon the third hypothesis has received only a little subsequent attention [3–5].

Here we report results on statistics of the Kolmogorov multipliers, determined using data from a direct numerical simulation (DNS) of the three-dimensional incompressible Navier-Stokes equation, as well as hot-wire data taken in a nominally steady atmospheric boundary layer. For the DNS, isotropic turbulence was simulated in a box with 512^3 mesh points under periodic boundary conditions. A statistically steady state was obtained using a low Fourier mode forcing [6]. The Taylor microscale Reynolds number was about 220. The experimental data were acquired at a height of 6 m above the smooth ground on a field in the Utah desert [7]. The Taylor microscale Reynolds number was about 10 500.

Not only the amplitude of the turbulent velocity increment is of interest, but also its sign. For example, the third-order skewness of longitudinal velocity increments must be negative to ensure a forward energy transfer (Ref. [5], Chap. 6). The statistics of sign as well as amplitude were implicit in Kolmogorov's definition of multipliers in Eq. (1), but we shall make explicit these two distinct aspects. For simplicity, we consider here multipliers defined only for longitudinal velocity increments, $\delta_\| v(\mathbf{x},\ell) = \delta_i v_i(\mathbf{x},\ell)$. We shall also employ for convenience a velocity increment normalized by the Kolmogorov 1941 prediction: $\delta u(\ell) = \delta_\| v(\ell)/(\varepsilon\ell)^{1/3}$, where ε is the mean of the rate of dissipation per unit volume. We then define the multiplier $w(\ell,\ell') = \delta u(\ell)/\delta u(\ell')$ as a slight modification of Kolmogorov's. We shall generally consider just a discrete sequence of length scales, $\ell_n = 2^{-n}\ell_0$, where ℓ_0 is a chosen large scale, e.g., the integral scale. Thus, n is the number of "cascade steps" by factors of 2 from ℓ_0 down to ℓ_n. We then define the multiplier for one cascade step by $w_n = w(\ell_n,\ell_{n-1})$. Of course, $w(\ell_n,\ell_{n'}) = \prod_{k=n'+1}^{n} w_k$. To unravel the separate dependencies on magnitude and sign we now define $|w_n| = e^{\sigma_n}$ and $\mathrm{sgn}(w_n) = (-1)^{\nu_n}$ where σ_n is real and $\nu_n = 0$ or 1. In that case, $|w(\ell_n,\ell_{n'})| = \exp(\sum_{k=n'+1}^{n}\sigma_k)$ and $\mathrm{sgn}[w(\ell_n,\ell_{n'})] = (-1)^{\sum_{k=n'+1}^{n}\nu_k}$. We have defined the new variables via exponentiation so that they are "additive," i.e., sums of terms rather than products. If we plot $\ln|\delta u(\ell_n)|$ versus ℓ_n, then σ_n is the slope between points ℓ_{n-1} and ℓ_n. The integer variable ν_n, on the other hand, just indicates whether there was a change of sign in $\delta u(\ell_n)$ between ℓ_{n-1} and ℓ_n: it is 1 if there was a sign change and 0 otherwise. In these new variables, Kolmogorov's hypothesis of scale similarity is equivalent to statistical homogeneity under discrete translations in the variable n. The third hypothesis of Kolmogorov is stated, somewhat loosely, as the requirement that $\sigma_{n'}, \nu_{n'}$ should be statistically independent of σ_n, ν_n for $|n' - n| \gg 1$.

Our first result is the probability density function (PDF) of the multiplier w_n in the inertial range, plotted in Fig. 1 for DNS $n = 4, 5$ and experiment $n = 11, 14$. We see that these distributions collapse quite well for different values of n, verifying the "translation-invariance" assumption or scale-similarity hypothesis made by Kolmogorov [1]. We also find that the plotted distributions are very close to those for multipliers defined for increments of fractional Brownian motion (FBM) with Hurst or Hölder exponent $h = \frac{1}{3}$. Such an agreement with FBM has been found previously for statistics of the ratio of a velocity increment and locally averaged dissipation, which appears in Kolmogorov's refined similarity hypothesis [8]. The multiplier distributions for increments of FBM are Cauchy, $P(w) = \frac{b}{\pi}\frac{1}{(w-a)^2+b^2}$, with $a = 2^{h-1}$ and $a^2 + b^2 = 1$. Kolmogorov himself had proposed that the distribution of the turbulence multipliers would be log normal [1], but this is clearly not so. The distributions are also strikingly different from those of the multipliers for volume-averaged dissipations [9], which, by definition, are compactly supported on the interval $[0, 1]$. Because of the Cauchy-like tails $\sim w^{-2}$ for $|w| \gg 1$, the distribution $P(w)$ has infinite moments $\langle |w|^p \rangle$ for all $p \geq 1$. This has the important implication that, to determine scaling exponents of velocity structure functions, *correlations between multipliers may not be neglected*. Indeed, if multipliers w_n for distinct n were independent, then there would be scaling for the velocity structure functions $\langle |\delta_\| v(\ell_n)|^p \rangle \sim \ell_n^{\zeta_p}$ with $\zeta_p = \frac{p}{3} + \log_2 \langle |w|^p \rangle$ (Ref. [5], Sec. 8.6.3). Our results in Fig. 1 clearly rule out the validity for $p \geq 1$ of such a formula, implied by independence. We shall see below that the scaling exponents can be inferred from the statistics of the Kolmogorov multipliers, but only from the joint statistics of multipliers at many scales and not from those at a single scale.

The results above have direct implications for the statistics of the sign and roughness variables. The probability for turbulent velocity differences to experience a sign change over one cascade step in the inertial range is $P(\nu_n = 1) \approx 0.27 \pm 0.0054$ from experiment and DNS. This is very close to the value for FBM, $P(\nu_n = 1) = \frac{1}{2} - \frac{1}{\pi}\arctan(\frac{a}{b}) \approx 0.283$ with $h = 1/3$. In Fig. 2 we plot the conditional distributions $P(\sigma_n|\nu_n)$ for the same cases as in Fig. 1. Except for a small nonzero mean value (see below), these distributions are also fitted rather well by the formula calculated for FBM with $h = \frac{1}{3}$: $P(\sigma|\nu) = \frac{b}{2\pi p_\nu}\frac{1}{\cosh\sigma - (-1)^\nu a}$ where $p_\nu = P(\nu_n = \nu)$. In particular, both distributions in Fig. 2 seem to consist of symmetric exponential tails $\sim e^{-|\sigma|}$ for $|\sigma| \gg 1$. These appear as the long straight lines with slopes ± 1 in the log-PDF plotted in Figs. 2(a) and 2(b), and correspond to $\sim w^{-2}$, $|w| \to +\infty$ and $\sim w^0$, $|w| \to 0$ for the distribution in Fig. 1. The symmetric tails can be easily explained, since they arise from the same events, namely, the "near-zero crossings" of the velocity increments. Because $\sigma_n = \ln|\delta u(\ell_n)/\delta u(\ell_{n-1})|$, the left tail for $\sigma_n \ll -1$ arises from events with $|\delta u(\ell_n)| \ll 1$ and $|\delta u(\ell_{n-1})| = O(1)$ while the right tail for $\sigma_n \gg 1$ arises from events with $|\delta u(\ell_{n-1})| \ll 1$ and $|\delta u(\ell_n)| = O(1)$. The statistics of such near-zero crossings can be inferred from the constancy of the PDF of velocity increments, $P(\delta u) = \text{const}$, near $\delta u = 0$ [10].

Despite the fact that there can be no strict independence of the random fields σ_n, ν_n, there is quite a significant decay of correlations. We define correlation functions $C_{\xi\eta}(n;k) = \langle \xi_n \eta_{n+k} \rangle - \langle \xi_n \rangle \langle \eta_{n+k} \rangle$ with $\xi, \eta = \sigma, \nu$. These are plotted in Fig. 3 for $n = 4$ in DNS and $n = 11$ in experiment. It is evident that the

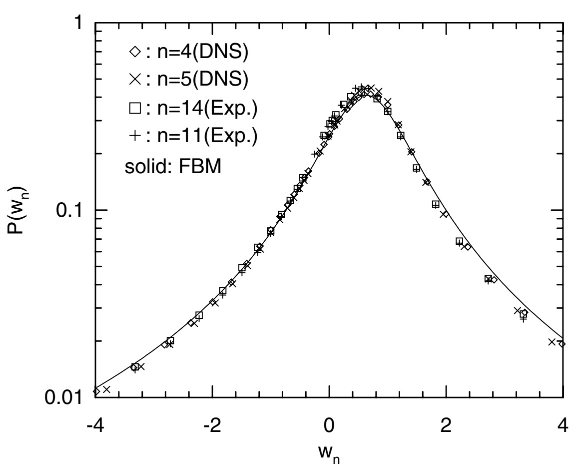

FIG. 1. Probabilities for w_n showing collapse for DNS, experimental data, and FBM.

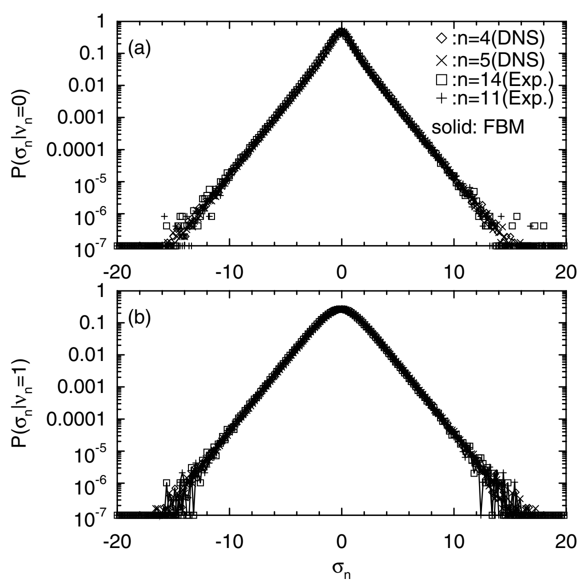

FIG. 2. Log-linear plots of the conditional distributions, (a) $P(\sigma_n|\nu_n = 0)$ and (b) $P(\sigma_n|\nu_n = 1)$.

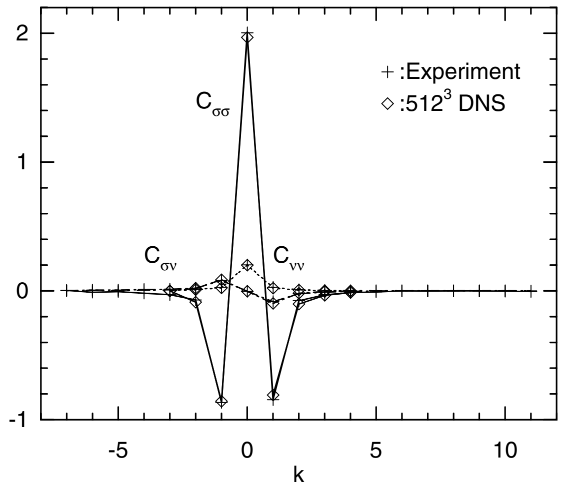

FIG. 3. The correlation functions $C_{\sigma\sigma}(4;k)$, $C_{\nu\nu}(4;k)$, and $C_{\sigma\nu}(4;k)$ for DNS and $C_{\sigma\sigma}(11;k)$, $C_{\nu\nu}(11;k)$, and $C_{\sigma\nu}(11;k)$ from experiment.

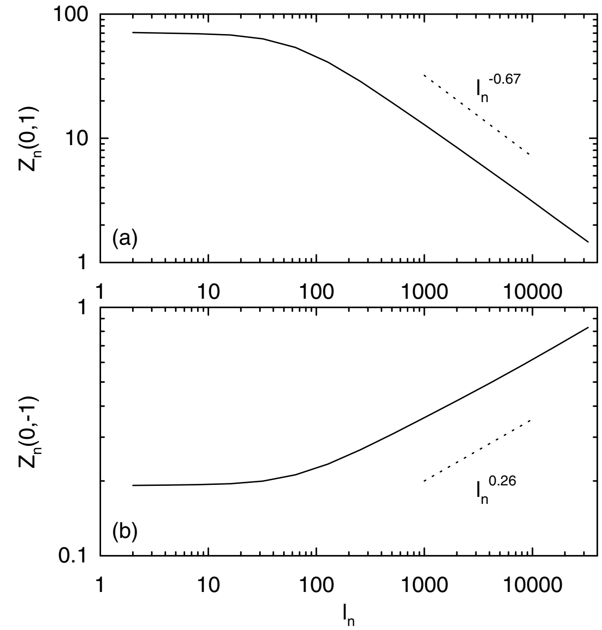

FIG. 4. Structure functions (a) $Z_n(0,1)$ (b) $Z_n(0,-1)$ from experimental data. The dotted lines are the fitting curves.

high-Reynolds-number results in experiment agree very well with the low-Reynolds-number results in DNS. It may be seen that the correlations decay quite rapidly, exponentially or as a large inverse power. This is similar to the behavior observed for another type of multiplier defined as ratios of the locally averaged dissipations [11].

Multifractal scaling laws for velocity structure functions will hold even if there are correlations between multipliers, so long as those decay sufficiently rapidly [3]. This may be seen from a standard thermodynamic analogy for multifractals (Ref. [5], Sec. 8.6.4; Ref. [12]). Indeed, $\langle|\delta_\parallel v(\ell_n)|^p\rangle = \ell_n^{1/3} Z_n(p,0)$ in terms of a "partition function" $Z_n(p,\mu) = \langle|\delta_\parallel v(\ell_0)|^p \exp[\sum_{k=1}^n (p\sigma_k + \mu\nu_k)]\rangle$. If correlations decay rapidly, there should exist a "Gibbs free energy" defined by a thermodynamic limit

$$g(p,\mu) = -\lim_{n\to\infty}(1/n)\ln Z_n(p,\mu). \tag{2}$$

The existence of such a limit is equivalent to validity of a multifractal scaling law, with exponents $g(p,\mu)$. For example, when $\mu = 0$, then $\langle|\delta_\parallel v(\ell_n)|^p\rangle \sim \ell_n^{\zeta_p}$, recovering the standard scaling law for amplitudes with $\zeta_p = \frac{p}{3} + \frac{g(p,0)}{\ln 2}$. The partition function above is a generalized structure function involving also sign statistics. The limit (2) holds if and only if there is a scaling law $Z_n(p,\mu) \sim \ell_n^{\delta\zeta(p,\mu)}$ for $\delta\zeta(p,\mu) = g(p,\mu)/\ln 2$.

As evidence of scaling, we present in Fig. 4 two typical partition functions (or structure functions) $Z_n(0,1)$ and $Z_n(0,-1)$ plotted versus ℓ_n. The power-law inertial ranges can be easily identified for both curves. We have determined the generalized scaling exponent from the experimental data, by means of the extended self-similarity procedure [13], in which $\ell_n^{1/3} Z_n(p,\mu)$ was plotted versus $\langle|\delta_\parallel v(\ell_n)|^3\rangle$. In Fig. 5 we show $g(p,\mu)$. This generalized scaling exponent or "free energy" contains information not only about the decrement in magnitude of velocity differences with decreasing space separations ℓ_n but also about the oscillations in sign. It therefore encompasses several exponents previously defined. For example, the "cancellation exponent" κ of longitudinal velocity derivatives that was defined and measured in [14] may be expressed as $\kappa = \frac{2}{3} - \frac{g(1,0)}{\ln 2}$. Likewise, the scaling exponents ξ_p of *signed* velocity structure functions $\langle(\delta_\parallel v(\ell_n))^p\rangle$ are given as $\xi_p = \frac{p}{3} + \frac{g(p,ip\pi)}{\ln 2}$, by an analytical continuation to imaginary $\mu = ip\pi$.

The concavity of $g(p,\mu)$ along the p direction has implications for the PDF of the "spin" σ. Although $P(\sigma)$ from Fig. 2 appears to be symmetrical under the spin flip $\sigma \to -\sigma$, we shall show that there is a small but nonvanishing mean $\langle\sigma\rangle$. A direct determination from the experimental and DNS data gives $\langle\sigma\rangle = -0.034 \pm 0.011$. In fact, this nonzero average is simply related to the fact

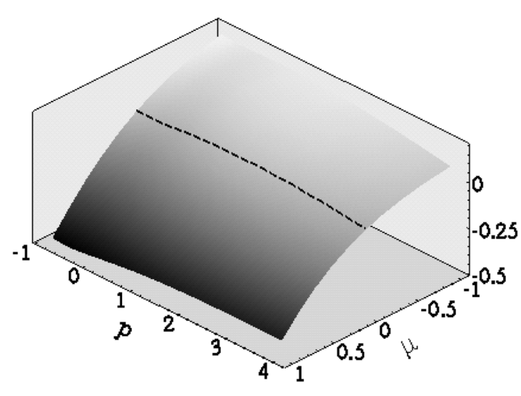

FIG. 5. "Gibbs free energy" $g(p,\mu)$ from experimental data. The line is for $\mu = 0$, $g(p,0) = \ln 2(\zeta_p - p/3)$. The concavity in the p direction is different from FBM, for which $g(p,\mu)$ is linear in p for each fixed μ.

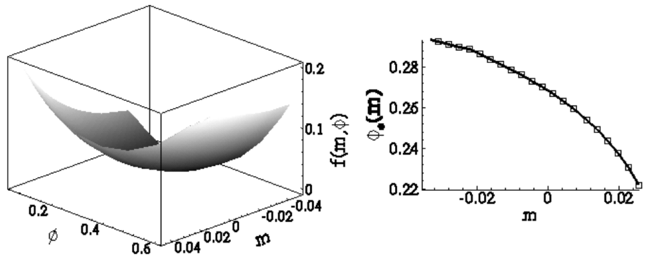

FIG. 6. Generalized multifractal spectrum $f(m, \phi)$ and conditional frequency of sign oscillations $\phi_*(m)$ from experiment.

that Kolmogorov's 1941 mean-field scaling of pth-order structure functions is not exact in the limit $p = 0$ [15]. The relation is easy to see from the thermodynamic analogy for the scaling exponents. Indeed, the average "magnetizations" should be obtained from the free energy by the derivative formulas $\frac{\partial g}{\partial p}|_{p=\mu=0} = -\langle\sigma\rangle$ and $\frac{\partial g}{\partial \mu}|_{p=\mu=0} = -\langle\nu\rangle$. Thus, the derivative with respect to p gives the average slope $m = \langle\sigma\rangle$, while the derivative with respect to μ gives $\phi = \langle\nu\rangle$, the average fraction of cascade steps n at which $\delta v(\ell_n)$ changes sign. In particular, the first of the above equations connects the nonzero mean $\langle\sigma\rangle$ to the anomalous scaling exponent at $p = 0$. A check of the equations by finite-difference approximation of the derivatives of g gives $\langle\sigma\rangle = -0.044 \pm 0.021$, $\langle\nu\rangle = 0.29 \pm 0.073$. Clearly, the thermodynamic calculation of the averages is consistent with the direct one (where $\langle\nu\rangle = p_1$, given earlier).

More precise information can be obtained from the convex Legendre transformed function $f(m, \phi) = \sup_{p,\mu}\{mp + \phi\mu + g(p, \mu)\}$, which corresponds thermodynamically to "Helmholtz free energy." One has the following "fluctuation formula"

$$\mathrm{Prob}\left(\frac{1}{n}\sum_{k=1}^{n}\sigma_k \approx m, \frac{1}{n}\sum_{k=1}^{n}\nu_k \approx \phi\right) \sim e^{-nf(m,\phi)}, \qquad (3)$$

which gives the probability to observe an empirical average slope m and the fraction of sign changes ϕ in the limit of a large number of cascade steps n. Note that $f(m, \phi)$ has its minimum value ($= 0$) at $m = \langle\sigma\rangle$, $\phi = \langle\nu\rangle$ so that those are overwhelmingly the most probable values to observe as n increases. However, other values of m, ϕ can occur, and $f(m, \phi)$ gives the (exponentially small) probability for that to happen. Notice that this Helmholtz free energy is a generalization of the usual Cramér function or $f(\alpha)$ spectrum in the multifractal formalism [5], with Hölder exponent $\alpha = \frac{1}{3} - \frac{m}{\ln 2}$. In Fig. 6 we plot the Helmholtz free energy corresponding to the Gibbs free energy shown above. The generalized multifractal spectrum $f(m, \phi)$ contains information not only about the local scaling $|\delta v(\ell_n)| \sim \ell_n^{\alpha}$ but also about the frequency of local oscillations in sign of $\delta v(\ell_n)$. For example, consider the function $\phi_*(m)$ plotted on the right in Fig. 6, which gives the minimizer of $f(m, \phi)$ at fixed m. It represents the frequency of sign oscillations conditioned on the value of the local Hölder exponent. Surprisingly, it is a decreasing function, indicating less frequent oscillations for more singular velocities. In this respect, turbulence is quite different from FBM, for which increasing roughness is instead associated with more frequent oscillations.

We tentatively interpret this result as a sign of the greater coherence of more singular structures in turbulent flow, such as intense vortex filaments. As one test of this idea we have randomized the phases in our DNS velocity field. This does not change the energy spectrum, but it eliminates all the coherent vortex structures. We have found that the scaling results for the phase-randomized turbulent field and for FBM are identical. In particular, the decreasing $\phi_*(m)$ curve in Fig. 6 disappears.

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