Asymmetry of Velocity Increments in Fully Developed Turbulence and the Scaling of Low-Order Moments

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A study is made of the scaling of the positive part (PP) and the negative part (NP) of velocity increments in turbulent pipe flow and in simulated homogeneous turbulence in a box. For moment orders above unity, the moments of NP are larger than those of PP for all separation distances, and the scaling exponents for PP are larger than those for NP. For moment orders below unity, the absolute value of the velocity increment and of NP, as well as PP, possess scaling exponents which vary linearly with the moment order q, though apparently greater than $q/3$. [S0031-9007(96)00976-3]

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The velocity fluctuation $\mathbf{u}(\mathbf{x};\mathbf{t})$ in a turbulent flow is a large scale feature and nonuniversal. Kolmogorov [1] introduced the so-called structure functions to study universal properties. Specializing to the velocity component u in the direction x, one defines (for integer q) the structure functions as

$$S_q(r) \equiv \langle \Delta u_r^q \rangle \equiv \langle [u(x + r) - u(x)]^q \rangle . \qquad (1)$$

It is expected that the structure functions in the inertial range [2] scale as some power of the separation distance r; that is, $S_q \sim r^{\zeta_q}$. A great deal of attention has been paid to the scaling exponents ζ_q.

Far less attention has been paid to another important aspect of structure functions, namely their asymmetry [3]. It was pointed out in Ref. [4] that the asymmetry is best studied by means of the plus and minus structure functions

$$S_q^{\pm}(r) = \langle \tfrac{1}{2} [\overline{S}_q(r) \pm S_q(r)]^q \rangle , \qquad (2)$$

where q is real and

$$\overline{S}_q(r) \equiv \langle |\Delta u_r|^q \rangle \equiv \langle |u(x + r) - u(x)|^q \rangle \qquad (3)$$

have been called [5] the generalized structure functions. Obviously, $\frac{1}{2}[|\Delta u_r| \pm \Delta u_r]$ is non-negative and represents, for the $+$ and $-$ signs, respectively, the positive part and the absolute value of the negative part of Δu_r. It was suggested in Ref. [4] that the plus and minus structure functions are basic to studying both classical and generalized structure functions (as well as others defined in Ref. [5]). On the basis of a ramp model for velocity, it was proposed in Ref. [4] that the scaling exponents of high order (i.e., large q) would be numerically larger for the plus structure functions than for the minus structure functions, and that the latter would eventually dominate the true scaling. In particular, it was argued that for the generalized dimensions [5–7], the relation

$$D_q^- < D_q^+ \qquad (4)$$

holds; here the $\pm$ signs correspond to plus and minus structure functions.

In this Letter, we obtain the plus and minus structure functions and show that, for all separation distances r, the minus structure functions are larger than the plus structure functions for $q > 1$, but smaller for $q < 1$. We provide tentative evidence that the plus structure functions possess larger scaling exponents than the negative ones for $q > 1$. The low-order generalized structure functions, as well as NP and PP, possess scaling exponents that vary linearly with the moment order q but are measurably larger than the classical value of $q/3$ even for q as low as 0.25.

Measurements were made in a pipe flow of water at a bulk Reynolds number $UD/\nu = 230\,000$, where U is the average velocity for the pipe cross section and D is the pipe diameter and ν is the kinematic viscosity coefficient. The pipe, constructed in cooperation with D.P. Lathrop (now at Emory University), had a diameter of 30.5 cm and a working length of about 27 m to allow for the flow to be fully developed. The probe was located on the pipe axis. The bulk mean velocity was 0.70 m/s. A boundary layer hot-film probe (Dantec 55R15), operated on Dantec Streamline anemometer system, was used for velocity measurements. The time sequence was treated as a spatial cut by invoking Taylor's hypothesis, and velocity increments over time intervals were assumed to be the same as those over space increments. The Taylor microscale λ was estimated from the root-mean-square velocity fluctuation u' and the mean energy dissipation rate $\langle \varepsilon \rangle$ to be 0.88 cm. The Taylor microscale Reynolds number $R_\lambda \equiv u'\lambda/\nu$ was 270. The Kolmogorov microscale $\eta = (\nu^3/\langle \varepsilon \rangle)^{1/4}$ was estimated to be 0.27 mm. Two data files of 4×10^6 points, sampled at a frequency of 4096 Hz, were obtained and processed. Moments up to order six converged well.

Simulations of homogeneous turbulence were made by solving the Navier-Stokes equations in a periodic box of dimension 512^3 by using a pseudospectral code [8]. To achieve statistically steady state, forcing was introduced in the first two shells of Fourier modes for the wave number $k < 2.5$. The total energy of all modes in each of the first two shells was maintained constant in time. Simulations were carried out for ten eddy turnover times. The velocity initial conditions were prescribed to have a Gaussian phase distribution with compact spectral support at low wave numbers. The microscale Reynolds number $R_\lambda = 220$. Taylor's hypothesis was not necessary.

The microscale Reynolds number is only moderately large in both experiment and simulations, and a critical question concerns the scaling range. A traditional way—see for example, Ref. [9]—is to obtain the scaling region from the flat part of $\langle \Delta u_r^3 \rangle / r$ versus r [3]. Unfortunately, it is not known if this procedure is valid exactly in the presence of strong anisotropies such as occur [10] in pipe flows, or if some nontrivial correction is needed [11]. We have examined the extent of scaling in the energy spectral density, considered the so-called extended scale similarity (ESS) [12], and the notion of relative scaling [13] and, in general, the sensitivity of the results to the scaling region used.

Figure 1 shows a plot of the compensated spectral density for the velocity data from the experiment; in accordance with Taylor's hypothesis, spectral frequency is treated as wave number. Scaling exists over a decade or so. We shall indicate this as the K range. Figure 2 plots the ratio $\langle \Delta u_r^3 \rangle / r$ against r for both the experiment and simulations. The two flows are at comparable Reynolds numbers, yet the scaling region (to be called the R range) is substantially smaller for the experiment than for simulations; it is definitely smaller than the K range. The cutoff at the small-scale end is roughly the same in all cases, but the R range in the homogeneous simulation as well as the K range in the experiment extend to much larger scales (or lower frequencies) than does the experimental R range. That the scaling in one-dimensional longitudinal spectrum extends to smaller wave numbers than one should expect has been discussed in Monin and Yaglom [14], p. 357, but it has not been noted before that different manners of forcing and consequent anisotropies can change the extent of the scaling so drastically. This matter will be discussed elsewhere in more detail. We have examined *many* structure function plots and consistently used least-square fits to the R range of Fig. 2 to obtain the numbers to be quoted below, and verified that the *relative* trends are robust even for the K range as well as for the ESS method.

One noteworthy feature of the plus/minus structure functions is shown in Fig. 3, which plots the logarithm of the ratio S_q^- / S_q^+ against $\log_{10} r$ for various values of q. It can be seen readily that the ratio S_q^- / S_q^+ is greater than unity for all $r < L$ whenever $q > 1$ and smaller than unity whenever $q < 1$. Here L is the so-called integral scale of turbulence characteristic of the large scale turbulence. By definition, the ratio should be exactly unity for $q = 1$. For one-dimensional data such as those considered here, it follows from the definition of generalized dimensions D_q that the ratio of the minus to plus structure functions scales as

$$(r/L)^{(q-1)(D_q^- - D_q^+)}.$$

For consistency with the observation that S_q^- / S_q^+ is greater than unity for $q > 1$ and less for $q < 1$, one should have

$$D_q^- < D_q^+$$

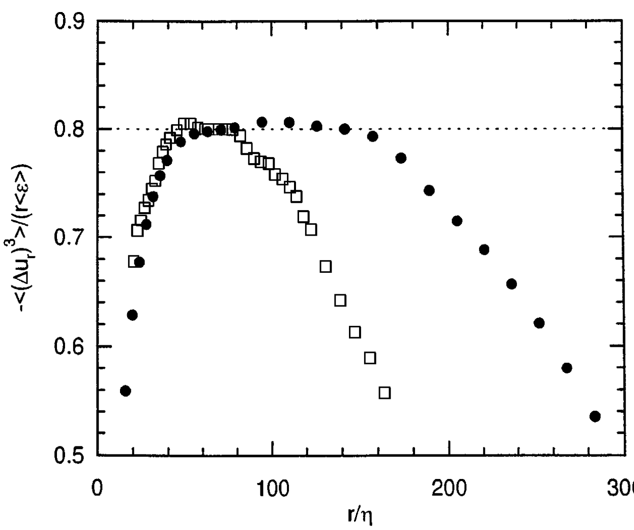

FIG. 2. The quantity $\langle \Delta u_r^3 \rangle / r$ as a function of r. Squares, experiment; circles, simulations; dots indicate Kolmogorov's $\frac{4}{5}$th law. It is believed that the slight bump in the left part of the experimental data is the bottleneck effect [see G. Falkovich, Phys. Fluids **6**, 1411 (1994); D. Lohse and A. Mueller-Groeling, Phys. Rev. Lett. **74**, 1747 (1995)]. While the bottleneck effects discussed in these two papers refer especially to second-order structure functions (or to energy spectrum), a similar effect is likely to exist for the third-order as well. This is typical of most measurements [see, for example, Y. Gagne, Docteur ès-Sciences Physiques Thèse, Université de Grenoble, France (1987)].

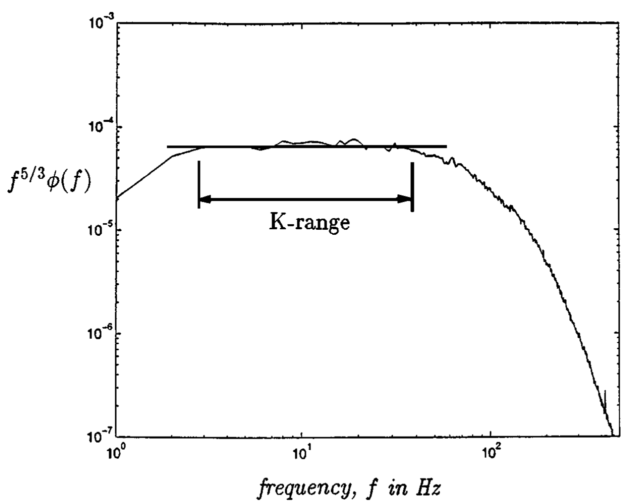

FIG. 1. The spectral density of u multiplied by $f^{5/3}$, where f is the frequency, plotted to show the flat region. Scaling occurs over a decade (the K range). There is no perceptible difference even when a power-law exponent slightly different from 5/3 is used to compensate for the frequency rolloff.

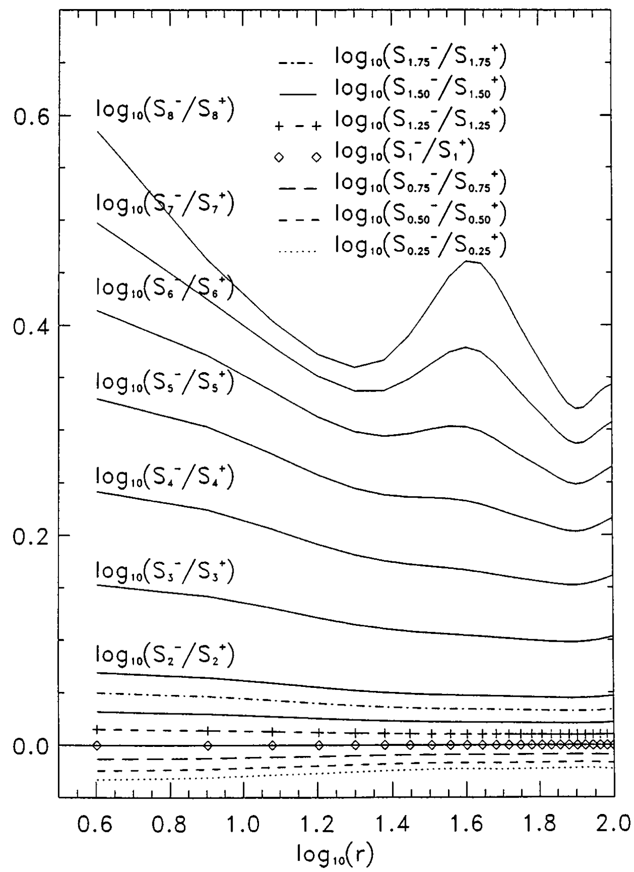

FIG. 3. The ratio S_q^-/S_q^+ plotted against r in log-log coordinates. Note that the ratio is less than unity for all r when $q < 1$ and greater than unity when $q > 1$. As argued in the text, this is consistent with ramplike structures in the velocity trace. The bump in high-order moments is weaker for simulations data.

for all r, which is precisely Eq. (4) inferred for the ramp model. This does not appear to be a coincidence. The physical interpretation of Δu_r^+ is that it corresponds to regions of acceleration while Δu_r^- corresponds to regions of deceleration, just as for the long and short legs of the ramp structure, respectively.

A second point to note is that all curves for $q > 1$ show, by and large, a decreasing trend with increasing r. If there were power laws to these curves, one may conclude that the scaling exponents for the plus structure functions are larger than for the minus structure functions. It is difficult to quantify this feature because of the uncertainty in identifying the scaling region precisely. Some additional information can be had by plotting the ratios $S_q/S_2^{q/2}$, $S_q^+/S_2^{q/2}$, and $S_q^-/S_2^{q/2}$ against $\log r$. This is shown in Fig. 4 for $q = 4$. The scaling region in each of these plots is somewhat different. However, if we use those regions, we note that the minus exponent is almost identically equal to that for the full structure function, whereas the plus exponent is larger than the minus exponent. Simulations data confirm this observation qualitatively.

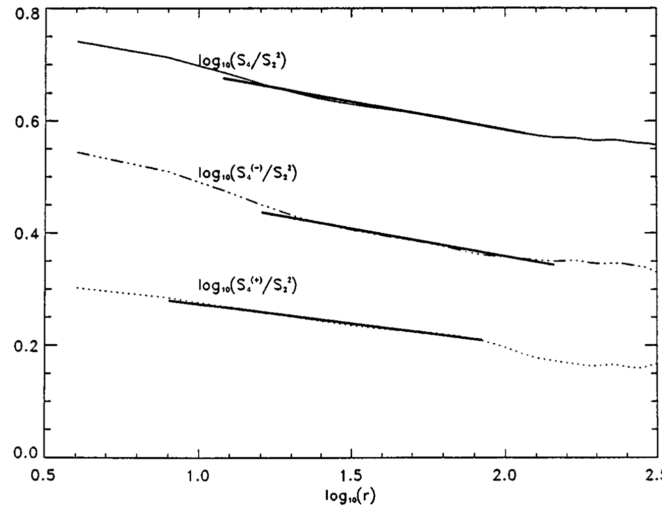

FIG. 4. The logarithm of the ratios S_4/S_2^2, S_4^-/S_2^2, and S_4^+/S_2^2 plotted against $\log r$. These ratios appear to scale, albeit in different ranges of r. From top to bottom, the slopes of the straight line fits and the least square errors are, respectively, -0.099 ± 0.003, -0.098 ± 0.004, and -0.070 ± 0.002. This shows that the plus exponent is larger than the minus exponent.

This conclusion is also apparent in the ESS plots; here, one plots the qth order structure function against the third-order structure function and determines scaling exponents. Our experience is that ESS in this form does not work to well for the pipe flow, consistent with our previous experience in another shear flow [15]. However, if the third-order structure function is replaced by the generalized third-order structure function (following Ref. [12]), there is much more convincing scaling, for which the plus exponent is, as already noted, larger than the minus exponent. Numerical data are in agreement. We have refrained from giving the precise exponents because the latter procedure cannot be justified easily. It would be most desirable to confirm these conclusions at higher Reynolds numbers. The atmospheric data available to us ($R_\lambda \approx 2000$, see [15]), for which the convergence is reasonably satisfactory for the fourth order, corroborate the basic conclusion.

The results so far suggest that there is a basic difference between the plus and minus parts of the velocity increments. The Kolmogorov law is an expression of part of this fundamental difference. The negative part becomes more dominant with the moment order and overwhelms the positive part asymptotically. Two related pieces of evidence presented in earlier work are worth recalling: (a) The odd exponents of high-order structure functions consistently lie above a smooth curve that can be drawn through even-order exponents [15]; in particular, they are different for the classical and generalized structure functions (Table I, Ref. [4]). (b) The ratio $|\langle \Delta u_r^n \rangle| / \langle |\Delta u_r|^n \rangle$ is comparable to unity for large odd n (Table II, Ref. [4]). Whether the emphasis on these differences will lead us to a better theoretical understanding of inertial scaling remains to be seen.

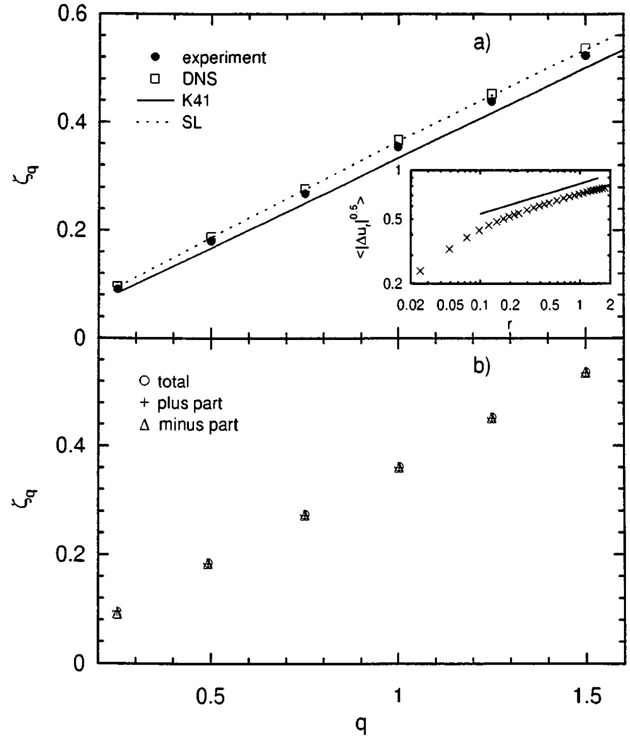

FIG. 5. (a) Low-order scaling exponents for generalized structure functions compared with $q/3$ as well as with outcomes of the intermittency models of [19] and [21]. The inset shows the scaling of the generalized structure function of order 0.5. The full line is for the model of Ref. [21]. (b) Comparison of the plus and minus exponents with those of generalized structure functions.

As already remarked, the present work reveals some correspondence with the ramp model for the velocity, and draws attention to a change in behavior of the *structure functions* at $q = 1$. This naturally suggests a potential connection to the work recently carried out for randomly forced Burger's equation [16], for turbulence without pressure [17], and for passive scalars advected by a rapidly varying velocity field [18]; these papers show that the scaling exponents transition from a regular behavior for small q to an anomalous behavior for large q. While there have been many measurements of exponents for large q, no such measurements exist for low q below unity. Here, we have obtained low-order scaling exponents (q as low as 0.25) for $\overline{S}_q(r)$ as well as the positive and negative structure functions. Figure 5(a) shows the simulations data. The inset is an example of the plots from which these exponents have been determined. Both experiment and simulations yield nearly identical results. The exponents vary linearly with q for low-order moments, but are measurably different from $q/3$ expected from Kolmogorov's universality [1]. Instead, they agree well with the refined similarity hypothesis [19] if one takes a value of 0.25 for the intermittency exponent [20], and with the intermittency model of Ref. [21]. In the limit $q \to 0$, the intermittency models just considered yield exponents about 10% different from $q/3$.

The fact that some intermittency models agree with measured exponents for q both below and above unity suggests that the *exponents* possess no transitional behavior envisaged in Refs. [16–18]. This is so even though the plus and minus structure functions themselves show a change in behavior at $q = 1$ (see Fig. 3). Figure 5(b) shows that the exponents for the generalized structure functions are not too different from those for the plus and minus structure functions (although, in general, the minus exponents are the largest and the plus exponents are the smallest).

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