The Utility of Dynamical Systems Approaches

Comment 3.

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Abstract

This is a commentary on the utility of the dynamical systems approach to the understanding of transitional and turbulent flows. After a few initial remarks on the position paper by Holmes, I present a summary of three aspects: Universality in transition to chaos in wake flows, the description and dynamics of intermittent fields in fully turbulent flows, and the nature of vorticity and scalar interfaces in turbulent free shear flows. I will show that novel techniques from low-dimensional chaos and fractal geometry yield new and useful information on quantities of central interest in turbulence. The claim is that the dynamical systems appraoch has made definite contributions, not merely enlarged our vocabulary, but the way ahead *vis-a-vis* the turbulence problem has remained hazy.

Between the idea
And the reality.....
Falls the shadow

From T.S. Eliot [1]

The nearer we come to the present, of course, the more opinions diverge. We might, however, reply that this does not invalidate our right to form an opinion ....

From J. Burckhardt [2]

1. Introduction

The so-called 'turbulence problem' is not a monolithic entity. Its three essential elements are the origin of turbulence, the dynamics of fully developed turbulence, and the control of turbulence – by which is meant 'making turbulent *flows* behave the way one wants'. The origin of turbulence may have some relation to the onset of complexity in nonlinear systems in general, and hence the currency for notions such as 'universality'. On the other hand, a universal solution to the turbulence control problem is unlikely to exist, because it is specific to a given set of flow and geometric constraints. Fully developed turbulence has a mix of the 'universal' (now in the slightly different sense of being common to a class of turbulent flows) and the particular, both of which are essential to predictive undertaking: The scaling properties of the turbulent energy dissipation and small scale mixing are examples of the former class while the variation with Reynolds number of the drag on a circular cylinder belongs to the latter. The different elements of the turbulence problem are all important in their own right, and the tools of trade are appropriately different. The corollary is that the mastery over no single set of tools will be adequate to address the problem in its full glory.

This section of the meeting was devoted to an assessment of the utility of the so-called 'dynamical systems approach' to the turbulence problem. Just as the turbulence problem is a diffuse one, so is the dynamical systems approach: As Holmes [3] points out, it is a loose but rich mixture of many tools including mathematical theorems, numerical work, experimental studies and model building. As a result of recent developments in dynamical systems, one is now in a position to say essentially everything of interest (at least in a certain parameter range) about the dynamics of some paradigmatic nonlinear systems possessing global universality – the period doubling system being the best known example. The situation is less satisfactory when such global universality is absent. The procedure then is to use the center manifold reduction and classify all possible phase portraits by unfolding the appropriate parameters [4]. In particular, one attempts to study all stable attracting sets. Except when the number of parameters needing unfolding is small (one or two), the possibilities are so huge that one is unlikely to succeed in any generality. One often strings together, by taking recourse to hindsight and symmetry, a number of local bifurcations 'explaining' a particular sequence. In rare circumstances, one has been able to generate dynamical equations by this knowledge.

Holmes [3] summarizes these essential points, and briefly surveys recent developments in 'closed' and 'open' flows; he devotes the remainder of his paper to the elucidation of a qualitative dynamical model [5] for the near-wall structure of the boundary layer at moderate

Reynolds numbers. His basic tool is the proper orthogonal decomposition introduced to turbulence literature by Lumley [6]. I have little to add to the specifics of his model, and so concentrate on a few issues to which he refers in passing or not at all.

2. A broad definition of the approach and the scope of the paper

Nothing will be said here on the control of transitional and turbulent flows. In the context of transition to chaos, there are essentially two points of interest: That chaos (or temporal complexity) does not require many degrees of freedom, and that transition to the chaotic state is universal in character. (This statement needs more precision, and we shall return to it later.) Predictions concerning universality (see [7] for the period doubling route to chaos, and [8] for the quasiperiodic route) have been tested in detail, and there is enough evidence now that low-dimensional temporal chaos has provided new as well as useful ideas and tools for analyzing early stages of transition in (some types of) flow systems which are closed [9] as well as open [10,11]. In Section 3, I present a summary of some recent findings in the wake of a circular cylinder to demonstrate the existence there of universal features.

Experience shows that temporal chaos is of restrictive value once spatial three-dimensionality and classical power-law behaviors set in. Fully developed turbulence is high dimensional [12] and has distinct spatial structures, the gap between what one *can* do with low-dimensional chaos and what one *needs* to do in fully developed turbulence being very wide indeed! One type of advance made in low-dimensional chaos that has found some application in turbulence is the invention of several dynamical measures of stochasticity such as Liapunov exponents [13] and scaling functions [6], or static measures such as the Hausdorff-Besicovich dimension [14], fractal dimension [15], generalized dimensions [16], multifractal spectra [17] and various entropies [18]. Such measures have been made accessible to an experimentalist because of the important notion that an attractor can be constructed by suitably embedding a time series in phase space [19], even though circumstances do exist in which the technique might not be accurate or even useful. These measures cannot be obtained in practice for high-dimensional systems, but progress has been made by treating an instantaneous realization of a flow as a kinematic object consisting of various *objects concentrated on fractal sets embedded in three-dimensional physical space*. I shall summarize this progress in sections 4-6, and remark on the possible utility of such measurements. Brief conclusions are set forth in section 7.

This paper is by choice a summary of results on all three aspects mentioned, rather than a detailed account of any single one. Much of this work has yet to be taken to its logical conclusion, but selective questions can already be asked and partially answered.

3. Universality in transition to chaos in wakes behind cylinders

Briefly, with increasing Reynolds number, the flow behind a stationary circular cylinder first undergoes a Hopf bifurcation [20,21] from the steady state to a periodic state characterized by the vortex shedding mode at a frequency f_o, say. We have shown [21] that the supercritical state (above the critical Reynolds number of 46 when based on the cylinder diameter D and the oncoming velocity) can be modelled by the Landau-Stuart equation, and experimentally determined the relevant constants. These details appear not to depend on the aspect ratio (the length to diameter ratio) if it exceeds about 60. I had shown earlier [10] that a quasiperiodic motion sets in at a somewhat higher Reynolds number, and that the onset of chaos follows. Of interest here are the universal features accompanying this onset of chaos.

A view has been expressed [22] that the quasiperiodicity observed in [10] could be the result of the aeroelastic coupling between the cylinder and the flow, but I must emphasize that no perceptible cylinder vibrations were present in our experiments; see Fig. 1. Recent work [23] has shown that the observed quasiperiodicity is due to the change in the spanwise direction of the vortex shedding pattern at low Reynolds numbers, and *its* onset (as opposed to the onset of vortex shedding) depends on the cylinder aspect ratio, its end conditions and other boundary effects, all of which are not totally under the control of the experimentalist. To observe universality, on the other hand, complete control must be maintained on the sources of quasiperiodicity. It is thus useful to explore quasiperiodic dynamics of the wake by transversely oscillating the cylinder in a controlled manner.

In the work described in [11], the cylinder was placed in a specially designed wind tunnel that allowed more than the usual degree of control on flow parameters, and was oscillated transversely (in the first mode) at various known amplitudes. The Reynolds number was fixed at some value in the supercritical state (55 being the Reynolds number for which the bulk of the data has been obtained). The flow velocity was monitored by a standard hot-wire placed approximately 15D downstream and 0.5D to one side of its rest position. The imposed modulation on the cylinder was at the desired frequency f_e, the amplitude of oscillation A being then a measure of the nonlinear coupling between the two modes. The system thus has two competing frequencies f_o and f_e, yielding two control parameters f_e/f_o and the non-dimensional amplitude of oscillation, A/D. Once the external modulation is imposed, we expect f_o to shift to f_o' (say). By mapping the entire plane of f_e/f_o' and A/D, one can observe many features

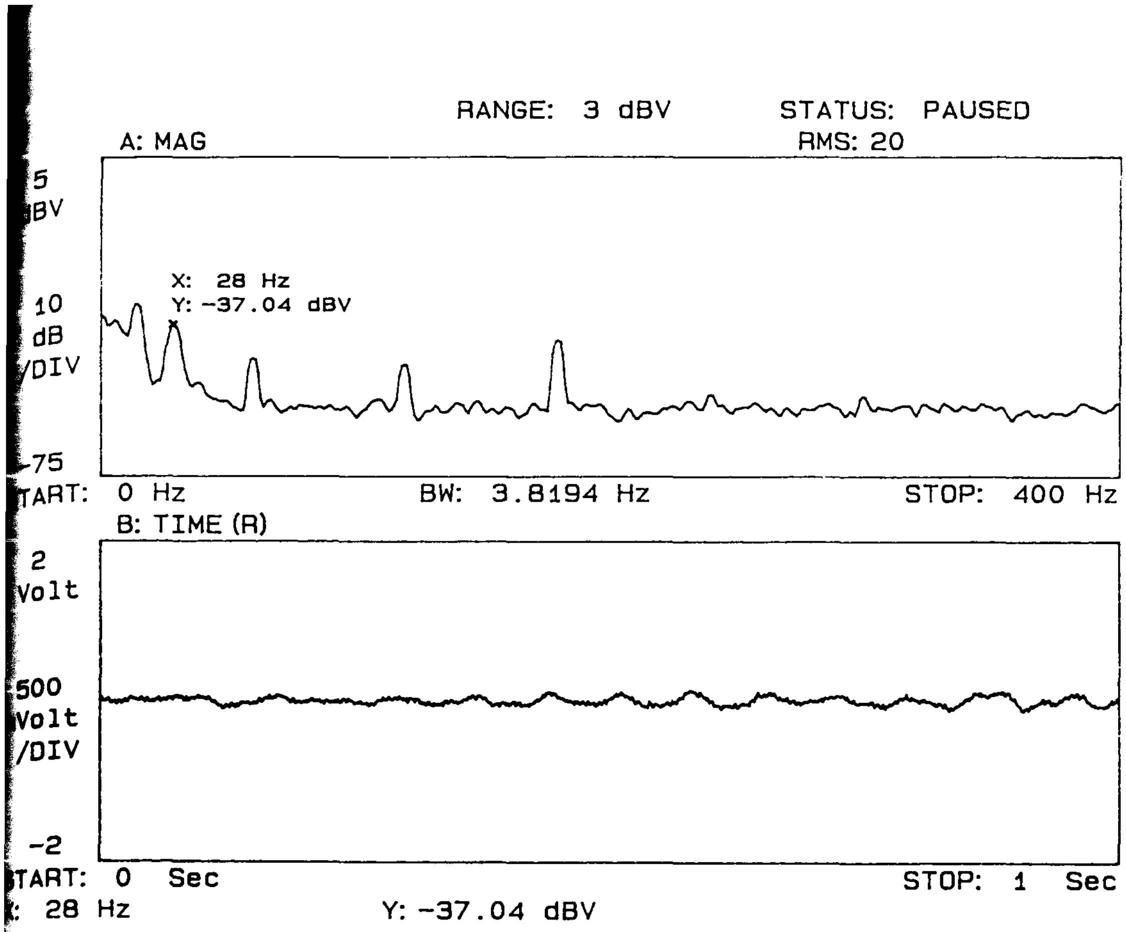

Fig. 1. The bottom figure is the time trace from the output of the optical probe (MTI Fotonic sensor) mounted to measure possible transverse displacements of the circular cylinder situated in uniform stream. The cylinder is 'shedding' vortices at a frequency of 287.5 Hz. and, as discussed in [10], the quasi-periodic behavior in temporal dynamics is evidenced by the presence of side-peaks with a difference frequency of 36 Hz. The power spectral density of the time trace is shown in the upper figure. The spectrum shows peaks at 14 Hz, 28 Hz, 60 Hz, 120 Hz and 180 Hz. The last three are related to the response of the optical probe to indicator lights of the electronic instrumentation in an otherwise darkened laboratory. The first two frequencies are related to floor (and thus tunnel) vibrations which have since been damped – the data presented here were obtained in 1985 – with no effect on the observed wake dynamics. More details are forthcoming in [23].

common to a class of nonlinear systems with two competing frequencies – no matter what precise differential equation governs the system. This is the spirit of universality.

In particular, these universal aspects of transition from quasiperiodicity to chaos have been worked out in detail for the sine circle map and are believed to be universal for any map with a cubic inflection point. In the sub-critical state, iterates of the map lock on to rational frequency ratios in the so-called Arnold tongues which increase in width as the nonlinearity parameter increases. At the critical state corresponding to the onset of chaos, universal features occur. To observe them, it is best to proceed without phase locking, this constraint naturally leading to the choice of the golden mean σ_G for the frequency ratios: Note that the irrational number σ_G is least well approximated by rationals (since it contains only 1's in its continued fraction representation), and hence is best suited for avoiding lock-ins – which in principle are possible at all rational frequency ratios.

Olinger & Sreenivasan [11] have demonstrated that the wake of an oscillating cylinder at low Reynolds number is a nonlinear system in which a limit cycle due to natural vortex shedding is modulated, generating in phase space a flow on a torus. They experimentally showed that the system displays Arnold tongues for rational frequency ratios, and approximates the devil's staircase along the critical line. At the critical golden mean point accompanying quasiperiodic transition to chaos, spectral peaks were observed at various Fibonacci sequences predicted for the circle map and, except for low frequencies, had the right magnitudes. A pseudo-attractor was constructed by the usual time delay and embedding methods [19] from the time series of velocity at the critical golden mean point, and Poincaré sections were obtained by sampling data at intervals separated by the period of forcing. The resulting Poincaré section was embedded in three dimensions (in which it was non-intersecting in all three views), and a smoothed attractor was obtained by performing averages locally. The data were then used to compute the generalized dimensions [16] by using standard box-counting methods; in the appropriate log-log plots, the scale similarity regime extended typically over two decades. The multifractal spectrum, or the $f(\alpha)$ curve, was then obtained *via* the Legendre transform discussed in [17]. The multifractal spectrum as well as the spectral peaks showed that the oscillating wake belongs to the same universality class as the sine circle map.

To push this correspondence further, consider the scaling function invented by Feigenbaum [7]. This compact way of describing the dynamics of the system contains complete microscopic description (not merely statistical averages) of the attractor – modulo the information on its embedding dimension. The attractor at the onset of chaos can be regarded as constructed by a process that undergoes successive refinement and leads eventually to the observed scale similar properties. Such a process can be mapped on to a subdividing tree structure whose branches now are the iterates of the dynamical system; the scaling function correctly organizes the intervals to be compared from one level of refinement to another. Extracting Feigenbaum's scaling function experimentally is a nontrivial task, but one can instead use a suitably modified version [24] based on comparing intervals within a single periodic orbit rather than of two different orbits at the same level of stability. Olinger et al. [11] have obtained this modified scaling function at several successive approximations to the critical golden mean point, and shown that it is in good agreement with that calculated for the sine circle map.

These are detailed tests of (metric) universality, comparable to that in some experiments in closed flows. *I find it remarkable that a complex flow such as the wake should conform so well to the predictions for the circle map. This does not mean, however, that transition to chaos and chaos itself can necessarily be defined in every flow equally neatly and in as much detail, or that they all belong to some universality class or another. Our knowledge of what special circumstances or features of the flow render such questions useful is accumulating only slowly, and there is room for further work. But it is clear that the dynamical systems approach is useful even for some class of open flows.* For further remarks on the implications of the results, I refer the reader to the papers cited.

In addition to demonstrating universality in a familiar flow, the work just summarized has the practical value of predicting the width of lock-in regions, and of organizing under a broad umbrella many isolated results on oscillating cylinders. I must remark, however, that *the relevance to fully turbulent state is unclear of these and similar demonstrations of universality accompanying transition to chaos; in fact, the role of chaos in bringing about turbulence is an outstanding question.* In simple dynamical systems – the circle map and the logistic map being two specific examples – one has been able to discover the underlying dynamical structure of the asymptotic state by unraveling the bifurcation sequence as the control parameter evolves. A lesson often emphasized [25] is that the ordering of the asymptotic state may be discerned by understanding the ordering during the evolution of the system to that state. This statement can at best be partially true, if that, for fluid turbulence. If the striking similarity between the coherent motion in a fully turbulent flow and the corresponding motion in its transitional stage is not accidental, it is probably true that the former can be understood in terms of transitional structures. On the other hand, it appears futile to seek the key to the understanding of universal aspects of fully developed turbulence in the transition process because different transition scenarios lead to the same end product. Some thoughts on these issues are taking shape, but I shall now move on directly to the fully turbulent state.

4. Internal intermittency in turbulent flows

As already remarked, an instantaneous realization of a fully turbulent flow such as a jet is an object I wish to study. In this section, I concentrate on aspects of internal structure which, in regions far away from the boundary, are statistically independent of the boundary and initial conditions; the specific quantities examined are the distribution of the turbulent energy dissipation, the 'dissipation' rate of the variance of a passive scalar, and absolute values of turbulent vorticity and the Reynolds shear stress. All these quantities are distributed in some complex way in the three-dimensional real space. Fig. 2 shows the distributions of one component of energy dissipation along a line in the fully turbulent part of a laboratory boundary layer at moderate Reynolds number and in the atmospheric surface layer at a much higher Reynolds number. It is clear that the spatial distribution becomes increasingly intermittent as the Reynolds number increases. The intermittency of the scalar 'dissipation' as measured on a planar cut is shown in Fig. 3.

Two points of interest here are the description of such intermittent distributions and the identification of the dynamics leading to them. Such highly intermittent processes cannot be described efficiently by the conventional moment methods known to be successful for Central Limit type processes. In particular, if a process is Gaussian, its mean and variance describe the process completely; for others close to Gaussian, a few low-order moments contain most of the information. On the other hand, for processes of the type shown in Figs. 2 and 3, the first few moments give little clue to the nature of the process.

Multifractal measures, as they are called in the present parlance of dynamical systems, have built-in intermittency which therefore makes it logical to examine their usefulness in our context; see Mandelbrot [16], Hentschel & Procaccia [16], and [17]. Essentially, multifractals are built up by a procedure (which is often rather simple) that proceeds from one scale (the parent scale) to the next smaller ones (the off-springs) in such a way that the measure (roughly, the amount of a positive quantity such as the rate of energy dissipation) contained in the parent scale is unequally divided among its off-springs. When this procedure repeats many times, the measure on the off-springs of each successively higher generation will become increasingly uneven. If the basic rule determining the unequal division from a parent scale to its off-springs is independent of the generation level, one expects certain scale-similar properties. Because the measure on an arbitrary off-spring at a given generation level is determined by the product of the multipliers (that is to say, numbers characterizing the unequal division of the measure) of all its fore-fathers, a multifractal can be associated with a multiplicative process. The first order of

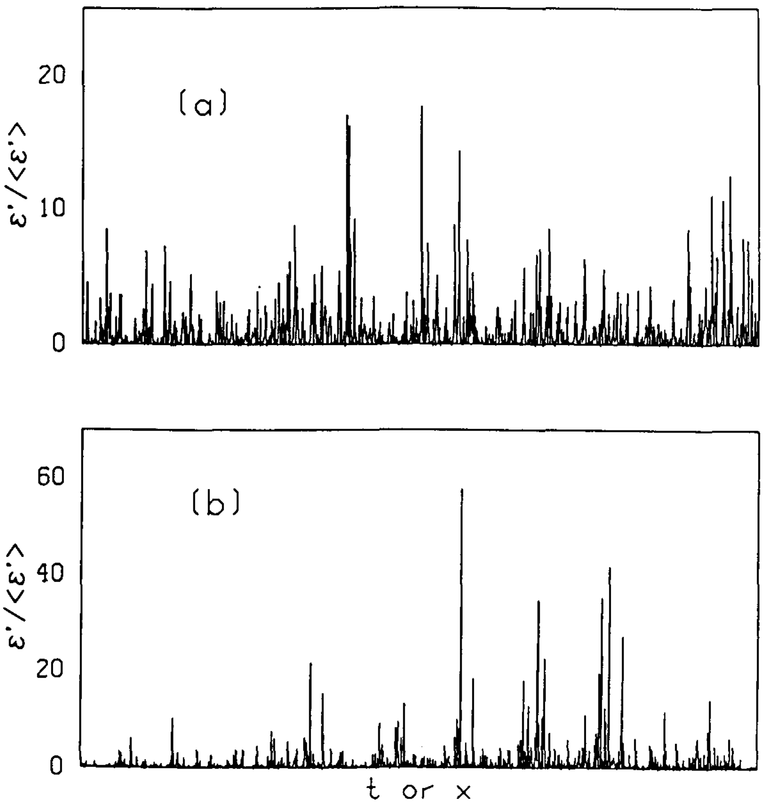

Fig. 2. Typical signals of $\varepsilon' = (du/dt)^2$ normalized by its mean. The upper trace (a) was obtained in a laboratory boundary layer on a smooth flat plate at the moderate Reynolds number R_λ of 150 (based on the Taylor microscale and the root-mean-square fluctuation velocity in the main flow direction). The lower trace (b) was obtained in the atmospheric surface layer a few meters above the roof of a four-storey building. The Reynolds number R_λ is about 1500. It is believed that the statistics of ε' are representative of those of the total energy dissipation. For a description of experimental conditions, see [26,28]. The figure is taken from [28].

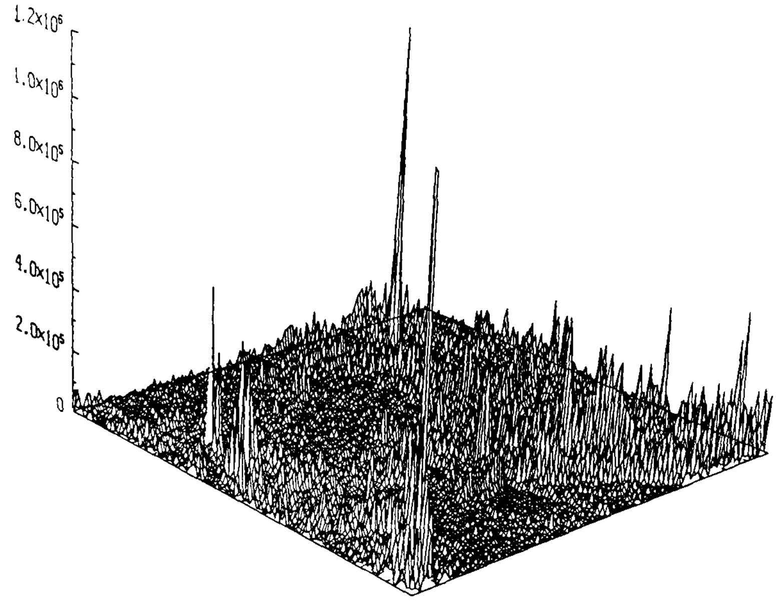

Fig. 3. The dissipation rate χ of the concentration fluctuation c as a function of the two coordinates x (axial) and y (radial) in the fully turbulent part of an axisymmetric jet. The figure covers a grid of 150 x 150 pixels. The nozzle Reynolds number is about 3600, and the center of the picture is about 15 nozzle diameters downstream. $\chi(c)$ was approximated by the sum $(dc/dx)^2 + (dc/dy)^2$, and $<\chi>$ is the average of χ. In [48] it has been shown that the addition of the third component $(dc/dz)^2$ to χ, z being perpendicular to x and y, does not affect the scaling properties. The figure is taken from Prasad et al. [30].

business would be to determine the multifractal properties of the intermittent process interesting to us, and, if possible, identify the associated multiplicative process – if one exists.

Because the measure concentrated on an off-spring at any level is the product of the multipliers of all its forefathers, it is clear that the scaling will only be local; there is therefore the expectation that many, in fact infinitely many, scaling indices will be required for a meaningful description. The purpose of analysis then is to quantify these various scaling indices and unravel their other properties. A possible vehicle for doing this is the $f(\alpha)$ spectrum [17] to which I have made reference already in section 3; the difference, however, is that we are now considering distributions in three-dimensional physical space rather than in a high-dimensional phase space. The $f(\alpha)$ curve has been measured for positive definite quantities characterizing small scale turbulence. (For the energy dissipation see [26-29], for the scalar dissipation rate see [30], for the squared and absolute vorticity see [31, 32], and for the absolute value of the Reynolds stress in the boundary layer see [33]. I invite the reader's attention especially to [28] where there is a detailed discussion of the measurement techniques, signal/noise ratios, the ambiguities in determining the scaling regimes and scaling exponents.) Typical $f(\alpha)$ curve for the energy dissipation is given in Fig. 4. Some of its salient features are the minimum value of α corresponding to the largest singularity in the distribution of ε, the maximum value of α corresponding to the least intense regions of ε, the maximum value of $f(\alpha)$ which is 3 (showing that there is some dissipation everywhere in the flow domain), the point $f = \alpha$ which corresponds to the fractal set on which all the dissipation is concentrated in the limit of infinite Reynolds number [34].

This description of intermittent quantities is more powerful than other descriptions, most of which turn out to be special cases of the present description. For example, Kolmogorov's space-filling dissipation [35] corresponds to the point (3,3) in the f-α plane. Similarly, the β-model [36], in which only the fraction β of the space is occupied by homogeneously distributed dissipative regions, corresponds to another point (D_β, D_β) on the plane depending on the precise value of β. If the $f(\alpha)$ curve can be approximated by a parabola, Kolmogorov's log-normal approximation [37] results. The random β-model [38] is also inadequate because the dimension of the support in that model is less than 3.

These measurements have allowed interesting inferences to be made, and I shall illustrate some of them presently (section 5). I must point out here that *the overwhelming conclusion of this work is that multifractals are a plausible vehicle for describing intermittent fields in turbulent flows.*

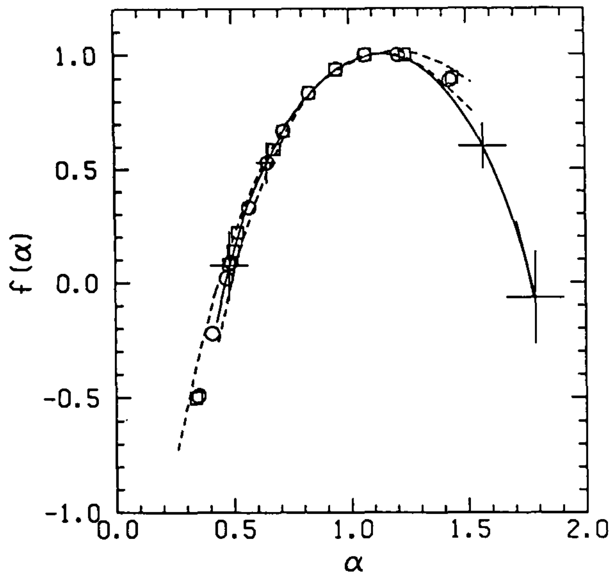

Fig. 4. The multifractal spectrum f(α) obtained from Legendre transforming the generalized dimension data; direct method of measurement [29] yields the same results. Circles correspond to a long data set (10^7 points) in the laboratory boundary layer on a smooth flat plate, and the squares to that in the turbulent wake behind a circular cylinder (5.10^6 points). Both flows have moderate Reynolds numbers (R_λ on the order of 200). The dashed lines represent results from ensemble averaging over many short data segments, each of which is of the order of ten integral scales. The error bars correspond to the standard deviation observed from the analysis of short records. The solid line is the average from [26] in various flows including the atmosphere. To transform f(α) values from one dimension to three dimensions, add 2 to the ordinate. The applicability of this additive law has been discussed in [28] from where this curve has been taken.

In the measurements mentioned above, the spatial resolution was limited to scales of the order of the Kolmogorov scale. If the Schmidt number σ (the ratio of the fluid viscosity to scalar diffusivity) is much larger than unity, the smallest scalar scale, the so-called Batchelor scale [39] $\eta_b = \eta\sigma^{-1/2}$, is much smaller. In such cases, there are two scaling regimes – roughly speaking, that between L and η and another between η and η_b – and it is possible to examine these two scaling regimes separately. In [40], we have explored the multifractal spectrum of the scalar dissipation in the scale range between η and η_b using one-dimensional sections. Finite Schmidt number corrections remain important even when σ of the order 1000, but the primary conclusion is that all generalized dimensions are essentially equal to the fractal dimension of the support in the limit of infinitely large Schmidt numbers or, equivalently, that the multifractal spectrum is simply the point (3,3) in the f-α plane. The natural conclusion is this: *It is not necessary to invoke multifractals – or even ordinary fractals – for describing scalar fluctuations in the range between η and η_b; classical tools should be quite adequate here.*

5. The thermodynamic formalism

Recall that the multifractal description of section 4, general though it is, is only a kinematic description. It is usually not possible to deduce dynamics from such static descriptions, but some headway can be made by noting that the multifractal description of dynamical systems is equivalent to thermodynamic description of statistical mechanical systems [41]. Specifically, then, the question is: Can one deduce something about dynamics given merely thermodynamic information?

Consider the multiplicative process in which each parent interval breaks up into 'a' number of new sub-intervals; we would have at the end of n stages of the cascade a^n pieces. The general procedure is to map such a process on to an a-state n-particle spin system and find, by taking recourse to the measured multifractal properties, the appropriate transfer matrix describing the transition from the n-th stage to the (n+1)-th stage, or from an n-particle system to an (n+1)-particle system. One then has a dynamic process yielding the observed thermodynamics. Usually, one can only get the leading approximations to the transfer matrix [41]. Chhabra et al. [42] have examined the issue in detail with particular reference to turbulence, and shown (not surprisingly) that the procedure yields non-unique solutions; that is, there are many dynamic processes which yield the same thermodynamics. However, a knowledge of the constraints on the system dynamics can render the choice much less ambiguous. Meneveau & Sreenivasan [43] have shown that the measured multifractal properties are in good agreement with a binomial cascade model (designated in [43] as the p-model) in which a parent eddy breaks up (in one dimension) into two equal-sized eddies such that the measure (say, the energy dissipation rate) is redistributed between its two offsprings in the ratio 7/3. (For general multiplicative processes in three dimensions, see [28].) This allows several quantitative predictions (such as the Reynolds number variation of the skewness and flatness factor of the velocity derivative) to be made. In [26,43], it has been shown that they are in good agreement with measurement. As an example of such comparisons, Fig. 5 shows the comparison between the measured and calculated spectral densities of $(du/dt)^2$. It is clear that the agreement is reasonable.

Keep in mind that the inversion is non-unique, which means that I cannot claim that p-models and other multinomial models represent true dynamics of energy cascade – despite impressive agreement with experiment. What I can say is that one can construct simple dynamical models whose outcome is statistically the same, up to some level of approximation, as those of the measured results. *To determine, among the host of existing possibilities, the true dynamic picture requires more information, and is an area of active research.*

Similar binomial and multinomial models have been constructed for other quantities mentioned earlier. Further, the multifractal formalism has been extended [33] to more than one coexisting multifractal measures. The primary motivation for this work is the realization that a high Reynolds number turbulent flow subsumes several intermittent fields simultaneously, and that they display different degrees of correlation among them. The formalism has been applied to simultaneous measurements in several classical turbulent flows of a component of the dissipation rate of the kinetic energy, the dissipation rate of the passive scalar, as well as the square of a component of the turbulent vorticity field. Several joint binomial models, also in agreement with measurement, can be deduced for these joint distributions. One further area of progress [44] has been the recognition that the $f(\alpha)$ curve possesses useful information on spatial correlations, and that it provides a vehicle for quantifying the relative importance to transport of large amplitude but rare events in comparison with small amplitude but ubiquitous events.

6. Interfaces in turbulent flows: An example of the 'outer' dynamics

We now turn to 'outer' dynamics of turbulence, typified by the vorticity interface (that is, the conceptual surface separating domains of intense and zero vorticity fluctuations). It is well-known [45] that an unbounded turbulent flow such as a jet develops at high Reynolds numbers 'fronts' across which vorticity changes are rather sharp on scales larger than the

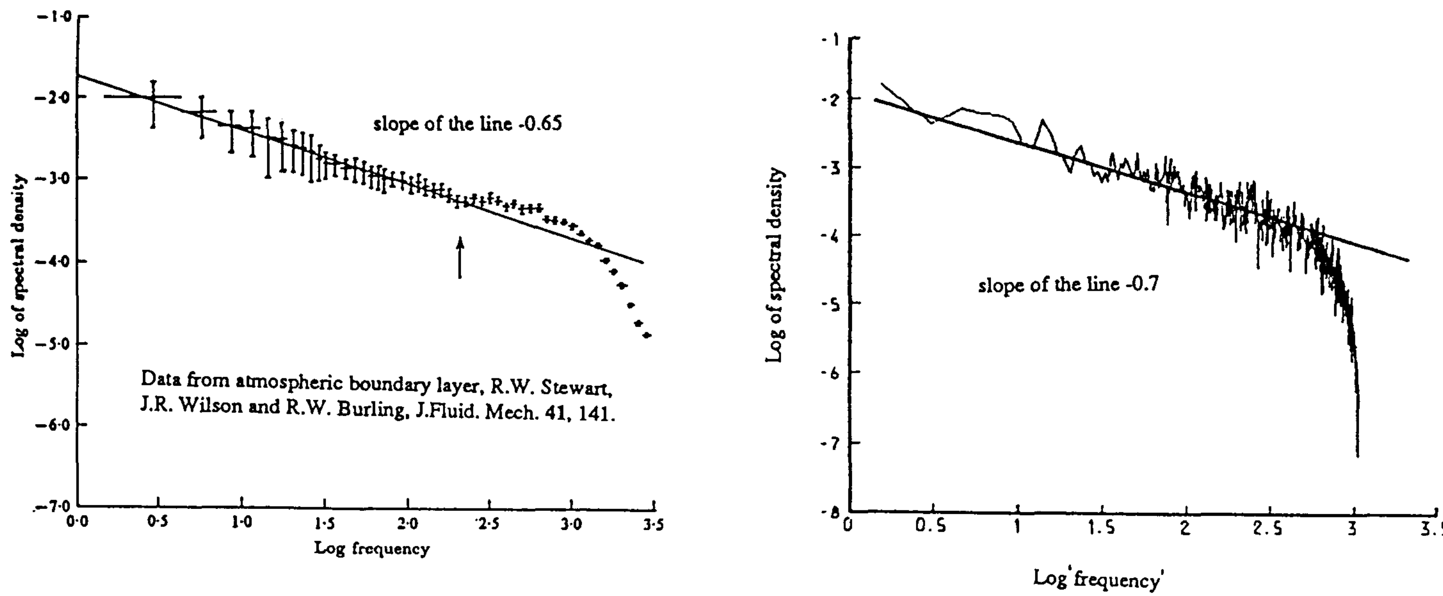

Fig. 5. The left figure is the power spectral density of $(du/dt)^2$ in the atmospheric surface layer. The figure on the right is obtained as follows. Starting with the uniform distribution of a measure on the unit interval, the first step of refinement is initiated by halving the interval and redistributing the measure in the ratio 7/3 (the p-model of [43]). This process is repeated until the smallest scale obtained is in the ratio η/L, η being the Kolmogorov scale in the flow to which the comparison is sought; this ratio is fixed by the Reynolds number. A meaningful comparison between the two requires, from the known Reynolds number of the flow, that 16 iterations be performed in the p-model. The resulting distribution is then Fourier transformed to obtain its spectral density (right figure). It is seen that a reasonable power-law index of −0.7 exists in the right plot, not very different from −0.65 of the left figure; it is possible to fit slightly different lines to the data in the two figures and come up with exactly the same slope.

characteristic thickness of the front. A passive scalar introduced in a fully turbulent flow gets dispersed by turbulence, but itself possesses a scalar interface.

Interfaces are complex objects residing in three-dimensional physical space, and convoluted over a range of scales which, according to conventional wisdom, may be statistically self-similar. It is therefore thought [15] that fractal description of such surfaces is possible. At high enough Reynolds numbers, the scale separation between the largest and smallest interface scales is rather large, and this allows the use of fractals in its characterization. Unlike a mathematical fractal, the scale-similar regime of the interface is bounded on both sides by physical effects: The upper cut-off occurs at around the integral scale of motion, this being comparable to (but distinctly less than) the gross size of the flow such as the jet width, whereas the inner cut-off occurs at a scale where the fluid viscosity is felt directly. This scale is approximately the Kolmogorov scale η (or some multiple of it). The fractal dimension thus characterizes scale similarity in the approximate range between L and η. For the scalar interface corresponding to large Schmidt numbers, there are (as described in section 4) two scaling exponents or fractal dimensions, the second one corresponding to the scale range between η and η_b.

A primary property of a fractal surface being its fractal dimension, much attention has been paid to measuring it. Fractal dimension measurements of vorticity interfaces have been made only in one-dimensional intersections [31,46], but those of the scalar interface in the scale range between L and η have been made using one, two- and full three-dimensional mappings of the scalar field [46-48]. A detailed discussion of the experimental procedures including considerations of the noise effects can be found in [47]. In the scale range between η and η_b, the requirement that the Batchelor scale be resolved allows only trivial extents of the flow to be mapped in two and three dimensions; one therefore has to resort only to one-dimensional intersections. Such measurements have been reported in [40].

Table 1 summarizes the results on the fractal dimensions of both scalar and vorticity interfaces. *One principal result is that the vorticity interface in several of the classical turbulent flows has a fractal dimension of 2.36 ± 0.05, and that the dimension of the scalar interface in the scale range between L and* η *is also the same*. This last result is consistent with the understanding that the scalar is dynamically passive. The fact that the dimension is independent of the flow is an indication that the scales that produce the fractal character (scales less than the integral scale) possess kinematic similarity at this level of description. In the scaling range between η and η_b, the dimension is close to 3, that is to say, fluctuations in this range are space-filling. This is consistent with the conclusion in section 4 that *one need not invoke fractals to describe these*

TABLE 1. Summary of fractal dimension measurements in classical turbulent flows; from [47].

Flow	scale range between η and L			scale range between[a] η and η_b
	Method of measurement			
	1-D cuts[b]	2-D images	3-D images	
round jet	2.36	2.36	2.36	2.7 (Sc = 1930)[c]
plane wake	2.40	2.36	2.36	2.7 (Sc = 1930)
plane mixing layer	2.39	2.34	--	--
boundary layer	2.40	2.38	--	--

[a]All measurements are from one-dimensional cuts, with Taylor's hypothesis

[b]These one-dimensional measurements for jets and wakes were made both with and without Taylor's hypothesis. Note that one-dimensional measurements often yield slightly higher values for the fractal dimension, but the experimental uncertainties preclude us from attaching much significance to it. The mean value and the statistical error bars, deduced from many measurements, are 2.36 ± 0.05. The slight difference from one flow to another may or may not be significant; the present thinking is that they are not.

[c]The Schmidt number is obtained from [49]. Typical error bars [40] for this estimate are ± 0.03.

sub-Kolmogorov scale fluctuations in scalars. I must remark that this is true only in the limit of infinite Schmidt numbers (if the Reynolds numbers is also proportionately higher), and finite Schmidt number effects reduce this value; for σ of the order of 1000, the measured dimension in this scale range is 2.7± 0.03; see [40].

One use to which the fractal dimension has been put is in the explanation of small scale turbulent mixing. The basic idea is that the properties of the scalar interface (say) and the mixing of the scalar with the ambient fluid are related; the area of the surface involves fractal dimension and the ratio of inner to outer cut-off scales [50-51]. The *amount* of mixing is governed by large eddies in the flow, but the small scale mixing is accomplished by diffusion across the surface whose geometry is determined exactly by this requirement. Thus, even though the process is initiated by large scales, one can legitimately try to understand small scale mixing by concentrating on the diffusion end. This approach neither minimizes the role of large eddies nor resorts to gradient transport models usually discredited in turbulence theory. It has been shown [50,51] that this approach yields results of some universality, simply because the small scale features of the flow are, to a first approximation, independent of configurational aspects of the flow. In particular, the arguments presented in [51] yield semi-theoretical estimates for fractal dimensions which agree well with measurements.

Preliminary measurements [52] suggest that the interface in supersonic turbulent flows has a lower dimension than that in incompressible flows. More work is needed before the implication of this result to mixing in compressible flows can be understood.

The outstanding question is to show that the dynamics of the Navier-Stokes equations (or, more precisely, that of the Euler equations, since the effect of viscosity is believed here to be benign in that it merely sets the inner cut-off) somehow imply that the features described here are indeed multifractal in nature – that is, they possess certain type of spatial scale similarity. In spite of much impressive work [52], these issues have remained essentially untouched and invite the attention of a talented reader. In particular, this question has to be reconciled with the existence or otherwise of scale similarity in temporal dynamics.

7. Conclusions

The dynamical systems approach has introduced a variety of new tools and ideas for analysing nonlinear systems. This is an outstanding accomplishment and deserves to become an integral part of the lore of nonlinear phenomena including turbulence. Whether all or some of them are useful for furthering our knowledge and predictive capabilities of transitional and turbulent flows depends to some extent on the ingenuity with which these new tools are applied. Just as initial exaggerations on the importance of the dynamical systems approach to the turbulence problem were unhealthy, so are gloomy thoughts that the approach is but a passing fad to be disparaged. The success one can have with the application of these new tools depends to some measure on how well one understands them, and on how well one can reduce sophisticated mathematical formalisms to realizable measurements. I have presented a summary of some results in transitional and turbulent flows of the open type. These measurements were inspired, in fact made possible, by recent advances in dynamical systems.

It is often believed that the success of the dynamical systems approach in open flows is much less tangible. This view is expressed, among others, by Holmes [3]. Just so, but a part of the reason is the culture of the community which works extensively with such flows. It is true that not all open flows in the early stages of transition can be usefully described by the dynamical systems approach, but so is it true that not all closed flows can be described in this manner! As mentioned already, our understanding of which type of flows can be so described is accumulating only slowly and its detailed discussion should therefore await another occasion; but it seems that the classification of flows as 'open' and 'closed' is probably not specific enough for present purposes. The papers presented in this section of the meeting give samples of the types of questions addressable at present by this approach. These same ideas may be helpful in other contexts such as noise reduction in experimentally obtained signals, an example of the current work being [54].

Mere description of turbulence is no end in itself, and one needs to make predictions by building working models. Some predictions rendered possible from the present work have been summarized at appropriate places; other papers in this section of the meeting present other avenues. Together, they constitute interesting, and possibly useful, contributions to turbulence dynamics. Turbulence has the peculiar status of being at once a classical problem and one at the forefront of physics. There is no complete agreement as to what information is essential and should be acquired, and there is room for fresh air here!

Acknowledgements

This work is the result of a collaborative effort with A.B. Chhabra, M.S. Fan, R.V. Jensen, A. Johnson, P. Kailasnath, C. Meneveau, D.J. Olinger, R. Ramshankar, R.R. Prasad and P.J. Strykowski; to them all, I am grateful. I have benefitted from discussions at various times with M.J. Feigenbaum, D. Gottlieb, B. Knight, B.B. Mandelbrot, M.V. Morkovin, M. Nelkin and L. Sirovich. I am thankful to A.B. Chhabra, B.-T. Chu, C. Meneveau, D.J. Olinger and P.P. Wegener for their comments on a draft and to M.J. Feigenbaum for his hospitality during its writing. Different aspects of this work were supported by grants from AFOSR, DARPA (URI), and NSF.

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