INDEPENDENT VELOCITY INCREMENTS AND KOLMOGOROV'S REFINED SIMILARITY HYPOTHESES

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ABSTRACT

Under the assumption of statistical independence of velocity increments across scales of the order of the Kolmogorov scale, it is shown that a modified version of Kolmogorov's refined similarity hypotheses follows purely from probabilistic arguments. The connection of this result to three-dimensional fluid turbulence is discussed briefly.

1. INTRODUCTION

In 1962, Kolmogorov[1] put forward a refinement of his earlier phenomenological theory[2] of high-Reynolds-number turbulence. This refinement has become a vital reference point in the research of locally isotropic and homogeneous turbulence. An important quantity in this description is the flux of energy ϕ_r transferred across scales of size r. Kolmogorov assumed that, in the inertial range, ϕ_r is the only relevant quantity upon which all other quantities would depend. Furthermore, he identified ϕ_r with $r\varepsilon_r$, where ε_r is the rate of energy dissipation per unit mass averaged over a volume of linear scale r. Kolmogorov's theory is made quantitative on the basis of the following two celebrated hypotheses.

The first similarity hypothesis: If $r \ll L$, where L is a measure of the large scale of turbulence, the probability density function (pdf) of the stochastic variable

$$V = \frac{\Delta u(r)}{(r\varepsilon_r)^{1/3}} \tag{1}$$

depends only on the local Reynolds number $Re_r = r(r\varepsilon_r)^{1/3}/\nu$, where ν is the kinematic viscosity of the fluid and $\Delta u(r)=u(x+r)-u(x)$, u being the x-component of the velocity vector $\mathbf{u(x)}$ and r is measured along x.

The second similarity hypothesis: If $Re_r \gg 1$, the pdf of V does not depend on Re_r either (nor on r, and is therefore universal).

Although it was shown recently[3] that the pdf of V depends on r as well, several aspects of these hypotheses have been verified experimentally[3,4] as well as by direct numerical simulations of turbulence[5]. Even before this verification, consequences of these hypotheses had been used extensively in the turbulence literature.

In spite of their widespread use, the hypotheses pose some troubling problems. For example, Kraichnan[6] has pointed out that, for r in the inertial range, $\Delta u(r)$ is a purely inertial range quantity, whereas $r\varepsilon_r$ is a mixed quantity (because ε_r is a dissipation quantity averaged over an inertial range scale), and so their ratio cannot be universal. Also, the notion of a cascade, where the energy is transferred locally in wavenumber space, has been criticized from time to time.

The primary shortcoming of Kolmogorov's hypotheses is that they have not yet been derived from basic principles. In this paper we wish to show that they can indeed be cast, under certain circumstances, in terms of general principles of stochastic processes. The physical picture of a cascade need not be assumed *a priori*, but rather as an *a posteriori* interpretation.

For convenience, we restrict our discussion to one-dimensional spatial cuts of the turbulent velocity fluctuation. In particular, we use the local isotropy approximation for the three-dimensional average dissipation rate, namely,

$$\varepsilon_r = 15\nu \frac{1}{r}\int_x^{x+r}\left(\frac{du}{dx}\right)^2 dx. \qquad (2)$$

Relaxing this assumption of local isotropy, Eq. (2), adds greater complexity to the proof given below, but it is believed that it will not affect its basic validity.

2. A CONVENIENT RESTATEMENT OF THE REFINED HYPOTHESES

The refined hypotheses are statements about the relation between the velocity increments

$$\Delta u(r) = \int_x^{x+r}\frac{du}{dx}dx \qquad (3)$$

and the energy dissipation rate in a segment of linear size r

$$r\varepsilon_r = 15\nu \int_x^{x+r}\left(\frac{du}{dx}\right)^2 dx. \qquad (4)$$

Given that both $\Delta u(r)$ and $r\varepsilon_r$ are functionals of the velocity gradient, they are (in general) correlated variables. Discretizing the integrals (3) and (4), and normalizing velocities by $(\eta\varepsilon)^{1/3}$ (where $\varepsilon = 15\nu\langle(du/dx)^2\rangle$ and η is the Kolmogorov scale given by $(\nu^3/\varepsilon)^{1/4}$), and lengths by η, we may write (3) and (4) respectively as

$$S_p = \sum_{i=1}^{p} X_i \quad \text{and} \quad Y_p^2 = \sum_{i=1}^{p} X_i^2, \qquad (5)$$

where $X_i=\frac{du}{dx}\eta/(\eta\varepsilon)^{1/3}$ and $p=r/\eta$. Physically, the X_i represent normalized velocity increments across a distance η. In these variables, one can write that

$$V = S_p / Y_p^{2/3}, \qquad (6)$$

and state Kolmogorov's similarity hypotheses as follows: the pdf of V depends only on $Re_r = pY_p^{2/3}$ and the variable p, and becomes independent of both when $p \gg 1$ and $Y_p \gg 1$.

The discretized equations (5) suggest that a more general formulation of the refined similarity hypotheses in terms of stochastic processes might be attainable. This is done in the next section.

3. REFINED SIMILARITY HYPOTHESES FOR BROWNIAN MOTION

Let us assume that the X_i are normally distributed independent random variables with zero mean and variance σ^2. We then have the following exact result.

Theorem: Given that $p \geq 2$ is an integer, the pdf of $\xi = S_p/Y_p^{2H}$ conditioned on Y_p is independent of Y_p only when H=1/2. In such a case, the conditional pdf assumes the form:

$$f(\xi|p,Y_p)=\frac{1}{\sqrt{p}\,\Omega_\theta(p)}\left(1-\frac{\xi^2}{p}\right)^{(p-3)/2}, \quad \xi^2<p \qquad (7)$$

where $\Omega_\theta(p)=\frac{\pi}{2^p}\frac{\Gamma(p-1)}{\Gamma(p/2)^2}$, $\Gamma(x)$ being the gamma function.

Proof: First we compute the joint pdf of S_p and Y_p for a given p as

$$P_2(S_p,Y_p|p)=$$

$$\int_{R^p} dX_1\ldots dX_p\,\Pi_X(X_1,\ldots,X_p)\,\delta\left(S_p-\sum_{i=1}^{p}X_i\right)\delta\left(Y_p-\left(\sum_{i=1}^{p}X_i^2\right)^{1/2}\right) \qquad (8)$$

where R^p is p-dimensional real space, $\delta(x)$ is Dirac's delta function, $\Pi_X(X_1,\ldots, X_p)$ is the joint probability of X_i and is equal to $(2\pi\sigma^2)^{-p/2}\exp(\sum_{i=1}^{p}X_i^2/2\sigma^2)$. To evaluate the integral we perform an orthogonal change of coordinates from $X\to U$:

$$U_j=\sum_{i=1}^{p}R_{ij}\,X_j, \qquad (9)$$

where R_{ij} is an orthogonal matrix with $R_{1j} = 1/\sqrt{p}$. The other rows of this matrix, R_{kj}, $2 \le k \le p$, can be computed using (for example) a Gram-Schmidt procedure, but they are irrelevant for our purposes here. As the Jacobian of this transformation is unity, the integral (8) can be written as

$$P_2(S_p, Y_p|p) =$$

$$\int_{R^p} dU_1 \ldots dU_p \, \Pi_U(U_1,\ldots,U_p)\, \delta(S_p - \sqrt{p}\, U_1)\, \delta\Big(Y_p - \big(\sum_{i=1}^{p} U_i^2\big)^{1/2}\Big), \tag{10}$$

where $\Pi_U(U_1,\ldots,U_p) = \Pi_X(X_1(U_1,\ldots,U_p),\ldots,X_p(U_1,\ldots,U_p))$.

The next step is to change to p-dimensional spherical polar coordinates:

$$\begin{aligned}
U_1(\rho,\theta,\phi_1,\ldots,\phi_{p-2}) &= \rho \cos\theta \\
U_2(\rho,\theta,\phi_1,\ldots,\phi_{p-2}) &= \rho \sin\theta \cos\phi_1 \\
U_3(\rho,\theta,\phi_1,\ldots,\phi_{p-2}) &= \rho \sin\theta \sin\phi_1 \\
U_4(\rho,\theta,\phi_1,\ldots,\phi_{p-2}) &= \rho \sin\theta \sin\phi_1 \cos\phi_2 \\
&\ldots\ldots \\
U_{p-1}(\rho,\theta,\phi_1,\ldots,\phi_{p-2}) &= \rho \sin\theta \sin\phi_1 \ldots \sin\phi_{p-3} \cos\phi_{p-2} \\
U_p(\rho,\theta,\phi_1,\ldots,\phi_{p-2}) &= \rho \sin\theta \sin\phi_1 \ldots \sin\phi_{p-3} \sin\phi_{p-2}
\end{aligned} \tag{11}$$

where $0 < \rho < \infty$, $0 < \theta < \pi$, $0 < \phi_i < \pi$ for $i=1,\ldots,p-3$ and $-\pi < \phi_{p-2} < \pi$. Noting that the Jacobian of this transformation is

$$\frac{\partial(U_1,\ldots,U_p)}{\partial(\rho,\theta,\phi_1,\ldots,\phi_{p-2})} = \rho^{p-1} (\sin\theta)^{p-2} (\sin\phi_1)^{p-3} \ldots (\sin\phi_{p-3}), \tag{12}$$

we can perform the integral in Eq. (10) in polar coordinates, to yield

$$P_2(S_p, Y_p|p) = \Omega_\phi(p) \left(1 - \frac{S_p^2}{pY_p^2}\right)^{(p-3)/2} \frac{\exp(-Y_p^2/2\sigma^2)}{(2\pi\sigma^2)^{p/2}}, \tag{13}$$

where $\Omega_\phi(p) = \frac{\pi(p-1)}{\Gamma((p-1)/2)}$ is the integral over ϕ_1 through ϕ_{p-2} of $(\sin\phi_1)^{p-3} \times \ldots (\sin\phi_{p-3})$ arising from the Jacobian (12). The probability of S_p conditioned on Y_p is computed as

$$P(S_p|p, Y_p) = \frac{P_2(S_p, Y_p|p)}{\int dS_p P_2(S_p, Y_p|p)}, \tag{14}$$

and we find

$$P(S_p|p, Y_p) = \frac{1}{\sqrt{p}\, Y_p \Omega_\theta(p)} \left(1 - \frac{S_p^2}{pY_p^2}\right)^{(p-3)/2}. \tag{15}$$

where $\Omega_\theta(p) = \int_0^\pi (\sin\theta)^{p-2}$ and is expressed in a closed form in the statement of the theorem. If we now change variables to $\xi = \frac{S_p}{Y_p^{2H}}$, we obtain

$$f_H(\xi|p, Y_p) = \frac{1}{\sqrt{p}\, Y_p^{1-2H} \Omega_\theta(p)} \left(1 - \frac{\xi^2}{pY_p^{2-4H}}\right)^{(p-3)/2}. \tag{16}$$

It is easily seen that the only value of H that renders $f_H(\xi|p, Y_p)$ independent of Y_p is $H=1/2$, and the form of $f_{1/2}(\xi|p, Y_p)$ is then the one stated in the theorem.

In fact, the theorem can be stated to appear closer to Kolmogorov's hypotheses:
H1) The pdf of the stochastic variable $\xi = S_p/Y_p$ conditioned on Y_p depends only on p.
H2) When $p \gg 1$, the pdf of ξ becomes independent of p; in fact, it tends to the normal distribution.

Plots of $f(\xi|p, Y_p)$ for different values of p are shown in Fig. 1. These functions are supported in the interval $(-\sqrt{p}, \sqrt{p})$. It can be seen that the distribution for p=2 is bimodal. The distribution for p =3 is uniform between $\sqrt{3}$ and $-\sqrt{3}$. For larger p,

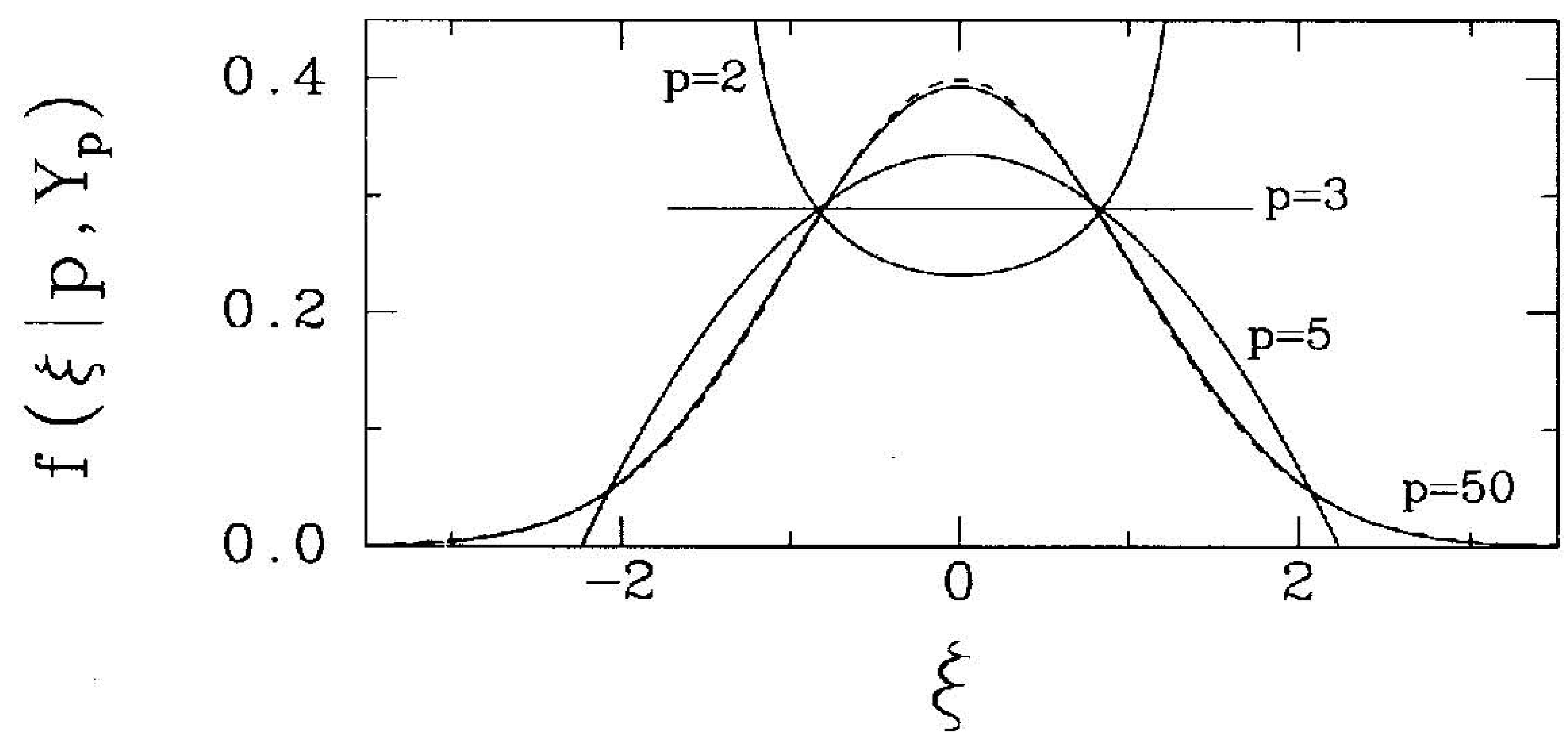

Fig. 1. Plots of $f(\xi|p, Y_p)$ for different values of the parameter p when the velocity increments X_i are assumed to be Gaussian.

the trend towards a Gaussian distribution occurs; for p=50 the departure from the Gaussian shape (dashed line) is negligible.

4. MORE GENERAL PROCESSES

It is useful to explore the consequences of relaxing the hypotheses that the X_i are normally distributed, and that they are independent. We will address the former issue here, and leave the latter for a forthcoming publication[7].

Let $\xi=S_p/Y_p$ and the X_i be independent random variables with zero odd-order moments and non-divergent even-order moments. From these conditions it is clear that for odd n, $<\xi^n|p,Y_p>=0$. We wish to show that for $p\gg1$ and even $n\geq2$

$$<\xi^n|p,Y_p> \approx (n-1)!!, \tag{17}$$

which implies that ξ is a normal random variable with zero mean and unity variance. In particular, for $p\gg1$ the pdf of ξ is independent of p and Y_p. We give an informal proof of this result below.

The n-th power of S_p can be written as

$$S_p^n = \sum_{i_1+...+i_p=n} \frac{n!}{i_1!...i_p!} X_1^{i_1}...X_p^{i_p} \tag{18}$$

where the sum extends over all the possible $0\leq i_j\leq n$ such that $\sum_1^p i_j=n$. It is clear from Eq. (18) that, if one computed $<S_p^n>$, all terms containing odd i_j will not contribute to the sum. This follows from the independence of the X_i and the assumption that odd-order moments of X_i are zero. Further, when $p\gg1$, most of the contribution to S_p^n is likely to come from terms containing even i_j because the terms with odd i_j (which, in general, possess both signs) are likely to cancel each other. Among all the terms with even i_j, the most numerous ones are those for which $\frac{n!}{i_1!...i_p!}$ takes a maximum value. This happens when i_j=0 or 2, and the number of such terms will be $\binom{p}{n/2} \sim p^{n/2}$. The next most numerous terms are those for which i_j=0, 2 or 4 (with 4 occurring only once), and there will be $\binom{p}{n/2-1} \sim p^{n/2-1}$ such terms. If all the terms are of the same order, it is clear that the terms with i_j=0 or 2 dominate the sum, so that we may rewrite Eq. (18) as

$$S_p^n \approx \sum_{i_1+...+i_p=n} \frac{n!}{2^{n/2}} X_1^{i_1}...X_p^{i_p}, \tag{19}$$

where i_j takes only the values 0 or 2. To make this statement more rigorous, it would be necessary to bound the errors made in this approximation, but this will not be attempted here.

We can make a similar analysis for Y_p and obtain, with i_j assuming only values of 0 or 1 in Eq. (20a) and values of 0 or 2 in Eq. (20b), that:

$$Y_p^n \approx \sum_{i_1+...+i_p=n/2} (n/2)!\, X_1^{2i_1}...X_p^{2i_p}, \tag{20a}$$

$$\approx \sum_{i_1+...+i_p=n} (n/2)!\, X_1^{i_1}...X_p^{i_p}. \tag{20b}$$

By comparing Eqs. (19) and (20b), we obtain

$$<S_p^n|Y_p> \approx \frac{n!}{2^{n/2}(n/2)!} Y_p^n. \tag{21}$$

Noting that $\frac{n!}{2^{n/2}(n/2)!} = (n-1)!!$, we can restate Eq. (21) as

$$<\xi^n|Y_p> \approx (n-1)!!, \tag{22}$$

for $p\gg1$, as required. We thus conclude that in this limit the pdf of ξ conditioned in Y_p is independent of p and Y_p. Actually, it tends to be Gaussian with zero mean and unity variance.

We now illustrate these results numerically. For definiteness, we will consider that the X_i are distributed according to an exponential density g, i.e.

$$g(X) = \frac{1}{2}\exp(-|X|). \tag{23}$$

For this distribution, $<X^2> = 2$, and hence $<Y_p>= 2p$. We computed the pdfs $h(\xi|p,Y_p)$ of $\xi=\frac{S_p}{Y_p}$ conditioned on Y_p for different values of p. The values of the conditioning parameter Y_p are taken as windows of size $<Y_p>/3$ centered at the values indicated in Figs. 2. We see from Fig. 2a that the distributions for p=3 are bimodal and show some dependence on Y_p. Recall that for p=3, the equivalent distribution for the case of Gaussian X_i is uniform; we conclude that the conditional distributions of ξ for small p do depend on the distribution of X_i. For p=10 (Fig. 2b), the distributions exhibit Gaussian-like behavior while for p=50 (Figs. 2c and 2d) they differ very little from the Gaussian. As p increases, the differences among the various curves corresponding to different values of Y_p tend to diminish. In particular, for p=50, all four curves coalesce quite well.

The behavior just discussed can be described in the same terms as the Kolmogorov's similarity hypotheses: the pdf of ξ conditioned on Y_p depends on Y_p and p. For $p\gg1$, it tends to a Gaussian with zero mean and unity variance. This behavior is independent of the particular distribution one chooses for X, although details for small values of p do depend on this choice.

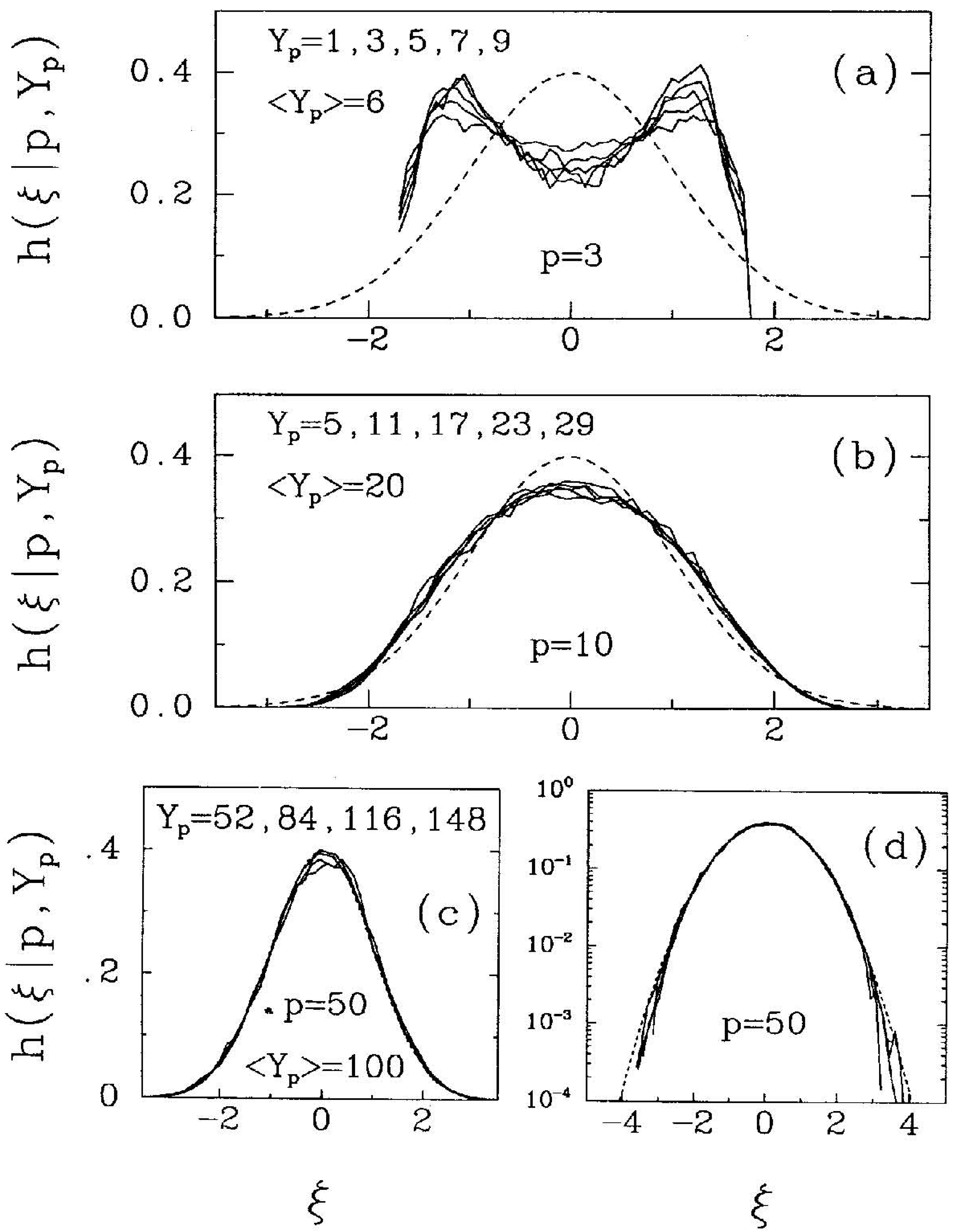

Fig. 2: Plots of the distribution of $\xi = S_p/Y_p$ conditioned on Y_p, $h(\xi|p, Y_p)$, for different values of the parameter p, when the X_i are exponentially distributed. (a) p = 3, (b) p = 10, (c) p = 50, and (d) p = 50, with logarithmic ordinate. The dashed lines in each of these figures is the normal density.

A final remark is in order. Even though this result looks similar to the central limit theorem, its says something more. While it is not difficult to derive the central limit theorem from it, as far as we are aware, this result cannot be derived from the central limit theorem.

5. CONCLUSIONS

We have seen that Kolmogorov's hypotheses can be phrased in a broader context than envisaged originally. The arguments leading to the hypotheses are taken from probability theory, and do not invoke any physical picture such as the cascade of energy. However, it is necessary to clarify the connection to the physics when dealing with real turbulence data. First of all, it is clear that in real turbulence data there exist correlations between the X_i. Therefore, the results discussed in this paper do not apply to turbulence in a straightforward fashion. In correlated processes where the X_i are not independent, the exponent H in

$$S_p = V\, Y_p^{2H} \tag{24}$$

will be different from 1/2. For turbulence we have to go to the Navier-Stokes equations to find the value of H. In the present notation, Kolmogorov's equation for the third-order structure function in the inertial range, $\langle \Delta u(r)^3 \rangle = -(4/5) r\varepsilon$, which is derived from Navier-Stokes equations with the additional assumptions of local isotropy and homogeneity, states that

$$\langle S_p^3 \rangle = -\frac{4}{5}(15^{1/3}) \langle Y_p \rangle. \tag{25}$$

If we assume that V is independent of Y_p for p»1, we are led to the choice H=1/3. Equation (24) is in this case the same as Eq. (6), and Kolmogorov's hypotheses follow. This also implies that $\langle V^3 \rangle$ is different from zero, and hence, the limiting pdf of V for p»1 is not a Gaussian. The detailed study of these issues for correlated systems - in particular for the fractional Brownian motion - and its comparison with atmospheric turbulence will be published elsewhere[7].

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