RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 41, NUMBER 4 15 FEBRUARY 1990

# Interface dimension in intermittent turbulence

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(Received 23 October 1989)

We elaborate on a result obtained in Sreenivasan, Ramshankar, and Meneveau [Proc. R. Soc. London Ser. A **421**, 79 (1989)] which explains the observations of a universal fractal dimension of interfaces close to $\frac{7}{3}$. Here we explicitly take into account the influence of local fluctuations in the Kolmogorov scale (due to the multifractal nature of the rate of dissipation) on the surface area. The resulting dimension is shown to be equivalent to the prediction of a simple argument involving coarse graining of the interface.

In this Rapid Communication we elaborate on a result obtained by Sreenivasan, Ramshankar, and Meneveau.[1] There it was shown that the experimental observations[2] of an apparently universal value for the fractal dimension of turbulent interfaces close to $\frac{7}{3}$ could be explained from basic principles in a simple fashion. Consider the situation (see Fig. 1) where an interface separates two distinct regions $A$ and $B$ of the flow. If the interface is a turbulent-nonturbulent interface, $A$ would be the (outer) region of irrotational nonturbulent fluid and $B$ would be the turbulent (vortical) region. Such a situation is typical in free shear flows such as boundary layers, jets, mixing layers, etc. When considering turbulent mixing of species or other passive scalars, $B$ could be a region where mixing has occurred, while $A$ is the unmixed region; the interface is then a scalar interface. It was shown in Ref. 1 that in order to relate geometric properties of the interface to the dynamics of turbulence, it is useful to focus on the flux of a given quantity through such interfaces. In the case of a turbulent-nonturbulent interface, the transportable quantities whose flux can be studied could be momentum, turbulent kinetic energy, mean-square vorticity, etc. In the case of scalar interfaces, the flux could be mass or heat flux, the flux of scalar fluctuations, etc. For now, we restrict our attention to transportable quantities whose diffusivity is equal to the viscosity. Extensions to cases where they differ can easily be performed, following Ref. 1.

The total diffusive flux of the transportable quantity across the interface will be given by the product of the appropriate diffusivity, the gradient of the transportable quantity across the interface, and the total area of the interface. In Ref. 1, it was argued that the thickness of the interface, being essentially given by a balance of viscous action and turbulent straining, has to be the Kolmogorov scale of the flow, $\eta$. This then implies that the gradients across the interface will be of the order $\Delta_{AB}/\eta$, where $\Delta_{AB}$ is the difference between the transportable quantity in regions $A$ and $B$. Furthermore, if the interface is fractal with dimension $D$, its total area $S(\eta)$ measured with a resolution equal to $\eta$ (convolutions of scales smaller than $\eta$ get smoothed out due to diffusion) will be of the order of (see Ref. 1)

$$S(\eta) \sim L^2(\eta/L)^{2-D} , \tag{1}$$

where $L$ is some outer (integral) length scale. The total flux $\Phi_{AB}$ will then be

$$\Phi_{AB} \sim \nu L^2(\eta/L)^{2-D}\Delta_{AB}/\eta . \tag{2}$$

Expressing this in terms of the Reynolds number $\mathcal{R}_L = u'L/\nu$, where $u'$ is the root-mean-square value of the turbulent velocity fluctuations, gives

$$\Phi_{AB} \sim \Delta_{AB} u' L^2 \mathcal{R}_L^{3/4(D-7/3)} , \tag{3}$$

where we have used $\eta/L \sim \mathcal{R}_L^{-3/4}$. As argued in Ref. 1, the total flux in fully turbulent flows is expected to be independent of the viscosity of the fluid, that is to say, independent of Reynolds number (the so-called Reynolds number similarity). Equation (3) then implies that $D = \frac{7}{3}$, in essential agreement with the experimental observations of Ref. 2. Reference 1 obtained corrections to this estimate by taking into account fluctuations in $\eta$ due to the intermittent (multifractal) nature[3,4] of $\varepsilon$, the rate of dissipation of turbulent kinetic energy. This was done by computing the mean value of the gradient $\Delta_{AB}/\eta$. The resulting correction to $D$ is of the form

$$D = \tfrac{7}{3} + \tfrac{3}{4}(3 - D_{1/4}) , \tag{4}$$

where $D_{1/4}$ is the "generalized dimension" $D_q$ of the dissipation $\varepsilon$ for $q = \frac{1}{4}$ (in the three-dimensional domain).

However, Ref. 1 did not make corrections to the estimate of the total area $S(\eta)$ due to fluctuations in $\eta$. This seems to be difficult at first glance, because the definition

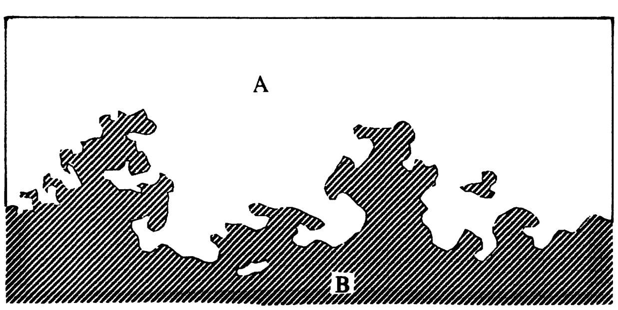


FIG. 1. Schematic diagram of an instantaneous planar section through an interface, separating region $A$ from region $B$.

 

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of $D$ via Eq. (1) already involves a spatial averaging and it is not clear which of the following length scales should be used: The spatial mean of $\eta$, $\langle\eta\rangle$, or the usual Kolmogorov scale $\eta^*$ defined in terms of the mean rate of dissipation $\langle\varepsilon\rangle$ according to

$$\eta^* = (\nu^3/\langle\varepsilon\rangle)^{1/4} . \quad (5)$$

In general, each of these scales are different. Here we show a means of circumventing this difficulty by explicitly integrating the flux over boxes along the interface, taking into account the multifractal nature of $\varepsilon$.

As before, consider covering the interface with cubic elements of size equal to the local thickness of the interface, which is assumed to be of the order of the local Kolmogorov scale $\eta_i$. The size of the $i$th cube along the interface is denoted by $\eta_i$. The local contribution of each such small element to the total flux will consist of the typical area of such elements $\eta_i^2$ times the local gradient $\Delta_{AB}/\eta_i$. The total flux will then be given by the sum of such contributions along the entire interface according to

$$\Phi_{AB} \sim \sum_i \nu\eta_i\Delta_{AB} \sim u'\Delta_{AB}L^2\mathcal{R}_L^{-1}\sum_i(\eta_i/L) . \quad (6)$$

The local value of $\eta_i$ can be expressed as a function of the local singularity strength $\alpha_i$ of the dissipation (see Ref. 5). This is done by recalling that in the multifractal formalism the dissipation averaged over a three-dimensional domain of size equal to $r$ varies as

$$\varepsilon_r \sim \langle\varepsilon\rangle(r/L)^{\alpha-3} , \quad (7)$$

where $\alpha$ is the local singularity strength of the dissipation. Further, the local value of $\eta$ can be estimated by

$$\eta_i = (\nu^3/\varepsilon_{\eta_i})^{1/4} , \quad (8)$$

where $\varepsilon_{\eta_i}$ is obtained by setting $r = \eta_i$ in Eq. (7). Solving for $\eta_i$ yields

$$\eta_i/L \sim (\eta^*/L)^{4/(1+\alpha_i)} , \quad (9)$$

where $\eta^*$ is the usual Kolmogorov scale of Eq. (5). Now we want to replace the spatial sum of Eq. (6) by an integral over all $\alpha$ values. For this we need to weight the integrand by the number of boxes having the same $\alpha$. In the entire domain, the total number of boxes of size $\eta_i$ where $\alpha$ has a particular value $\alpha_i$ [corresponding to $\eta_i$ through Eq. (9)] scales as[5]

$$N(\alpha_i) \sim (\eta_i/L)^{-f(\alpha_i)} \sim (\eta^*/L)^{-4f(\alpha_i)/(1+\alpha_i)} . \quad (10)$$

Let us now assume that along the interface the dissipation $\varepsilon$ displays basically the same multifractal behavior (at least for low-order moments) as everywhere else in the fully turbulent domain. In Ref. 1 it was shown experimentally that this is a reasonable assumption. Now we use the additive properties of codimensions,[6] and argue that $3-f(\alpha_i)$ (the codimension of an iso-$\alpha_i$ set in three-dimensional space) must be equal to the sum of the codimension of the interface $(3-D)$ and the codimension of an iso-$\alpha_i$ set in *along the interface only*. It follows that the dimension of such a set is equal to $f(\alpha_i)-3+D$. Therefore, the total number of boxes where $\alpha$ has a certain value $\alpha_i$ along the interface scales as

$$N(\alpha_i) \sim (\eta^*/L)^{-4[f(\alpha_i)-3+D]/(1+\alpha_i)} \quad (11)$$

Now, we can replace the sum over all boxes of Eq. (6) by an integral over all possible $\alpha$ values and write

$$\Phi_{AB} \sim u'\Delta_{AB}L^2\mathcal{R}_L^{-1}\int(\eta^*/L)^{-4[f(\alpha)-4+D]/(1+\alpha)}d\alpha . \quad (12)$$

This integral is evaluated using the method of steepest descent in the limit of small $\eta^*/L$. The extremum of the exponent is given by the condition

$$df/d\alpha = [f(\alpha)-4+D]/(1+\alpha) . \quad (13)$$

We now use the usual relations[7] between the $f(\alpha)$ curve and the moment exponents $D_q$ and designate by $Q$ the value of $q = df/d\alpha$ at which Eq. (13) is satisfied. Using $\eta^*/L \sim \mathcal{R}_L^{-3/4}$ the flux can then be written as

$$\Phi_{AB} \sim u'\Delta_{AB}L^2\mathcal{R}_L^{-1+3Q} , \quad (14)$$

and condition (13) can be rewritten as

$$D = 4+Q+(Q-1)D_Q . \quad (15)$$

Again, the independence of the flux on the Reynolds number now implies, from Eq. (14), that $Q = \frac{1}{3}$. It then follows from Eq. (15) that the dimension of the interface is

$$D = \tfrac{7}{3} + \tfrac{2}{3}(3-D_{1/3}) . \quad (16)$$

Thus the intermittency correction obtained by this approach involves the moment exponent of order $\frac{1}{3}$ instead of the one of order $\frac{1}{4}$ that was obtained in Ref. 1 by considering fluctuations in $\eta$ through the gradients only.

Finally, we point out that the result (16) can also be obtained by two different arguments. One of them, considering the transport due to relative velocity fluctuations over distances of the order of $\eta^*$, was presented in Ref. 1 and will not be repeated here. An alternative argument[8] considers a systematic coarse graining of the interface shown in Fig. 1. One coarse grains the interface up to a scale $r$ (where $r$ is a distance pertaining to the scaling range) and computes the flux across this coarse-grained interface. Let $S(r)$ be its area measured with resolution $r$. The gradient will be estimated as $\Delta_{AB}/r$. The eddy viscosity $\nu(r)$ that accounts for the convective action of all scales smaller than $r$ is usually estimated in terms of typical turbulent velocity differences $\langle|\mathbf{u}(\mathbf{x}+r)-\mathbf{u}(\mathbf{x})|\rangle = |\Delta u_r|$ as

$$\nu(r) \sim r|\Delta u_r| \sim u'r(r/L)^{\xi} . \quad (17)$$

Here $\xi$ is the scaling exponent of $|\Delta u_r|$, which in the Kolmogorov 1941 theory is $\xi = \frac{1}{3}$. In the presence of intermittency, $\xi$ can be related[4] to the moment exponents of $\varepsilon$ according to

$$\xi = \tfrac{1}{3} + \tfrac{2}{3}(3-D_{1/3}) . \quad (18)$$

The total flux computed from a coarse-grained interface is thus

$$\Phi_{AB}(r) \sim u'\Delta_{AB}S(r)(r/L)^{\xi} . \qquad (19)$$

If the eddy viscosity consistently takes into account the effects of the coarse-graining procedure, and if the surface area $S(r)$ is correctly evaluated as a function of $r$, one would expect the estimate for the flux to be independent of the resolution $r$ chosen to formulate the problem. This immediately implies that $S(r) \sim r^{-\xi}$, which means it is fractal. Furthermore, since for fractal surfaces of dimension $D$ one has $S(r) \sim r^{2-D}$, it is clear that

$$D = 2 + \xi . \qquad (20)$$

Using Eq. (18), this result agrees with Eq. (16).

This work was performed with financial support from U.S. Defense Advanced Research Projects Agency University Research Intiative and U.S. Air Force Office of Scientific Research.

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