Properties of Velocity Circulation in Three-Dimensional Turbulence

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Properties of velocity circulation in three-dimensional turbulence are studied using data from high-resolution direct numerical simulation of Navier-Stokes equations. The probability density function (PDF) of the circulation depends on the area of the closed contour for which circulation is calculated, but not on the shape of the contour. For contours lying within the inertial range, the PDF has a Gaussian core with conspicuous exponential tails, indicating that intermittency plays an important role in circulation statistics. The measured scaling exponents are anomalous and substantially smaller than those implied by Kolmogorov's phenomenology.

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Motivated principally by the phenomenological picture proposed by Kolmogorov [1,2], the scaling properties of quantities such as velocity increments and locally averaged dissipation in three-dimensional turbulence have been studied extensively (e.g., [3–6]). Much effort has been devoted to the understanding of the fundamental physics associated with the scaling behavior. On the other hand, vorticity-related quantities such as circulation have not attracted much attention until very recently [7–9]. In this Letter, we study various properties of circulation, with emphasis on scaling, using velocity data from high-resolution direct numerical simulation of Navier-Stokes equations in a periodic box [10].

The circulation, Γ_A, of the velocity field $\mathbf{v}$, around a closed contour surrounding an area A, is defined as

$$\Gamma_A = \oint \mathbf{v} \cdot d\ell = \int_A \boldsymbol{\omega} \cdot \mathbf{n} dA \,. \tag{1}$$

Here $\mathbf{n}dA$ is an element of an open surface bounded by the closed curve, the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, and the second equality in Eq. (1) follows from Stokes theorem. Using the loop equation, Migdal [7] pointed out that the probability density function (PDF) of Γ_A allows a scaling solution in the variable Γ_A^3/A^2 [11], where A is the minimal area of the surfaces enclosed by the loop. Since Γ_A is a local average of the vorticity field, Migdal [7] and Sreenivasan, Juneja, and Suri [8] invoked the law of large numbers [12] to suggest that the PDF of circulation should be close to Gaussian if A is large enough, such as in the inertial range [13]. Further, Sreenivasan, Junega, and Suri [8] used the velocity data in a plane of a low-Reynolds-number turbulent wake, obtained by particle image velocimetry, and showed that the PDF was nearly Gaussian, though not exactly so. Numerical simulation data at low Reynolds numbers have confirmed the approximate Gaussianity of the PDF [14].

If the circulation is distributed exactly as a Gaussian, its scaling should be close enough to that implied by Kolmogorov's arguments, namely,

$$\langle (\Gamma_A)^n \rangle \sim A^{2n/3} \sim r^{4n/3}, \tag{2}$$

where we have used a square contour with $A = r^2$. (In this situation, one may refer to Γ_A also as Γ_r without ambiguity.) However, the experimentally measured [8] scaling exponents of Γ_r are substantially smaller than this phenomenological result. Sreenivasan, Junega, and Suri argued that the discrepancy could be attributed to the dominant effect of the mean shear on vorticity characteristics at the low Reynolds number of the experiment ($R_\lambda \le 40$ [15]). They also argued that the circulation scaling can be related to the transverse velocity scaling: the latter attains its asymptotic value in shear flows only at a very high Reynolds number (R_λ of the order of 1000; see [16]). To see in a different way how deviations might arise from Kolmogorov scaling, assume that the loop integration is carried out along a square box in the X-Y plane with area $A = l^2$. The coordinates of the four corners of the box are $(0,0)$, $(l,0)$, (l,l), and $(0,l)$. The circulation around the loop can be easily written as $\Gamma_A = \int_0^l [v_2(l,y) - v_2(0,y)]\,dy - \int_0^l [v_1(x,l) - v_1(x,0)]\,dx$, where $v_1(x,l)$ represents the x-direction velocity along the top boundary of the box and other symbols have similar meanings. This formula can be rewritten using the mean value theorem as

$$\Gamma_A = l[v_2(l,y^*) - v_2(0,y^*)] - l[v_1(x^*,l) - v_1(x^*,0)], \tag{3}$$

where x^* and y^* are some middle points with $0 \le x^* \le l$ and $0 \le y^* \le l$. We notice that the same value of x^* (y^*) on the upper and lower (left and right) sides of the box can be assured by the theorem. If $v_2(l,y^*) - v_2(0,y^*)$ and $v_1(x^*,l) - v_1(x^*,0)$ scale as in Kolmogorov's theory [1], Eq. (2) is essentially exact. However, as we can see from the procedure just discussed the statistics on points x^* and y^* are only a subset of the original field. Moments of Γ_A in Eq. (3) are equivalent to conditional statistics of

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the transverse velocity difference on a given local mean. Since this mean varies in space, it does not follow that the conditional statistics should have the same scaling as the original field.

The present research is related directly to that of Refs. [7,8]. Our goal is to study the properties of circulation using data from three-dimensional direct numerical simulation of isotropic turbulence. The improvements over the previous studies are the precise control of isotropy (and absence of shear) and the higher Reynolds number than was possible in Ref. [8]. Our major results are as follows: (i) The PDF of circulation depends solely on the magnitude of area around closed contours, with little dependence on whether the area is a square or a rectangle. (ii) The form of PDF varies as a function of magnitude of area, and has significant exponential tails in the inertial range. These tails can be fitted by a stretched exponential with the exponent nearly equal to unity for small areas and to 2 for large areas. The form of the PDF varies also with R_λ. (iii) The inertial-range scaling of circulation is anomalous and the exponents are smaller than those estimated using Kolmogorov scaling [1]. In fact, circulation appears to be more strongly affected by intermittency than velocity increments, and thus shows a new class of anomaly.

Direct numerical simulation of the Navier-Stokes equations with normal viscosity [10] was carried out with 256^3 and 512^3 mesh points on the CM-5 machine at Los Alamos and the SP machines at IBM. To obtain a steady state, a low mode forcing scheme ($k < 3$) was used [17]. The Reynolds number was controlled by varying the viscosity of the system. Analysis was carried out for statistically steady states at $R_\lambda = 101$, 181, and 216. The first two Reynolds numbers were obtained using 256^3 lattices, and the highest Reynolds number was obtained with 512^3 mesh points. Circulation was calculated according to the second equality in Eq. (1). For simplicity, only surfaces parallel to axes were used. Statistical average was taken over the whole simulation space and along all directions. The PDFs of circulation were also computed using loop integration in Eq. (1), and excellent confirmation of the area-integral results was obtained.

One fundamental assumption in Migdal's paper is that the statistics of circulation along a loop depend only on the minimum area surrounded by the loop. To verify this area rule, we have calculated the PDF of circulation along rectangles with different aspect ratios but with the same area. One set of these PDFs (for $R_\lambda = 181$) is shown in Fig. 1 with both sides of each rectangle lying within the inertial range. It is clear that the area rule is a good approximation. One may argue that the rule applies in the inertial range because of universality of that scale range, but it holds empirically even when r_1 and/or r_2 spill over to the dissipation range as well. These results are not shown here.

Figure 1 shows that the PDFs have Gaussian core but stretched tails. Departures from Gaussianity were not ruled out in Migdal's analysis [7]. To quantify departures

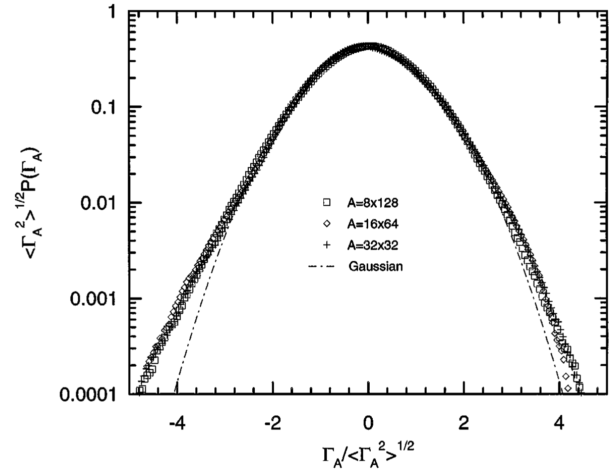

FIG. 1. Normalized PDFs of Γ_A when $A = 1024$ (in lattice units) for different aspect ratios of rectangular contours, $R_\lambda = 181.32$ lattice units correspond to $r = 0.785$.

from Gaussian more clearly, we have plotted in Fig. 2 the flatness of circulation (for square contours) as a function of the linear scale r for $R_\lambda = 101$ and 216. The flatness tends to the Gaussian value only when r approaches the integral scale, and is substantially larger than 3 in the inertial range [18] ($0.2 \leq r \leq 0.8$ where $r = 2\pi$ corresponds to the box length). For comparison, we also plot the flatness of Δu_r, the longitudinal velocity increment (with the separation distance r matched to $A^{1/2}$). Flatness of Γ_r appears to be larger than that of Δu_r over a considerable range of r, indicating that the former is actually more intermittent than the latter. There is some tendency for the lower Reynolds number data to be closer to Gaussian, roughly consistent with the observations of Refs. [8,14].

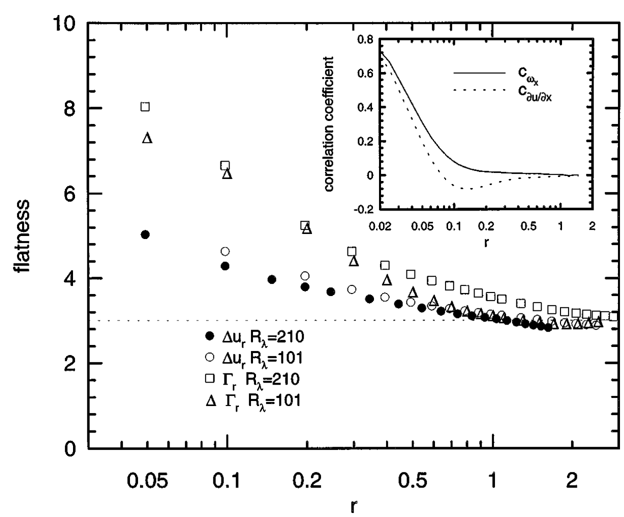

FIG. 2. Flatness of circulation and the longitudinal velocity increment as a function of r for $R_\lambda = 101$ and 216. Both circulation and the velocity increment show intermittency effects in the inertial range. The inset shows the correlation coefficient of vorticity and velocity derivative as a function of r. In these units, $r = 2\pi$ is the box length.

The non-Gaussianity of the PDFs of Γ_r and Δu_r is probably related to the intermittency of vortex structures and energy dissipation in three-dimensional turbulence. This is not difficult to understand given that Γ_r and Δu_r are local averages of vorticity and pseudodissipation, respectively [19]. Direct visualization [20] of three-dimensional vorticity and dissipation fields of high amplitude shows that the vorticity field is more intermittent than the dissipation field, consistent with the flatness data just discussed.

The basic argument for the Gaussianity of the circulation PDF is the law of large numbers, one of whose requirements is the independence among the random variables being summed. The vorticity field in three dimensions shows very rich structures, some of them in the form of tubes with lengths often comparable to the integral scale. This leads to the decorrelation length comparable to the inertial range scale. To quantify this, we have plotted in the inset to Fig. 2 the autocorrelation coefficients for vorticity component ω_x and the longitudinal velocity derivative $(\partial u/\partial x)$, both for $R_\lambda = 216$. For vorticity, $C_{\omega_x} = \langle \omega_x(y)\omega_x(y + r)\rangle/(\omega_x)^2_{\rm rms}$, with separation in the transverse direction; for the velocity derivative, the separation is along the longitudinal direction, i.e., $C_{\partial u/\partial x} = \langle[\partial u/\partial x(x)][\partial u/\partial x(x + r)]\rangle/(\partial u/\partial x)^2_{\rm rms}$. Although the correlation is negligible inside the inertial range, the decorrelation length is comparable with, at least not substantially smaller than, the inertial-range scales. Therefore, the argument of Gaussianity based on the law of large numbers is questionable in the inertial range, due to the fact that the circulation Γ_A consists of vorticity with all the scales up to $r = A^{1/2}$.

The PDFs of Γ_r with $R_\lambda = 216$ at three typical scales in dissipation, inertial, and integral scale regions are given in Fig. 3. The tails of the PDF are fitted to the stretched exponential of the form $P(\Gamma_r) = A\exp(-\beta\Gamma_r^\alpha)$, and the stretching exponent α is plotted in the inset as a function of r, where $\alpha = 2$ corresponds to a Gaussian distribution. The qualitative variation of $P(\Gamma_r)$ with r is similar to that of $P(\Delta u_r)$ [17,21]. On the other hand, the skewness of Γ_r appears to be smaller than that of Δu_r.

Since the PDF of Γ_r is non-Gaussian, one may expect an anomalous scaling of Γ_r. To display the scaling behavior of Γ_r, we have plotted the moments $\langle\Gamma_r^n\rangle$ for $n = 2$ and 4 at $R_\lambda = 216$ as a function of r in Fig. 4, both of which display nice power laws in the inertial range, as in Ref. [8]. The local scaling exponents for the two Reynolds numbers are presented in the inset to Fig. 4. They are constants to a better approximation at the higher Reynolds number. In the inertial range, it appears that scaling exponents at the higher R_λ are somewhat higher, but this increase is not significant. It is thus generally in support of the claim that the smaller exponent values observed in Ref. [8] might be principally due to shear effects.

In Fig. 5 we present the scaling exponent for circulation, λ_n, as a function of power index n. Here the scaling exponent is defined through $\langle|\Gamma_A|^n\rangle \sim A^{\lambda_n/2} \sim r^{\lambda_n}$. The other two symbols correspond to results from Refs. [1,2], the latter with the present estimate of the intermittency exponent $\mu = 2/9$ (see also Ref. [22]), assuming that the circulation scaling can be related by Eq. (2) to that of the velocity increments. The scaling exponents are smaller than those obtained on the basis of Ref. [2]. The discrepancy between simulation and phenomenological models increases as n increases. This might reflect real physical difference between the original field and the conditional statistics of a field, as discussed above. On the other hand, since the Reynolds numbers in numerical simulations are not as large as one would desire, one cannot exclude the possibility that the present exponents are not in the asymptotic region and that intermittency models may predict the scaling exponents well for circulation at very large Reynolds number.

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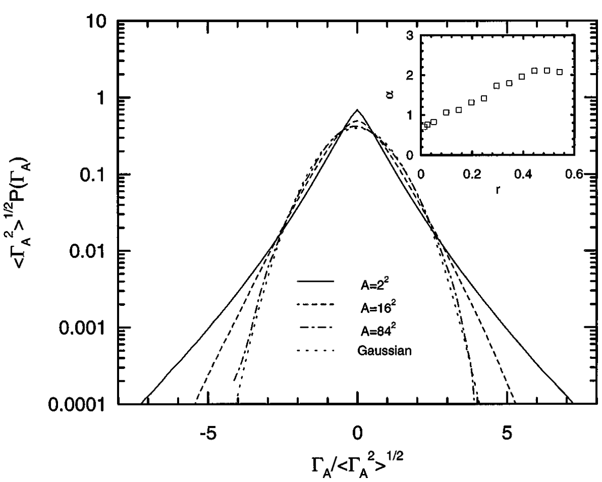

FIG. 3. Normalized PDFs of Γ_A at $R_\lambda = 216$ for $A = r^2$ (square loop in lattice units) for $r = 4$ (dissipation), $r = 16$ (inertial), and $r = 84$ (integral), corresponding to $r = 0.049$, 0.196, and 1.031, respectively. The inset shows the stretching exponent for the tails of the PDF as a function of r.

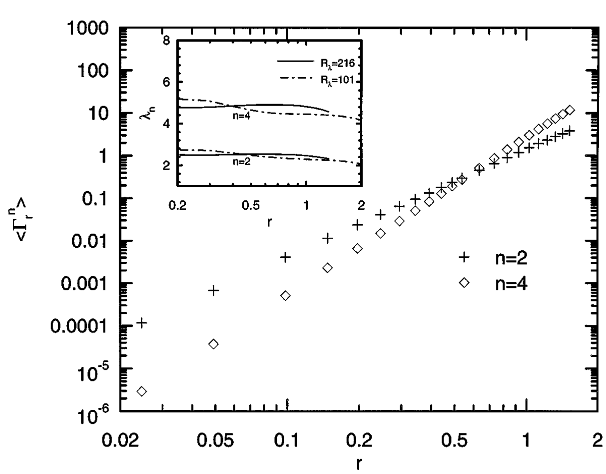

FIG. 4. Second- and fourth-order moment of circulation as a function of r, for $R_\lambda = 216$. The inset shows local scaling exponents of circulation for $R_\lambda = 216$ and 101.

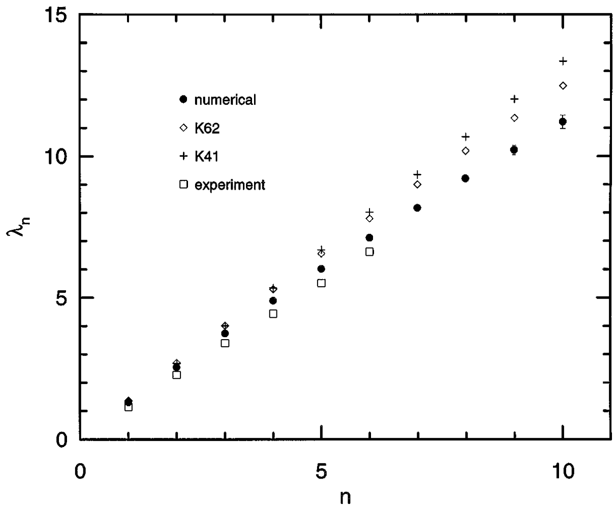

FIG. 5. Scaling exponent for circulation as a function of the moment order, n, compared with K41 and K62, the latter with intermittency exponent $\mu = 2/9$, calculated from the same numerical data as Fig. 4. Typical error bars are shown for the ninth and tenth order moments. □ are experimental data from Ref. [8].

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