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Accumulation Rates of Spiral-Like Structures in Fluid Flows

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Accumulation rates of spiral-like structures in fluid flows

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Spiral-like structures in the passive scalar fields of typical fluid flows are examined with special reference to their rate of accumulation. The study was motivated in the particular context of the Kolmogorov capacity of such structures and their role in potentially dominating the measured dimension of iso-scalar contours in fully turbulent flows, but it may have broader implications. A specific conclusion is that the structures occurring in these flows are better modelled by a logarithmic spiral rather than a power-law spiral, and therefore do not make significant contributions to the empirical estimates of dimensions by box-counting methods. Another conclusion is that, at least for the Reynolds numbers examined here, the spiral-like structures (such as they are) rarely possess more than four or five turns.

1. Introduction

The first serious consideration to spiral structures in turbulence was given by Lundgren (1982), who showed that the axially stretched rolled-up spiral vortices could result in Kolmogorov's $-\frac{5}{3}$ power law for the inertial range in the spectral density of velocity fluctuations. Moffatt (1984, 1990) recognized that the accumulation points of velocity discontinuities (at scales larger than the viscous cut-off scales) associated with spiral structures could produce non-integer powers in the spectral roll-off rates, and demonstrated this by some simple and illuminating calculations. Gilbert (1988) examined, in two dimensions, the winding up of vortex patches due to strong concentrated vorticity in the vicinity and showed that the energy spectral density in two-dimensional turbulence could have fractional roll-off rates between three and four.

Another development concerning spirals in turbulence has recently occurred in the course of explaining the measured fractal characteristics of interfaces in turbulence. Measurements in several turbulent flows (Sreenivasan & Meneveau 1986; Prasad & Sreenivasan 1990) as well as a theoretical analysis of the convection-diffusion equation governing a passive scalar (Constantin *et al.* 1991) show that, in a certain range of scales, the self-similar interfaces of the scalar mixed by turbulence are usefully characterized by a fractal dimension. Typically, these measurements involve obtaining thin two-dimensional sections of the interface embedded in the three-dimensional space, and running a 'box-counting' algorithm: If $N(\epsilon)$ is the number of boxes of size ϵ containing the (section of the) interface, the dimension is obtained by the relation $N(\epsilon) \sim \epsilon^{-D}$. By invoking the method of intersections – for whose justification see Mandelbrot (1983) in a broad context and Prasad & Sreenivasan (1990) in this particular context – the dimension of the surface itself was obtained by $D+1$.

In interpreting this observation, Vassilicos & Hunt (1991) emphasized that the

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Figure 1

Figure 2

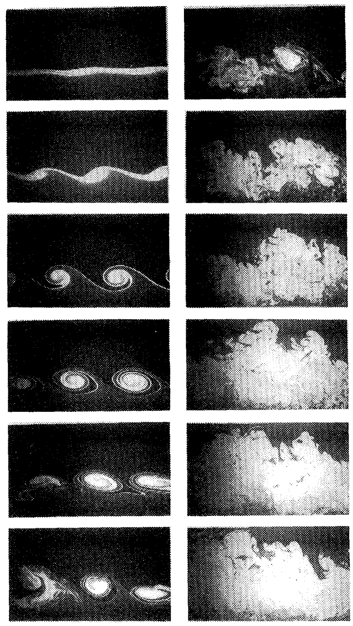

Figure 1. A section of a water jet emerging through a circular nozzle. The jet is made visible by laser induced fluorescence. A beam from a pulsed Ng:YAG laser was shaped into a sheet by a combination of lenses and was directed towards a jet (Reynolds number 3900) mixed with a fluorescing dye. The light sheet excited fluorescence radiation in the plane of intersection, which was then captured onto a camera. The image region extends from 8 to 24 nozzle diameters. The laser sheet thickness was on the order of the average Kolmogorov scale. The spiral-like structures present in the jet section are the object of attention here.

Figure 2. The temporal evolution of the shear flow between two countercurrent streams was visualized by inserting small amounts of the fluorescing material between the streams, and using the laser-induced-fluorescence technique already described (for details, see Sreenivasan *et al.* 1989). As a consequence of the Kelvin–Helmholtz instability, the shear layer rolls up in the form of spiral-like structures before attaining a more complex state. Time increases from top left to bottom right. Characteristics of some of these structures are examined.

box-dimension defined above is not the Hausdorff dimension but the Kolmogorov capacity dimension (Kolmogorov 1958). To understand the significance of this observation, it is necessary to consider briefly the definition of the Hausdorff dimension. Consider a covering of a geometrical object, embedded in d-dimensional space, with d-dimensional boxes of sizes ϵ_i, and define the quantity

$$H(\epsilon) = \inf \sum \epsilon_i^D. \tag{1}$$

The infimum in (1) extends over all possible countable coverings of the object subject to the constraints that $\epsilon_i \leqslant \epsilon$. In the limit $\epsilon \to 0$, the quantity $H(\epsilon)$ is zero or infinite according to whether D is larger or smaller than a critical value D_{H}. This critical value is the Hausdorff dimension of the object. In general, the box dimension $D \geqslant D_{\mathrm{H}}$. Thus, even though the box dimension of an object may be non-integer and larger than its topological dimension, it may well be that the Hausdorff dimension is the same as the topological dimension, and the object is not a true fractal in the sense of its Hausdorff dimension being larger than the topological dimension. It follows that, for the interfaces, the fact that the measured non-integer box dimension is greater than their topological dimension of 2 is not a sufficient condition for inferring the existence of an underlying fractal. Vassilicos & Hunt noted that a spiral

$$r(\theta) = C\theta^{-\alpha}, \tag{2}$$

(where (r, θ) are polar coordinates and $C, \alpha > 0$ are constants) has the Kolmogorov capacity

$$D = 2/(1+\alpha), \tag{3}$$

while its topological dimension and the Hausdorff dimension are both unity. (If a spiral possesses only a few turns, say less than five, Vassilicos & Hunt proposed a different expression, but it is unclear that a theoretical discussion of aspects such as Kolmogorov capacity and Hausdorff dimension of spirals with very few turns is interesting.) These authors speculated that the measured fractal dimension may indicate the presence of spirals in the turbulent interface (rather than its being a fractal). The argument seemed to derive additional strength from Moffatt's observation that a single spiral structure could produce Kolmogorov-type inertial range spectrum (even though a more complete explanation would probably require a hierarchy of spiral structures at different length scales). It is attractive to think that a well-defined classical object could at once explain non-integer dimensions as well as non-integer spectral roll-off rates.

It should be noted that, in contrast to the power-law spiral (2), the Kolmogorov capacity of a logarithmically accumulating spiral,

$$r(\theta) = C\,\mathrm{e}^{-\alpha\theta}, \tag{4}$$

is unity. The logarithmic spiral accumulates onto its centre too fast to be space-filling. Therefore, a logarithmic spiral does not contribute to box-counting calculations or affect the measured value of the interface dimension.

This report examines the spiral structures found in passive scalar fields of fluid flows with the specific purpose of determining their accumulation rates in turbulent jets, developing mixing layers, as well as smoke rings. Some interpretations of the data are presented.

2. Experiments and measurement methods

The bulk of the search for spiral structures in turbulence was made in the longitudinal and transverse slices of the concentration field of axisymmetric water jets (Prasad & Sreenivasan 1990). The slices were obtained by the laser-induced fluorescence technique. Experimental details are given by Prasad & Sreenivasan and will not be repeated here. The nozzle Reynolds number was 3900 and the Schmidt number of the dye was of order 1000, so that the dye may be regarded as non-diffusing and therefore marking the nozzle fluid. Two longitudinal images covering the region approximately 8–24 nozzle diameters downstream from the nozzle, together with six transverse sections (that is, sections perpendicular to the jet axis) taken at or around 18 nozzle diameters downstream, were examined.

Some evidence of spiralling structures is visible in the concentration field, $c(\boldsymbol{x})$, of the jet (see figure 1). A discerning eye may even see a preponderance of such structures. Even those regions of the jet which appear to be smeared out reveal, on closer inspection, structures which look like spirals. Generally, however, these spirals possess only a few turns, often only three or four. The physical mechanism for their

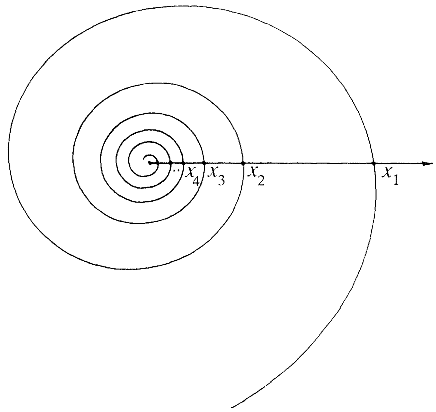

Figure 3. A schematic of a spiral. The intersections along the x-axis are indicated by x_n, with the suffix n indicating the number counted from the outermost intersection.

generation is not clear, and it is possible that they are generated by mechanisms other than the classical Kelvin–Helmholtz instability. In short, the relation between the spiralling structures in the scalar field – henceforth termed spiral-like structures – and true spiralling vortices which accumulate at the origin is unclear. We have examined the spiral-like structures here with the expectation that their behaviour may shed some light on the asymptotic behaviour presumed to be characteristic of very high-Reynolds-number shear flows.

It happens that the scalar 'dissipation' field, proportional to $|\nabla c|^2$, accentuates the structures more clearly than the concentration field, and so measurements were made from the 'dissipation' field. Note, however, that the finest resolution of the measurements was comparable to the Kolmogorov scale, so that the 'dissipation' field used here is a coarse-grained version of the true dissipation; in particular, no information on the sub-Kolmogorov scales could be inferred.

Some developing flows were also considered. Figure 2 shows two-dimensional sections of temporally developing countercurrent shear layer resulting from Kelvin–Helmholtz instability. The shear layer was set up (Sreenivasan *et al.* 1988) using a larger version of the tilting apparatus originally used by Thorpe (1968). Two streams moving in opposite directions were generated by making use of a slight density difference (of the order 2.5 %) between the top and bottom layers of the fluid. It should be remarked that the density difference is immaterial to the dynamical evolution of the flow. The bottom four figures in the left column show some spiral-like structures.

A few flows generated by other investigators were also examined. Details will be given as appropriate.

As pointed out earlier, our primary interest here is in determining the accumulation rates of the spirals. Figure 3 shows a schematic of an inward-winding spiral. The intersections of a line (which may be taken to be the x-axis) with the spiral are denoted by x_n, x_1 being the intersection furthest from the centre. If the spiral accumulates algebraically according to (2), a plot of x_n against n on doubly logarithmic scales would lie on a straight line. In contrast, x_n derived from a logarithmic spiral would lie on a straight line when plotted against n on semi-logarithmic scales.

Figure 4*a* and *b* shows two typical examples, respectively from the transverse and longitudinal sections of the jet, of spiral-like structures near to the outer periphery

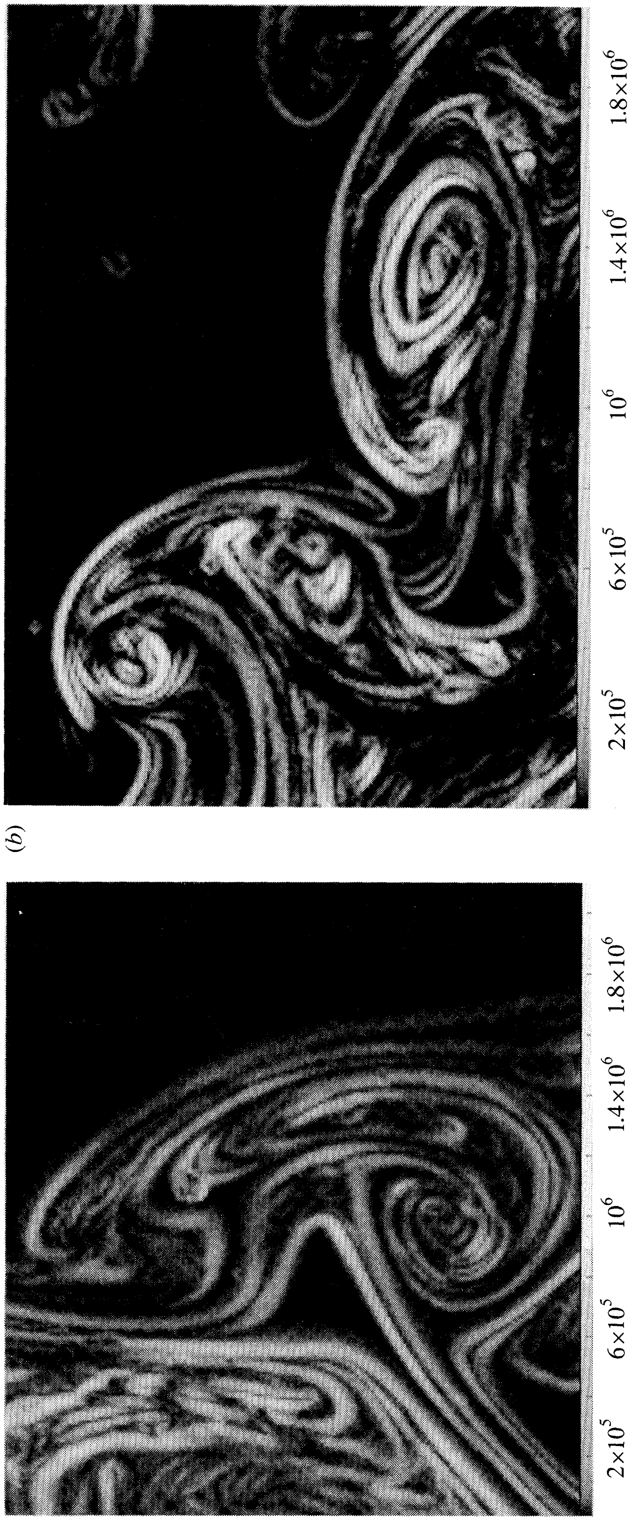

Figure 4. (*a*) Close-up of a typical spiral-like structure in a transverse section of the jet. (*b*) Similar close-up from a longitudinal section.

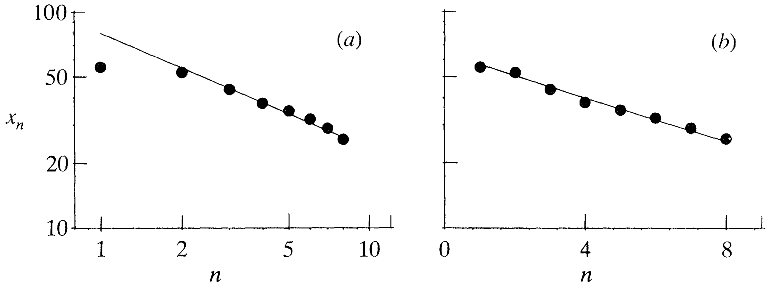

Figure 5. The log–log (*a*) and semi-log (*b*) plots of the distances x_n as a function of the intersection number n for a typical transverse section. These particular sections had several identifiable turns of the spiral. Different lines drawn in different orientations through the presumed 'centre' show slight differences among them. In this figure and elsewhere, several line intersections through the 'centre' were examined. The uncertainty is one realization is of the order of the size of the symbol. Here and elsewhere, full lines are least-square fits to *all* the data points shown.

of the jet. That from the transverse section has a larger number of turns. The figures are representative of the quality of data, and illustrate the difficulties inherent in the characterization of spiral-like structures. One is faced with difficulties in locating the 'centre' of the spiral and in dealing with imperfections of the spirals. Two initially separate arms may sometimes come close together making their subsequent identification ambiguous; equally likely are cases where an arm of the spiral appears to split into two parts. The spiral-looking structures may consist of foliations (that is, a number of different thin structures banded in an apparent spiral). As described below, the present analysis was mindful of these limitations.

Typically, the best position for the 'centre' was chosen by eye, different lines were drawn in different orientations through this presumed 'centre', and the intersections x_n were measured. In most cases the spiral-like structures did not possess windings very close to the 'centre', and would therefore not allow it to be established with accuracy. If the spiral is logarithmic this error may be removed since the differences

$$\Delta_n = x_n - x_{n+1} = C(1-\mathrm{e}^{-2\pi\alpha})\,\mathrm{e}^{-\alpha\theta} \tag{5}$$

also lie on a straight line when plotted against n on semi-logarithmic scales. This, however, is not the case for power-law spirals. To achieve a better fit for data assumed to be derived from power-law spirals, a small offset δ was added to each of the intersections x_n:

$$w_n = x_n + \delta. \tag{6}$$

Best-fit straight lines to log w_n against ln n were then found by allowing δ to vary over a small range. Both least-squares and 'robust' estimators (Emerson & Hoaglin 1983) which are tolerant of outliners were used. These methods were successful in recovering power-law variations from simulated data.

Since the outer arms of the spiral are likely to be affected by their interactions with other structures in the flow, one may expect a sizeable shift in the origin of θ. The effects of substantial offsets on θ were also studied.

The intersections were measured manually with a ruler and pencil in enlarged images, as well as (in most cases) by using a computer program on digitized images. Because of the finite thickness of the arms of the spiral, the intersections were defined in a self-consistent way, either by noting the point at which the line enters the arm of the spiral or leaves it; each of the present authors independently assessed the reasonableness of the fits to the data.

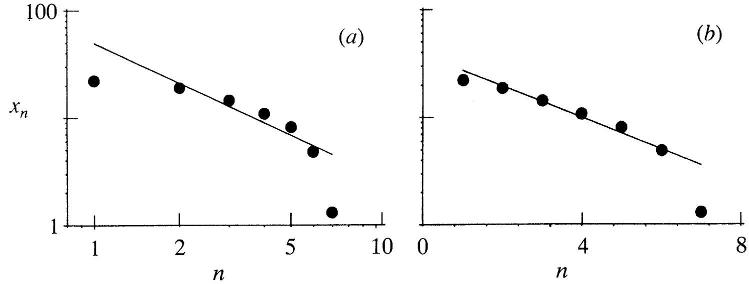

Figure 6. The log–log (*a*) and semi-log (*b*) plots of x_n against n for the spiral-like structures in a longitudinal jet section.

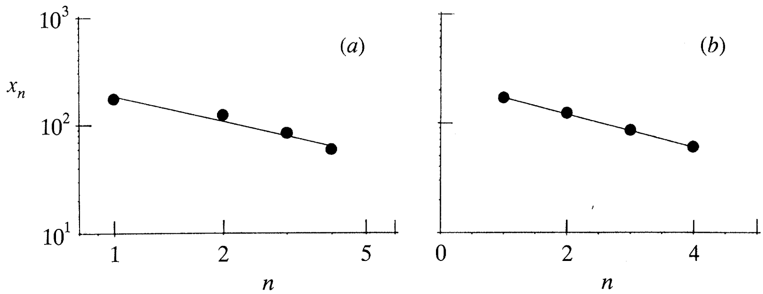

Figure 7. The log–log (*a*) and semi-log (*b*) plots corresponding to the middle structure in the second row from bottom of the left column of figure 2. Again, a logarithmic spiral seems to be a better fit to the data.

3. Results

(*a*) *Turbulent jets*

Figures 5 and 6 show, respectively for the transverse and longitudinal sections of the jet, the log–log and semi-log plots of x_n against n for a typical spiral-like structures. The estimation of 'error bars' on the data is uncertain and difficult to obtain formally, and the size of the symbols in the figures is a representative error for a given realization, not the statistical uncertainty associated with many intersections. In both figures 5 and 6 one can fit a power law by ignoring one or two data points on either end, but we saw no obvious justification for doing so. A logarithmic law appears to be a better fit. For the 30 or so spiral-like structures we have examined, the logarithmic model is generally superior, although by no means completely unequivocal.

Shifts in the angle θ are equivalent to shifts in the index n, and it is natural to expect them to be positive. Adding a value 2 (for example) to n produces fits which are qualitatively similar to those already discussed; one can again fit power laws to a subset of the data, but not to the full set.

In each of these respects, the spiral-like structures near the jet axis are quite similar to those near the outer periphery.

(*b*) *Developing flows*

Returning to the temporally developing shear layer (figure 2), there are very few turns and the location of the 'centre' is again difficult. Even so, it may be noted that the data (figure 7) appear to favour a logarithmic model.

Spirals with many turns may be found in the 'smoke ring' experiments performed

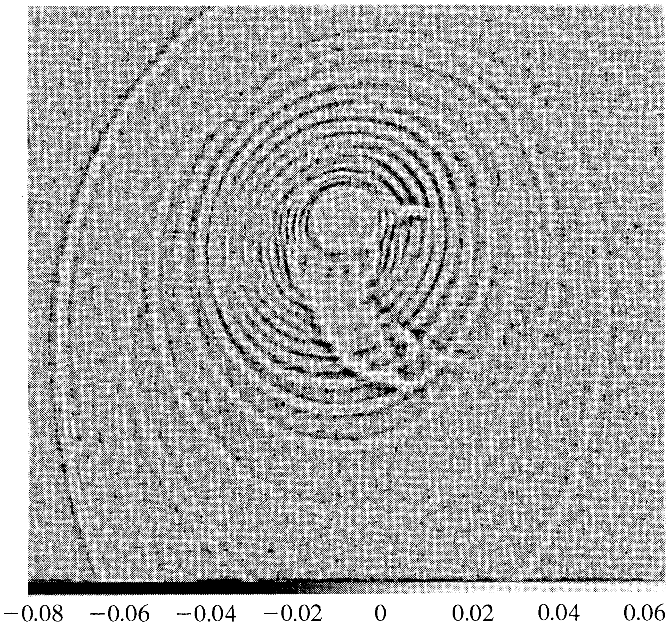

Figure 8. A digitized version of a smoke ring from Magarvey & MacLatchy (1964).

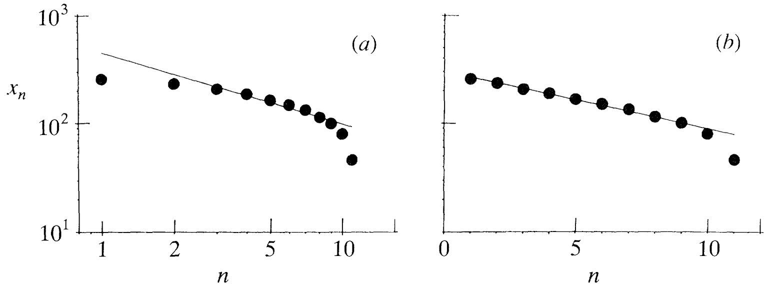

Figure 9. Shows clearly that the logarithmic model is a better fit to the spiral-like structures in a smoke ring. The number of intersections as well as the fits in the semi-log plot are unambiguous in this particular example.

by Margarvey & MacLatchy (1964). Though the flows in their experiment are far from turbulent, they do provide examples of spirals with many turns and are therefore helpful in establishing the nature of spirals unequivocally. Margarvey & MacLatchy investigated the formation of smoke rings by tracing the motion of fluid emerging from a tube in the vicinity of a smoke source. In figure 8, a spiral obtained from their figure 3 is shown. The data plotted in figure 9 clearly support a logarithmic description of the object shown in figure 8.

(c) *Spiral-like structures in numerical simulations*

Personal communications from Mr G. Ruetsch of the Division of Applied Mathematics (Brown) and Mr M. Meneguzzi of Centre d'Etudes (France) suggest that well-formed spirals do not occur in simulations of isotropic, homogeneous turbulence. We know of no well-resolved simulations of inhomogeneous flows such as jets. Numerical simulations of periodic vortex sheet roll-ups (Krasny 1986) show that a logarithmic fit is better than a power-law fit (see figure 10).

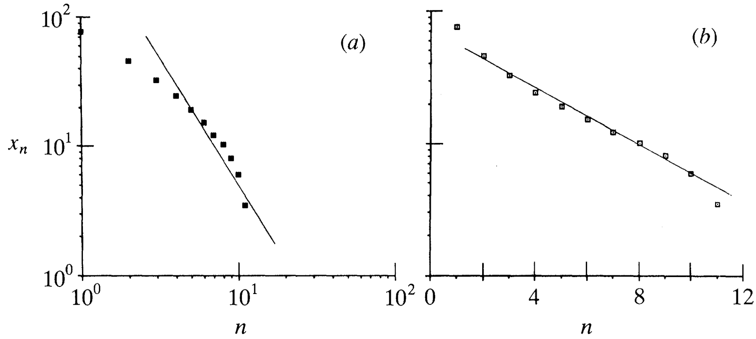

Figure 10. The log–log (*a*) and semi-log (*b*) plots of the distances x_n as a function of the intersection number n for the spiral vortex numerically simulated by Krasny (1986) according to a desingularized prescription. (Briefly, a small parameter is introduced in the numerical calculations. One recovers the inviscid limit if the constant vanishes identically.) The superiority of the logarithmic fit over a power-law fit seems clear. The line drawn in (*a*) corresponds to a quadratic power-law, which is the expectation according to the formula of Vassilicos & Hunt for spirals with small number of turns. A different power-law exponent is not out of the question if a few data points are ignored.

Note, however, that power law fits can indeed be obtained for Krasny's data by ignoring the smallest few intersections. It is unclear whether this step is justified on the grounds of dubious numerical accuracy of spirals near the accumulation point. Further, Krasny's (1987) simulations of the roll-up of tip-vortex of an elliptically loaded wing do suggest the formation of a power-law spiral with an exponent of about 0.75, in reasonable agreement with Kaden's (1931) spiral. However, this latter situation is quite different from the Kelvin–Helmholtz instability.

4. Conclusions

In the concentration field of shear flows like jets, spiral-like structures are common but well-formed spirals are rare. The vast majority of these structures form part of a complicated interface, rather than being isolated entities. The outer radii of such structures range from approximately one jet diameter to a few tens of Kolmogorov scales. This is quite comparable to the scaling range observed for the interface. While this may encourage the view that spirals may indeed be relevant to the observed fractal scaling, *all* the cases examined were better fitted by a logarithmic spiral rather than a power-law spiral. As already remarked, logarithmic spirals do not contribute to the box-counting calculations. Box-counting of individual spiral-like structures yields results consistent with the dimension of the interface measured as a whole, though the paucity of data for individual spirals makes these measurements rather imprecise. It may therefore be concluded that the observed scaling in the iso-scalar surfaces asymptotically represents an underlying fractal-like structure.

Two cautionary remarks should now be made. First, the experimental spirals do not in general have more than a four or five turns. It is not clear whether this is an artefact of the moderate Reynolds number of the flow. If so, the present data do not rule out the occurrence of spirals whose asymptotic form is algebraic. Alas, we do not foresee the possibility in the near future of experiments at significantly higher Reynolds numbers keeping the resolution relative to the Kolmogorov scale the same as in present experiments. Second, in the majority of instances examined, the spiral-like structures have been subjected to some sort of deformation by the shear. While

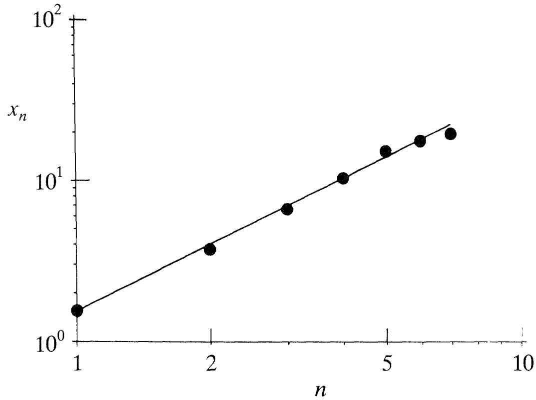

Figure 11. A replot of the data of figure 5 to emphasize that an outward spiralling model fits the data fairly well. The quality of power-law fits to other spiral-like structures is not always as good, especially for the smoke-ring of figure 8.

measurements have been made of the intersections along lines in which shearing was judged to be most constant – and variants of this procedure make little difference to the primary result of this paper – one could legitimately wonder about the effect of this deformation. On the other hand, one could also take the view that the deformation of the spiral is intrinsic to shear flows, and that is should not be regarded as an artefact needing compensation. In any case, the proper nature of the compensation (if one is required) could not be assessed.

The precise mechanism for creating logarithmic spirals in fluid flows is not clear. In potential flow, the superposition of a vortex and a sink produces logarithmically spirally streamline patterns. The inherent three dimensionality of turbulent motion could lead to a situation roughly resembling the sink-vortex kinematics locally.

Finally, we considered the possibility that the spiralling structure (both algebraic and logarithmic) may be outwards rather than inwards. Here, intersections with an axis would be labelled from the centre of the spiral, and α in (2) and (4) would be negative. Figure 11 shows the data of figure 6 replotted appropriately. It is seen that the data fit to the outward spiralling model is good. This is often the case for the spiral-like objects seen in jets, but not for the smoke-ring spirals. It should be noted that outward spirals do not accumulate at any point and therefore cannot explain the measured non-integer dimensions of interfaces or the Kolmogorov-type scaling in the inertial range. It is hard to envisage a physical mechanism in fluid flows which might produce outward spirals, whether they are logarithmic or algebraic.

In the light of all the evidence presented here, it would be interesting to understand the relevance of the theoretical work of Lundgren (1982), Moffatt (1984, 1990), Gilbert (1988), and Vassilicos & Hunt (1991) which purports to show that some properties of two- and three-dimensional turbulence are consistent with power-law spirals. It should be emphasized that the present conclusion refers to the scalar field while the theoretical work is concerned with the vorticity field. To date, our velocity measurements in a plane have not had sufficient resolution to say much of substance. Conversely, under certain assumptions, one may well be able to infer some properties of the vorticity field from details of the scalar field. A preliminary attempt in this direction has been made by P. Similon (personal communication). A conclusion of this effort is that the spiral-like structures seen in the scalar field are consistent with local singular-like distributions of vorticity. This may well be the most important ingredient of some of the theoretical work mentioned above. If so, it is more appropriate to describe the distribution of vorticity by a superposition of a number of singular distributions of the sort visualized typically in multifractal formalisms (see, for example, Sreenivasan 1991).

The jet data were obtained by R. R. Prasad to whom our thanks are due. We are grateful for thoughtful correspondence on a preliminary draft to M. E. Fisher, R. Krasny, H. K. Moffatt (also for sending a copy of his DAMTP report) and J. C. Vassilicos. We acknowledge useful comments by M. Meneguzzi, J. Saylor, P. Similon and G. Reutch. The work was financially supported by a DARPA (URI) grant and the National Science Foundation.

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Figure 1

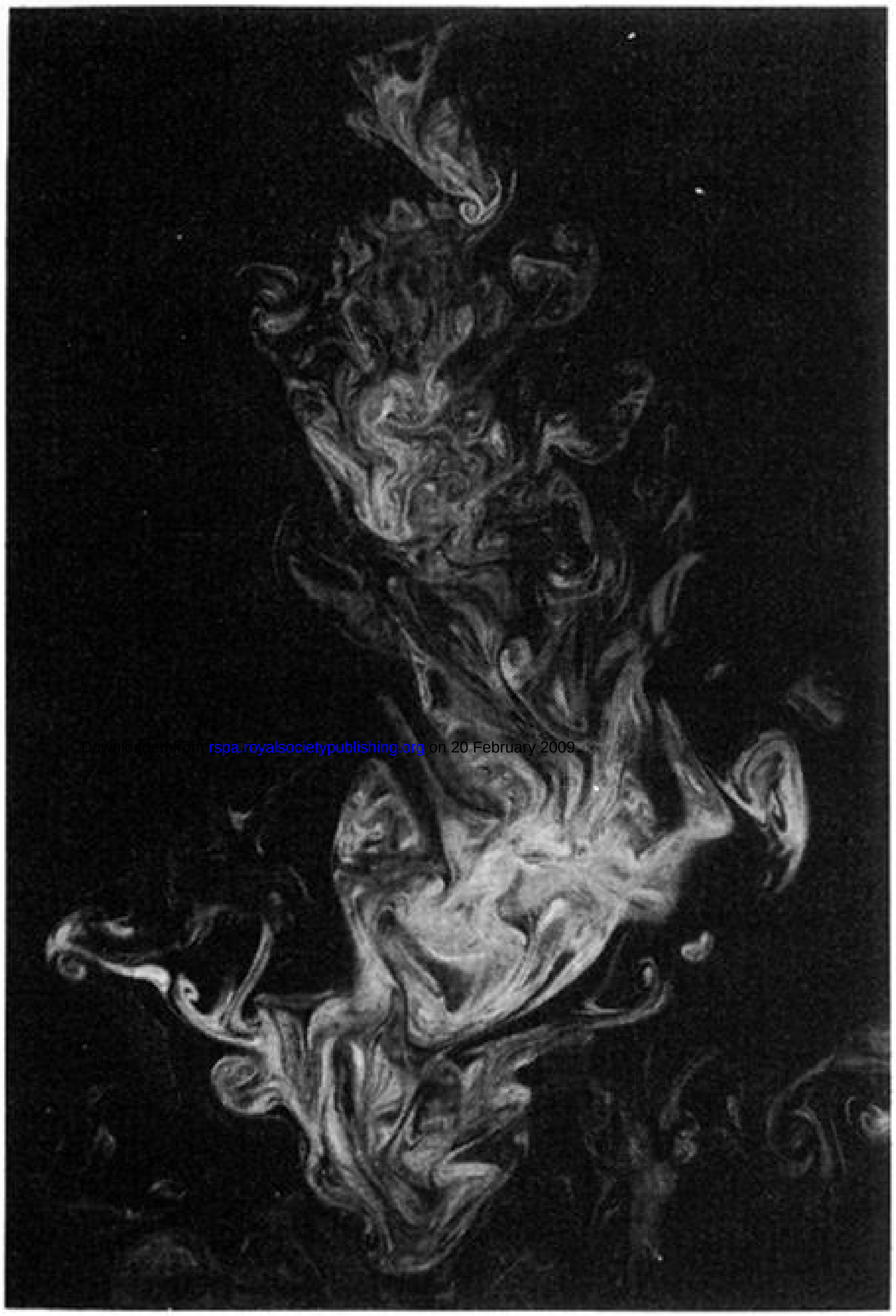
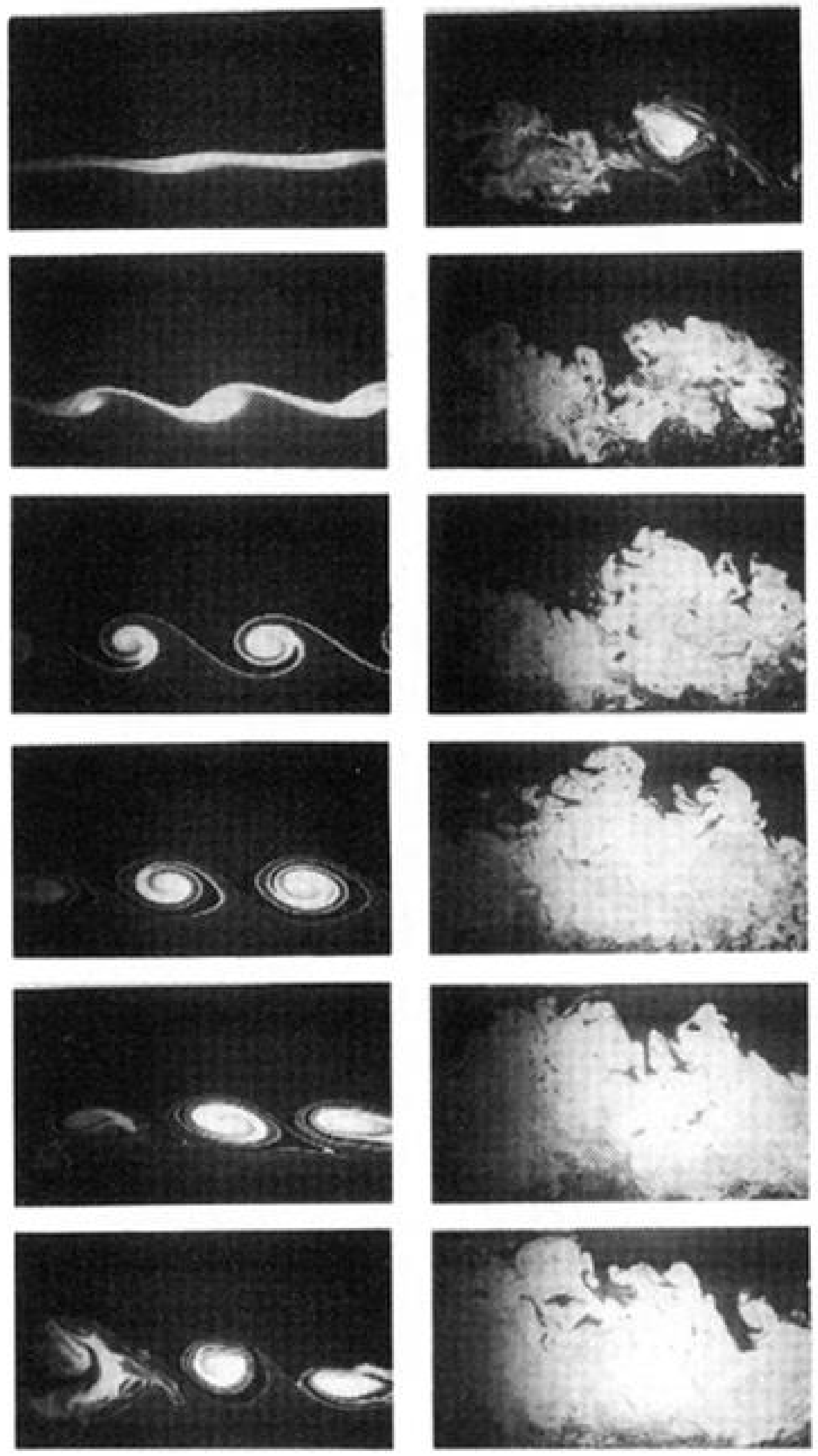


Figure 2



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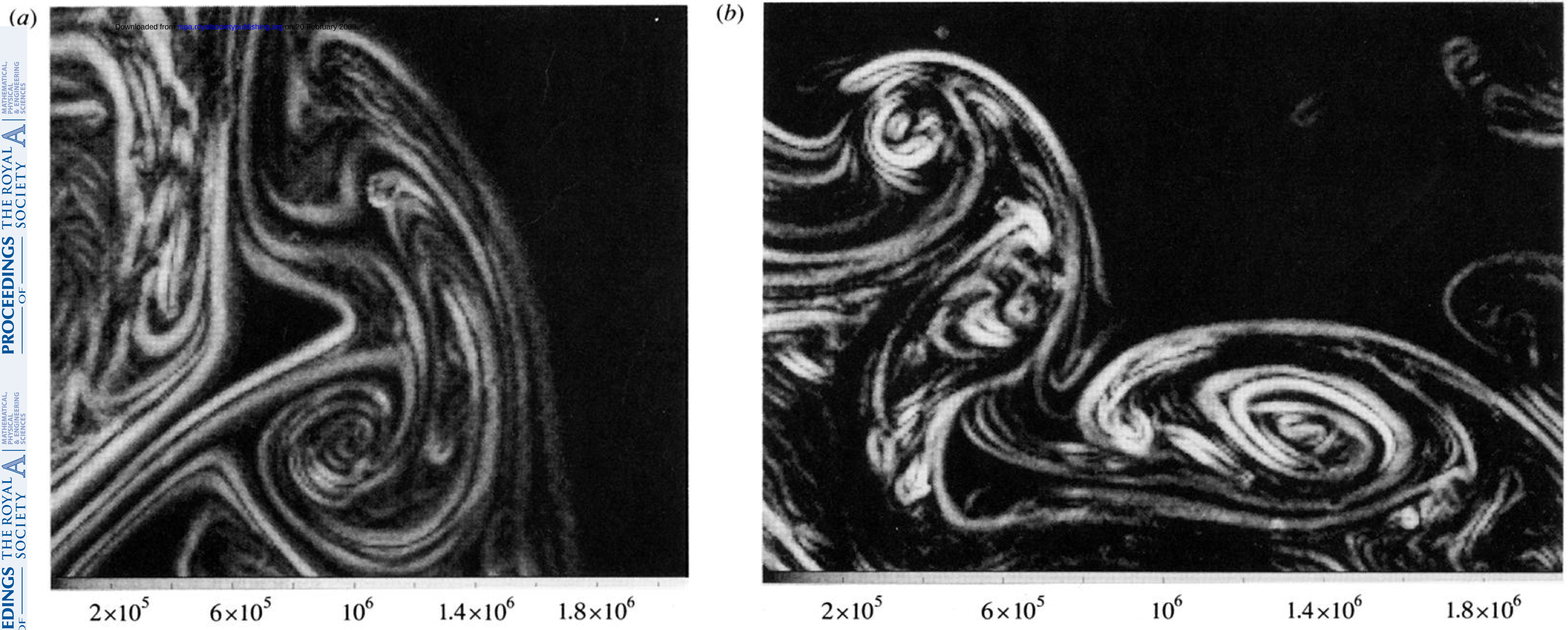
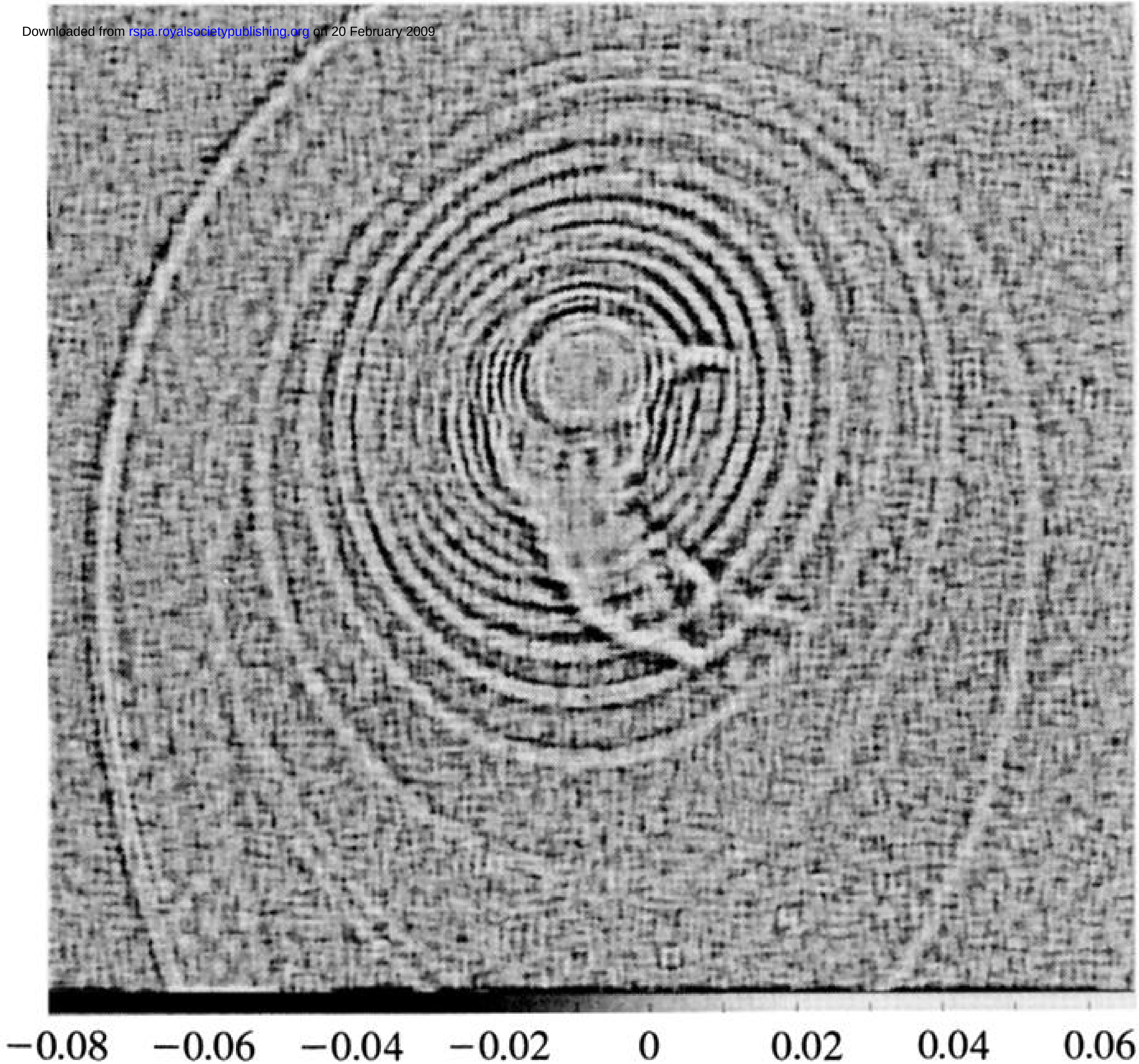


Figure 4. (*a*) Close-up of a typical spiral-like structure in a transverse section of the jet. (*b*) Similar close-up from a longitudinal section.



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