Fractals, Vol. 2 No. 2 (1994) 253–263

Flow

FRACTALS IN FLUID MECHANICS

K. R. SREENIVASAN
Mason Laboratory
Yale University
New Haven, CT 06520-8286, USA

The basic concepts of fractal geometry are relatively simple. Although they are not entirely new, the recognition that these simple notions form a unified language for a variety of disciplines in natural science is due to Mandelbrot.[1] Our objective is to assess briefly the role of fractals and multifractal measures in fluid flows broadly, including turbulence and combustion. As applications have yet to mature, the report captures a snap-shot of the changing scene. We focus on activities that are common to both fluid dynamics and fractals and ignore some isolated aspects; we also omit comments on possible fractal structure obtained in chaotic mixing. Finally, we emphasize the question of how fractals enter physical problems, not the classical results. Much of the material to be covered below can be found in refererences cited in the bibliography.[2–7] Other references cited are not meant to be exhaustive.

1. AGGREGATION IN PARTICLE-LADEN FLOWS

Fluid dynamics of particle-laden flows is replete with applications. If interparticle interactions are ignored, the essential problem is one of understanding how various physical properties of the flow (such as effective viscosity) are altered by particle loading; alternatively, the effect of the flow on particle motion is also of interest. Under circumstances often dictated by hydrodynamics, inter-particle interactions may become important and lead to the formation of aggregates (that is, structures in which particles stick together irreversibly).

There are two basic aspects to the study of aggregation: kinetics and geometry. Kinetics involve the quantitative description of the time evolution of aggregates and their size distribution, whereas geometry is concerned with quantitative description of the structure of aggregates. The first aspect has been studied for a long time, whereas the second aspect, which used to be the backwaters of aggregation studies until recently, has taken on a life of its own since the advent of fractals.

Aggregation can occur in a variety of ways: electrodeposition (induced by electric field), sedimentation (induced by gravity), filtration (caused by the particle motion stopped by small pores), and so forth, and can be either of the particle-cluster type or cluster-cluster type. A simple kinetic model[8] called the diffusion-limited-aggregation (DLA) has been studied extensively and thought of (with some minor modifications) as a paradigm for a number of processes such as protein aggregation, colloid clusters of gold and silica, soot formation, viscous fingering in porous media (at least when the flow rates are high), dielectric breakdown, dendritic solidification, and so forth. The common feature among many of these phenomena is that a suitably defined potential governed by the Laplace equation can be defined[a]. It is therefore worth examining DLA briefly.

In the two-dimensional version of DLA, one considers a lattice on a plane and first chooses the origin for the cluster. A particle, the "seed" for the aggregate, is placed at the origin. One then considers a large circle of radius R, centered at the origin, and chooses a point at random on this circle. A particle is released at a site nearest to this point and is allowed to execute a random walk on the lattice. If the random walker reaches a site nearest to the origin, it stops and stays stuck to the seed. (If the particle exits the circle without getting close to the origin, it is abandoned.) Another particle is released from a lattice point close to another randomly chosen point on the circle, and allowed to stick to the seed or the two-particle cluster (as the case may be). The process is continued until a cluster of the desired size is reached. The algorithm can be extended to any higher dimension.

The DLA aggregates are fractal structures with a dimension of about 1.7 in two dimensions and 2.5 in three dimensions.[2,9] The growth of the cluster is governed by the so-called harmonic measure which is the probability that a random walker approaching the cluster from the far-away circle hits the cluster in a certain infinitesimal interval along the boundary. The harmonic measure is a solution of Laplace's equation for the electrostatic potential when the cluster boundary is taken to be at zero potential and the circle at a potential of unity. It is intermittent in appearance and amenable to multifractal analysis.[10,11] As remarked in Sec. 3, the basic evidence for the self-similarity in the DLA structure comes from the self-similarity of the multiplicative process.[11]

Most fractal and multifractal characteristics of DLA have been extracted from computer simulations[2,11]; to our knowledge, there are no exact analytical results.

2. APPLICATIONS IN POROUS MEDIA AND VISCOUS FINGERING

2.1 Porous media

The Stokes equation governing the fluid motion in porous media is linear. It can be reduced to the Darcy law (according to which the velocity is linearly proportional to the pressure

[a]Just as the Ising model describes the essential physics of a wide variety of materials near the critical point, the hope has been expressed that DLA would describe a variety of growth processes. DLA and turbulence (see Sec. 5) are often thought to be the paradigm problems for serious multifractal applications.

drop) by using appropriate assumptions of homogeneity. Equivalently, this can be written as the Laplace equation for the pressure. This suggests analogies with DLA, except that the boundary conditions in the porous media are more difficult to assess.

The difficulty lies in describing the boundary between fluid-filled regions and solid-like regions. One procedure is to model porous media as a network of capillary tubes. This model conceives of the solid phase as being continuous with interconnected fluid-filled pores running through it. A major simplification is achieved in this way because the Poiseuille law valid for laminar flow through pipes holds for each capillary. Alternatively, one assumes the fluid phase as continuous and the solid particles as obstacles for the flow.

Fractals appear in studies of flow through porous media because of the random character and the extreme variety of shapes encountered. The pore space, the solid phase and the solid-pore interface could all exhibit fractal scaling[b]. Various estimates for the fractal dimensions of these three aspects have been made.[3,4]

In practical applications, the one unknown is the permeability of the medium which, in general, is a tensorial quantity. The permeability in most cases is measured by pressure drop experiments or estimated empirically. A useful goal would be to relate the permeability of a medium to its characteristic fractal dimensions. This has not yet been accomplished.

2.2 Viscous fingering in the Hele-Shaw cell

When a low viscosity fluid is pushed into a high viscosity immiscible fluid, Saffman-Taylor fingers develop[12]; these fingers occur singly, and are broad and smooth in shape. The equation governing viscous fingering in the Hele-Shaw cell is formally the same as that for flow through porous media, except that:

(a) the permeability in the former is not real but related to the gap between the plates, while that in the latter depends on the local volume fraction;

(b) the former has a well-defined surface tension at a normal fluid-fluid interface, while the use of surface tension in the latter is rather murky.

The Saffman-Taylor fingers correspond to the wavenumber with maximum instability, and their width varies as the square root of the surface tension between the two fluids (if all other conditions are held fixed). As the surface tension is lowered, the fingers split more and more; but there is a practical limit to how low the surface tension can get. If the high-viscosity fluid is a miscible colloidal solution with shear-dependent viscosity, the interface grows to be fractal-like in appearance[13] even when the capillary number[c] is moderately high; it appears that the more non-Newtonian the solution, the more the tendency to fractal fingering. In spite of recent studies to model this behavior,[14,15] the basic physics of fingering in non-Newtonian fluids is not well-understood.

It has been argued[16] that, if interfacial tension is ignored[d] the viscous fingering problem is analogous to DLA; indeed, the viscous fingering patterns in radial Hele-Shaw cells (which

[b] All porous media may not have fractal pore space, but the surface of the grains is very often fractal due to long-term chemical or sintering processes.

[c] The capillary number is the ratio $\mu U/\sigma$, where μ, U and σ are, respectively, the viscosity coefficient for the driven fluid, fluid velocity and interfacial tension.

[d] In practice, this is far from being correct. The arguments postulating similarity between viscous fingering and DLA must be examined critically in spite of the resemblence of the observed fractal patterns.

do not have the anisotropic constraining effects of rectangular cells) are similar to the DLA structure and possess roughly the same fractal dimension. The growth of viscous fingers depends on the pressure gradient, which therefore plays a role analogous to the harmonic measure for DLA. Since the pressure field is not easily measured, related growth measures have been defined and characterized by multifractals.[3] If the medium in the Hele-Shaw cell is porous, one obtains fractal structure even for Newtonian fluids and the capillary number is moderately high.[3] For a review of viscous fingering, see Homsy.[17]

2.3 Percolation and diffusion

Percolation involves the spreading of a fluid in a random medium, where the words "fluid" and "medium" are used in a certain general sense. The role of randomness in percolation is quite different from that in diffusion. In the latter, the Brownian particle executes a random walk, whereas percolation deals with randomness that is frozen into the medium. While any position in the medium can be reached by diffusion, the spreading in percolation is confined to finite regions except when the so-called "percolation threshold" is exceeded. In model studies, one considers a square lattice whose sites are either randomly occupied (with probability p) or empty (with probability $1-p$). Occupied sites represent parts of the medium through which the fluid can percolate while empty sites represent those parts that cannot be invaded by the fluid. Connected sites form clusters. On an infinite sample, all clusters remain finite below the percolation threshold, $p = p_c$. For $p > p_c$, infinite clusters appear with a finite probability, and the probability of this occurrence varies with p as $(p - p_c)^b$. Percolation clusters are self-similar and possess,[18] in the limit of large clusters, a fractal dimension of 1.89. The "hull" of the diffusion front is also a fractal with a dimension of 1.75, whereas its external perimeter has a dimension[19] of about 1.37.

One should also mention here the "invasion percolation" (applicable for low flow rates) where the water displacing oil in porous rocks may trap regions of oil.[20] Randomness encountered by the invading fluid would now also depend on the trapped regions.

Branched polymers have size distributions that are self-similar,[21] and are therefore candidates for fractal description. Some useful analogy exists between polymers and percolation studies.[22] As in percolation, the fractal dimension of branched polymers can be related to other indices characteristic of the polymer size above a "percolation threshold".

3. ONSET OF CHAOS IN NEWTONIAN FLUID FLOWS

Multifractals have played a powerful role in characterizing universality at the onset of chaos in low-dimensional systems. The renormalization theory has been worked out for the onset of chaos for period-doubling[23,24] and quasiperiodic cases.[25,26] Experiments in forced Rayleigh-Benard convection[27,28] and the near-field of oscillating cylinders at low Reynolds numbers[29,30] strongly support the universality theory: the so-called $f(\alpha)$ curve describing the non-uniform distribution of the invariant measure on the attractor at the onset of chaos agrees well with that calculated for one-dimensional circle maps. This is the power of universality, and illustrates an application of multifractals where powerful theory and imaginative experiments have come together satisfactorily.

The multifractal spectrum or the $f(\alpha)$ curve provides a thermodynamic — and hence degenerate — description of the dynamical system. Even so, it has been possible to develop[31] a basis for extracting (up to a level of detail which depends on our knowledge of the system in terms of other statistical measures, such as the similarity structure exhibited by the power spectral density) the multiplicative process leading to the observed multifractal state.

It should be noted that dynamical universality was indeed known before the multifractal formalism came to the fore. For example, the Feigenbaum number in period doubling bifurcations[23,24] was experimentally observed in convection experiments.[32] Furthermore, it was also known that the microscopic information about a deterministic dynamical system and its scaling properties could be characterized in detail by the scaling function.[23,24] However, the experimental measurement of the scaling function is quite difficult, and the advent of multifractals (albeit statistical) made the search for universality significantly easier.

As a final note, we wish to emphasize that chaos (which involves temporal complexity) is quite different from turbulence (which involves spatial as well as temporal complexity); however, transition to chaos is sometimes relevant to early stages of transition to turbulence.

4. NON-REACTING AND REACTING TURBULENT FLOWS

High-Reynolds-number turbulent flows consist of a wide range of interacting scales. The ratio of the largest to the smallest scale increases roughly as the $3/4$ power of the large-scale Reynolds number. The conventional wisdom is that statistical similarity prevails over a range of intermediate scales; the precise form of this similarity and the scale-range over which it holds are matters of much interest. It has been thought that fractals and multifractals provide proper tools for better description, and better understanding, of aspects of turbulence. In the following summary statements, we indicate the degree of our confidence by FC (for fairly certain) or P (for provisional); the latter means that the results come essentially from one laboratory. For references to original sources and further discussions, see Sreenivasan.[6]

4.1 Fractal scaling of flames, iso-surfaces and interfaces

The main question here is whether these objects can be treated as thin surfaces with many (essentially) self-similar convolutions. It is useful to quote a few results in some detail for a paradigm problem in turbulence.

(a) For the scalar interface (i.e. outer boundary of scalar-marked regions in unbounded free shear flows), fractal scaling occurs over much of the interval between the integral scale and the Kolmogorov scale. The dimension in this scale range is 2.35 ± 0.05 (FC).

(b) In fully turbulent parts of shear flows, iso-scalar contours possess a fractal scaling with a dimension of 2.67 ± 0.05. The scaling range is smaller than that for (a), and the inner cut-off occurs at some multiple of the Kolmogorov scale; the latter can be estimated *a priori*[33] (FC).

For both (a) and (b), data exist on the Reynolds number variation of the dimension (P). On the basis of the fractal structure, efforts have been made to model the interface as a chaotic system.[34,35]

The original heuristic explanation[6] for these observations relied on Reynolds number similarity. In Constantin,[33] by combining the "co-area formula" culled from measure theory with the convection-diffusion equation governing the scalar evolution, it has been possible to obtain the fractal dimensions of iso-scalar surfaces and interfaces in turbulent flows. The only information about the velocity field that enters the calculation is the scaling of the first-order structure function. The dimensions so obtained are in excellent agreement with experiment.

(c) For the range between the Batchelor scale and the Kolmogorov scale, the fractal dimension of iso-scalar surfaces as well as interfaces approaches 3 as the Schmidt number approaches infinity. The finite Schmidt number correction appears to be logarithmic (P). Newer theories have also been developed[36,37](P).

(d) The results (a) and (b) hold also for vorticity interfaces and iso-vorticity contours (P).

(e) The dimension of flame surfaces depends on the ambient turbulence level, but the flame front in both diffusion and premixed flames has a fractal dimension of 2.35 for large turbulence levels (FC). Note that, in contrast to high-Reynolds-number isothermal flows, the scale range of convolutions in high temperature flames is small, except when the turbulence levels are high. Several fractal-based closure models have been attempted in combustion (P).

4.2 Results from time series analysis

The difficulty in the determination of the fractal structure of a time series lies partly with the definition of a suitable cover, and partly with proper recognition of the cut-offs between global, local and latent dimensions.[38] These artifacts are now moderately well-understood, and the fractal structure of a time series of velocity or temperature fluctuations in high-Reynolds-number turbulent flows has been explored.[39] These time traces resemble fBm traces with the exponent $H \approx 0.35$, and the (local) fractal dimension $D = 2-H \approx 1.65$. This is consistent with the classical theory of Kolmogorov,[40] and is comparable in the quality of scaling.[41] An implication is that the dimension of iso-velocity and iso-temperature surfaces in fully developed turbulence is about 2.65, consistent with the result (b) in Sec. 5.1. At moderate Reynolds numbers, the scaling is better for the spatial aspects and ambiguous for temporal data with the exception of those taken for interfaces, for which modest scaling occurs even at moderate Reynolds numbers.

4.3 Multifractal scaling

The evidence is strong that multifractals are useful tools for describing scaling properties of structure functions,[42,43] and of turbulent energy dissipation rate and scalar dissipation rate in turbulent flows[6,44,45]; provisional evidence[6] suggests that other positive definite quantities, such as the square of turbulent vorticity, can be described similarly. This type of work has produced the following results.

(a) Phenomenological models for small-scale intermittency, with outcomes consistent with experiment, have been constructed.[46] These models have yielded correct intermittency corrections in the inertial and dissipation ranges, produced a refinement of the scaling of the power spectra in the dissipation range,[47] generated stochastic signals which do not differ from the real turbulent signals in most respects,[48,49] and so forth. The latter idea has been used for providing realistic initial conditions for large-scale turbulence calculations.[50]

(b) The nature of the observed near-singularities appears consistent with the mathematical result[51] on the partial regularity of weak solutions of the Navier-Stokes equations.

(c) By assuming that the scalar as well as vector fields are fractal graphs with the measured dimension, a broad theoretical apparatus has been constructed[52] to explain the Kolmogorov scaling in the inertial range and its various modifications.[53,54]

5. GEOPHYSICAL PHENOMENA

Geophysical fields (such as cloud radiance, rainfall, temperature and pollution records, sea surface infrared reflectivity, sea surface geometry, lightning paths, and so forth) are a result of nonlinear processes involving different fields at widely varying scales. In each case, the statistical invariance of a wide scale range suggests that fractal concepts may be useful. One could ask, for instance, if lightning paths and cloud boundaries are fractal and, if so, measure their fractal dimensions. Among the first fractal measurements made in geophysics was the dimension of cloud boundaries[55]: the dimension of fair-weather clouds as well as large clouds is about 2.35 (the same as that of scalar interfaces in turbulent flows), whereas clouds strongly affected by the mean wind shear possess smaller dimensions.[6]

In geophysics, there are numerous candidates which are potentially fractals. As already remarked in Sec. 3, it is not enough to seek information about the binary picture of whether or not there is rainfall (for example), but one needs to know something about the different rainfall rates. The variability and intermittency of rainfall records suggest that multifractals could be quite useful. Preliminary multifractal analysis has been made for rainfall rates, cloud radiance and other geophysical fields. For a survey of articles on these topics, see work by Schertzer and Lovejoy.[7]

Because accurate computations of high-order statistics require extraordinary amounts of data, geophysical measurements have generally been restricted to low-order multifractal measures.

It is claimed that high-order moments may diverge in geophysical situations.[7,56] However, controlled measurements in the atmospheric surface layer[57] suggest that the apparent divergence is an artifact of relatively small data records.

6. APPLICATIONS FOR IMAGE COMPRESSION AND DATA INTERPOLATION

The application of fractals in the construction of natural scenery such as mountains, clouds, lakes, and so forth is well-known.[2,58–60] This technology has found application in diverse ventures such as movie-making, art and music. Since they are far from fluid dynamics, they will not be discussed here.

Fractals find a useful application in image compression and reconstruction. Lovejoy and Mandelbrot[61] constructed cloud images using the model that a rain field is composed of self-similar pulses; the areas of these pulses had power-law distributions and rainfall rates were random. The resulting pictures looked quite realistic even though real clouds are stratified in the vertical direction, and the governing principle for reconstruction should be self-affinity rather than self-similarity.

Image compression by taking advantage of fractal structure is the subject of Barnsley[62]; related applications involve fractal interpolation schemes by minimizing the Hausdorff distance. However, extensive application of these ideas to turbulence data has not been made. It is worth emphasizing that image compression on the basis of self-similarity or self-affinity is not especially the domain of fractals alone; in fact, applications based on wavelet theory work quite well.[63]

7. A QUALITATIVE ASSESSMENT OF PAST ACCOMPLISHMENTS

It is sometimes implied that fractals embody one of the basic truths about complexity: that all known truths about Nature are expressible in the form of some generalized concepts, and that fractals represent one such generalized concept. *This is the appeal of fractals.*

This philosophical appeal has occasionally produced an exaggerated response,[64] and the "wheat" cannot always be separated from "chaff". *Note, however, that other glamorous concepts such as catastrophe theory never came close, even in their heydays, to enjoying the degree of appeal that fractals possess.* (Whether popular appeal is always correlated with scientific significance is another matter.)

Fractals have been taken seriously in mainstream science for something like fifteen years. We hope that the summary given above makes clear the impressive fact that fractals have had much impact on providing descriptive and incremental understanding in many fields.

To the examples already cited, we can add a few more here. Suppose we need to model the spread of forest fires, or the seepage of ground water or radioactive substances. While exact formulation of these problems is in principle possible, it would be futile to take this approach because of their extreme complexity. Instead, simulations of the type carried out in percolation studies can be quite useful in providing an overall picture (even if incomplete). Similarly, a host of other phenomena such as rainfall rates can be modelled by multifractals. *Fractals provide tools for modeling a variety of complex systems with some realism.*

Even though the evidence is still not compelling, it is strong enough to think that fractals are well-suited for handling scaling phenomena. *There is a certain tangible benefit of unity that fractals have brought to apparently unrelated areas of science.*

Much of the work using fractals has so far occurred in the physics community (or those small pockets of fluid mechanics community with relatively strong ties to physics), and the focus has not been engineering applications. It can be said that a serious beginning has been made. However, *for fractal-based models to solve complex problems at the level of engineering utility, such as producing new materials or new numerical codes for computing high-Reynolds-number turbulent flows, it is essential for the engineering community to take sustained interest in these tools.*

Whenever fractal (and multifractal) scaling is observed in fluid flows, it is nearly always statistical. This means that they provide only partial information, even if valuable and unique. Therefore, as with all partial information, *the degree to which one can make sense of an observation depends on the ingenuity of the individual trying to extract it. This is the ultimate constraint.*

Fractal-related work falls into three broad classes: description, explanation and prediction. To-date, description and measurements of fractal dimensions have consumed most of the energy. This effort was not trivial because it involved an understanding of the potential of fractals when they were still not commonplace, as well as the improvisation of theoretical, experimental and computational techniques. As for the second aspect, in percolation, multifractal measures, scalar interfaces in turbulence, and in other areas, a variety of results exist in which phenomenology and rigor play complementary roles. More such results are likely to emerge as fractals integrate increasingly with applications-oriented research. As regards predictions, a few powerful ones have already been pointed out in Sec. 5, even though their engineering consequence is still unclear. *On the whole, the past efforts have been quite rewarding.*

8. FUTURE PROSPECTS

The question of what constitutes future opportunities in fractals may well be thought to belong to the domain of mathematics. That, however, is not very useful. All opportunities in specific fields of fluid mechanics will have to be assessed in the context of those specific fields. That would be a Herculean task for this brief report, and it therefore seems best to restrict attention to a few broad questions common to most applications of fractals.

(a) The situation typical of most fractal-related studies is that much of our knowledge comes from experiment and simulations. While simulations and experiments are very useful, they are often limited by approximations, finite size effects, noise, and other artifacts, and cannot supplant theoretical results. *There is at present a big backlog of observations without the backup of solid explanations.*

(b) Much of the physics in problems that possess scale-similar phenomena is hidden in the cut-off scales. *It is essential in practice to pay greater attention to cut-off scales and cross-over phenomena.*

(c) It would be essential to know, at least for one hard problem like turbulence, the relation between dynamics on the one hand and fractal geometry and multifractal measures on the other. (Eyink[65] claims to have done just that recently.) What aspects of the hydrodynamic equations yield the fractal structure of its solutions? How may one show, without empirical input, that the turbulence structure is fractal, obtain dimensions of its various facets and generate a closed list of fractal dimensions to define turbulence uniquely? *Without such knowledge, it is difficult to make a case for the inevitability of fractals as the tool of choice for studying large classes of nonlinear problems.*

(d) There are practical issues that appear to be within the realm of near-term possibility. For example, it appears possible to construct a much better model for flame speeds in premixed flames; model the gross spread rates of jets; generate a robust large-eddy-simulation model based on multifractals. Tentative progress has already occurred on these fronts. Such efforts should be driven more by expertise in the respective fields rather than in fractals *per se.*

REFERENCES

1. B. B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, San Fransisco, 1982.
2. P. Meakin, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz, Academic, 1987.
3. J. Feder, *Fractals*, Plenum, 1988.
4. D. Avnir, editor, *The Fractal Approach to Heterogeneous Chemistry*, Wiley, 1989.
5. T. Vicsek, *Fractal Growth Phenomena*, World Scientific, 1989.
6. K. R. Sreenivasan, Annu. Rev. Fluid Mech. **23**, 539 (1991).
7. D. Schertzer and S. Lovejoy, editors, *Nonlinear Variability in Geophysics*, Kluwer, 1991.
8. T. A. Witten and L. M. Sander, Phys. Rev. Lett. **47**, 1400 (1981).
9. F. Argoul, A. Arneodo, G. Grasseau, and H. L. Swinney, Phys. Rev. Lett. **61**, 2558 (1988).
10. A. Amitrano, A. Coneglio, and F. Di Liberto, Phys. Rev. Lett. **57**, 1016 (1986).
11. B. B. Mandelbrot and C. J. G. Evertsz, Nature **348**, 143 (1990).
12. P. G. Saffman and G. I. Taylor, Proc. Roy. Soc. Lond. **245A**, 312 (1958).
13. J. Nittmann, G. Daccord, and H. E. Stanley, Nature **314**, 141 (1985).
14. P. G. de Gennes, Europhys. Lett. **3**, 195 (1987).
15. S. D. R. Wilson, J. Fluid Mech. **220**, 413 (1990).
16. L. Patterson, Phys. Rev. Lett. **52**, 1621 (1984).
17. G. Homsy, Annu. Rev. Fluid Mech. **19**, 271 (1987).
18. A. Aharony, in *Directions in Condensed Matter Physics*, edited by G. Grinstein and G. Mazenko, World-Scientific, 1986.
19. B. Sapoval, M. Rosso, and J. F. Gouyet, Phys. Rev. Lett. **46**, L149 (1985).
20. D. Wilkinson and J. F. Willemsen, J. Phys. A. **16**, 3365 (1983).
21. M. Daoud and J. E. Martin, in *The Fractal Approach to Heterogeneous Chemistry*, edited by D. Avnir, Wiley, 1989.
22. S. Havlin and D. B. Avraham, Adv. Phys. **36**, 695 (1987).
23. M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978).
24. M. J. Feigenbaum, J. Stat. Phys. **21**, 669 (1979).
25. M. Feigenbaum, L. Kadanoff, and S. Shenker, Physica D **5**, 370 (1982).
26. S. Ostlund, D. Rand, J. Sethna, and E. Siggia, Physica D **8**, 303 (1983).
27. J. Stavans, F. Heslot, and A. Libchaber, Phys. Rev. Lett. **55**, 596 (1985).
28. M. H. Jensen, L. Kadanoff, A. Libchaber, I. Procaccia, and J. Stavans, Phys. Rev. Lett. **55**, 2798 (1985).
29. D. J. Olinger and K. R. Sreenivasan, Phys. Rev. Lett. **60**, 797 (1988).
30. D. J. Olinger, PhD thesis, Yale University, 1990.
31. M. J. Feigenbaum, M. H. Jensen, and I. Procaccia, Phys. Rev. Lett. **57**, 1507 (1986).
32. A. Libchaber and J. Maurer, J. de Physique **41**, c3 (1980).
33. P. Constantin, I. Procaccia, and K. R. Sreenivasan, Phys. Rev. Lett. **68**, 2762 (1992).
34. Y. Tsuji, K. Honda, I. Nakamura, and S. Sato, Phys. Fluids **A3**, 1941 (1991).
35. C. Meneveau, Phys. Fluids **A4**, 1587 (1992).
36. S. Grossmann and D. Lohse, preprint, 1994.
37. R. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
38. B. B. Mandelbrot, in *Fractals in Physics*, edited by L. Pietronero and E. Tosatti, pages 3–28, Elsevier Publishing Company, 1986.
39. K. R. Sreenivasan and A. Juneja, submitted to J. Fluid Mech., 1992.
40. A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941).
41. A. A. Praskovzky, J. F. Foss, S. J. Kleis, and M. Y. Karyakin, Phys. Fluids A **5**, 2038 (1993).
42. U. Frisch and G. Parisi, Fully developed turbulence and intermittency, in *Turbulence and Predictability in Geophysical Fluid Dynamics*, edited by M. Ghil, page 84, North-Holland.
43. R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984).
44. C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).
45. C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).
46. K. R. Sreenivasan and G. Stolovitzky, (1994), to appear in J. Stat. Phys.
47. B. Castianga, Y. Gagne, and E. Hopfinger, Physica D **46**, 177 (1990).
48. J. Eggers and S. Grossmann, Phys. Rev. A **45**, 2360 (1992).
49. A. Juneja, D. Lathrop, K. Sreenivasan, and G. Stolovitzky, Phys. Rev. E **49** (1994).

50. A. Juneja and G. Ertebacher, to be published, 1994.
51. L. Caffarelli, R. Kohn, and L. Nirenberg, Commun. Pure. Appl. Math. **35**, 771 (1982).
52. P. Constantin and I. Procaccia, Phys. Rev. E **47**, 3307 (1993).
53. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics II*, M.I.T. Press, 1971.
54. F. Anselmett, Y. Gagne, E. J. Hopfinger, and R. A. Antonia, J. Fluid Mech. **140**, 63 (1984).
55. S. Lovejoy, Science **216**, 185 (1982).
56. B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1972).
57. M. Fan, PhD thesis, Yale University, 1990.
58. R. F. Voss, in *Fundamental Algorithms in Computer Graphics*, edited by Earnshaw, pages 805–835, Springer, 1985.
59. R. F. Voss, Physica D **38**, 362 (1989).
60. H.-O. Peitgen and P. Richter, editors, *The Beauty of Fractals*, Springer, 1986.
61. S. Lovejoy and B. B. Mandelbrot, Tellus **37A**, 209 (1985).
62. M. Barnsley, *Fractals Everywhere*, Academic, 1988.
63. L. Zubair, K. R. Sreenivasan, and V. Vickerhauser, in *Studies in Turbulence*, edited by T. Gatski, S. Sarkar, and C. G. Speziale, Springer, 1992.
64. L. P. Kadanoff, Phys. Today **39**, 6 (1987).
65. G. L. Eyink, submitted to Phys. Fluids, 1994.