TRANSITIONAL AND TURBULENT WAKES AND CHAOTIC DYNAMICAL SYSTEMS

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Recent studies of the dynamics of simple nonlinear systems with chaotic solutions have produced very interesting and (perhaps) profound results with several implications in many disciplines. However, the interest of fluid dynamicists in these studies stems primarily from the expectation that they will help us better understand the process of transition and turbulence in fluid flows. At this time, much of this expectation remains untested, especially in 'open' or unconfined fluid systems. This work is aimed at filling some of this gap.

We have measured in the wake behind a circular cylinder, chiefly about 5 diameters behind it, the spectral density of streamwise velocity as a function of the Reynolds number. If the free stream turbulence is low and devoid of any discrete frequency, the signal/noise ratio is large (as in our experiments where the peak signal/noise ratio is of the order of 10^6 or more), and the FFT has adequate resolution, it can be seen that the transition to chaotic state (broad-band spectrum) is characterized by the following stages. As the Reynolds number is increased:

(a) there is first only one basic frequency f_1 (and its harmonics) arising from vortex shedding;

(b) this is followed by the appearance of a second frequency f_2, incommensurate with the vortex shedding frequency and the various combinations of the two frequencies;

(c) a third incommensurate frequency appears (with several combinations of the three frequencies);

(d) at a slightly higher Reynolds number, the spectrum has a broad-band character, although the peak corresponding to the vortex shedding remains.

Phase diagrams and Poincaré sections, as well as calculations of the dimension of the attractor, confirm the existence of these stages, which are much like those indicated by the Ruelle-Takens-Newhouse picture.

However, with further increase in Reynolds number, there is a re-emergence of order, appearance of a fourth independent frequency, and a return to chaotic state; we emphasize that there is a stage in which there are four independent degrees of freedom with no chaos. From this second chaotic state, one can discern the re-emergence of order and return to chaos once again; we suspect that there are many windows of chaos and order - much as in several dynamical systems. It appears that the discontinuity in the vortex shedding frequency at Reynolds numbers of about 80 and 130 is related to the appearance of chaos and order. We have shown that the dimension of the attractor is truly representative of the number of degrees of freedom in the early stages of transition characterized by discrete frequencies. If this same interpretation of the dimension is true also in the chaotic state, then the relatively low dimension of the attractor even at Reynolds number up to about 10^4 suggests that the number of degrees of freedom in turbulent flows far past transitional stages is not high, and some kind of slaving principle or renormalization theory ought to be brought to bear in the reformulation of the "turbulence problem".

1. INTRODUCTION

The equations governing the (incompressible) motion of fluids are

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{1.1}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{1.2}$$

where we have restricted ourselves to body-force-free situations and used suitably normalized quantities, and Re is the Reynolds number. Observations have shown that for the given boundary conditions (and external forces, if applicable), the flow is unique and steady for $Re < Re_{cr}$, where Re_{cr} is a certain critical value of Re; this is the steady laminar motion. As Re increases, the fluid motion may first become periodic, quasi-periodic, etc., and "eventually" chaotic and irregular such that the details of this state of motion are not reproducible. This state is not necessarily "turbulence" as generally understood - whatever this precisely means - but it is believed that the turbulent state is reached if the Reynolds number is high enough. The traditional goal of the stability theory is to describe the evolution from the laminar to the turbulent state, and the goal of all turbulence theories is to understand the (fully) turbulent state itself.

In the recent past, claims have been made that autonomous dynamical systems with small number of degrees of freedom, typified by

$$\frac{db_i}{dt} = f(b_i; \epsilon_i) \,, \tag{1.3}$$

where i is small and ϵ_i are the control parameters, help us towards attaining both of the goals mentioned above.

Several questions arise immediately. One natural question is to what degree dynamical systems with small number of degrees of freedom are relevant to fluid flows. To elucidate the concept of "degrees of freedom in fluid flows", let us approximate

$$u_i = \sum_{\underset{\sim}{k}} a_i(\underset{\sim}{k};t)e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \tag{1.4}$$

which, with (1.1) and (1.2), yields equations of the type

$$\frac{\partial a_i}{\partial t} = F(a_i;Re), \quad i = 1, N \ \text{(large)}. \tag{1.5}$$

The number of the coefficients a_i, which, for given boundary conditions for the fluid flow, are capable of variation in time, can now be called the degrees of freedom of the fluid flow governed by (1.1) and (1.2). Since the laminar flow is uniquely specified by the boundary (and external force) conditions, this number is zero. If Re increases past Re_{cr}, only a finite number of degrees of freedom are excited, and hence it appears that, at least in the transcritical regime, consideration of a small number of degrees of freedom is adequate.

An interesting hypothesis, which we shall examine in this paper, is that the number of degrees of freedom (not necessarily in the sense described above) remains small even in high Reynolds number turbulence.

Assuming that the number of degrees of freedom excited in the neighbourhood of the critical state is indeed small, we must ask if the behaviour in the transcritical regime is independent of the precise nature of the right-hand side of equations (1.3) and (1.5). The reason most often cited in support of the belief that the detailed structure of f_i in (1.3) is immaterial in understanding the evolution of chaotic state in dynamical systems, is the RUELLE-TAKENS theorem [1], which states that chaos (or strange attractor) sets in abruptly, following a few HOPF bifurcations, and that this behaviour is "typical". (In a later paper, NEWHOUSE, RUELLE & TAKENS [2] consider motion on a three-torus (i.e., quasi-periodic motion with three incommensurate frequencies) and introduce a small nonlinear coupling among the three oscillators. They argue that to produce a broadband spectral density, it is enough to have a weak coupling among the three oscillators.). Whether or not fluid flows are "typical" in the sense that RUELLE & TAKENS discuss is not clear at all, and one should attempt to answer this question by looking at the specific form of F in (1.5) and by observing the actual bifurcations in experiments on laminar-turbulent transition.

Finally, one must mention the predominant role played by spatial chaos (and order!) in turbulent flows of fluids. Autonomous dynamical systems, on the other hand, do not contain any space information. While spatial order and chaos in fluid turbulence may in some way be related to temporal chaos and order, it is clear that there is little that (autonomous) dynamical systems can say directly about the former.

Several beautiful experiments that have been carried out in the Taylor-Couette flow (e.g., Refs. 3,4) and the convection box (Refs. 5,6) have lent support to the idea that fluid flows bear a close correspondence to dynamical systems. This in itself is undoubtedly remarkable, but it should be remembered that these two flows are special in the following sense. In all "closed flow" systems - of which the convection box and the Taylor-Couette flow are popular examples - each value of the control parameter (for example, the rotation speed of the inner cylinder in the Taylor-Couette problem) characterizes a given state of the flow globally. At least in principle, one can follow the various stages of transition to turbulence in as much detail as possible by exercising infinitely fine control over the control parameter. This is not necessarily true for another class of flows, which we may call "open systems", e.g., channel flows, wakes, jets, boundary layers, etc.. Consider the channel (or the plane Poiseuille) flow. For a given value of the control parameter Re, the flow can be laminar at one location, transitional at another, and turbulent at yet another (downstream) location; the same is true of jets, wakes, for example. As a result, at least two complications arise. First, in open systems, observations cannot be made with such exactitude as in closed flow systems. Second, there could in principle be a strong coupling between different phenomena in different spatial locations in a way that is peculiar to the particular flow in question.

On balance, all these considerations suggested to us that it is desirable to look at some open flows to determine the extent to which (low-dimensional) dynamical systems can assist us in our goals of understanding transition and turbulence in fluid flows. This is the motivation for the work described in this paper, which is to be viewed more as a progress report than as a complete account; obviously much more remains to be done. Our approach is to select well-known flows and follow the bifurcations as closely as possible. Surprisingly, while much work has been done on these flows in the past, an amazing amount of new information can still be acquired that will facilitate clarifying the relation between chaotic dynamical systems and fluid flow transition and turbulence.

2. EXPERIMENTS

Our first attempts were (for historical reasons) on flow in a coiled pipe, SREENIVASAN & STRYKOWSKI [7]. Spectral measurements indicated that transition to turbulence occurred somewhat similarly to the RUELLE-TAKENS picture; that is, with increasing Re, the power spectral density of the streamwise velocity fluctuation shows essentially a single peak, two peaks and then three peaks immediately followed by the onset of a broad-band component. This behaviour might suggest the presence of a strange attractor. Our subsequent evaluation of the "dimension" (see section 4) of the attractor indicated that this quantity was small (not greater than about 6), at least at Reynolds numbers not too far from the transition value. Our calculations at much higher Re were inconclusive, due to various computational and instrumentation resolution problems; it was also displeasing that spectral peaks were not as sharp or as narrow as desired. A further problem seemed to be the somewhat unusual flow configuration, which itself led to many physically unfamiliar behaviours, making interpretations of results somewhat difficult. Although our further work has led to a better understanding of that flow, it seemed necessary to make measurements in other less unfamiliar flows of common occurrence. We decided to make measurements in a two-dimensional wake, covering a Reynolds number range from the onset of vortex shedding to an "essentially turbulent" state.

All experiments were done in a 70 cm × 50 cm suction-type wind tunnel with speed control obtained by varying the armature current of the d.c. motor driving the fan. At the speeds of the experiments, the free-stream turbulence level (including the wind tunnel unsteadiness) was less than about 0.2 % - neither very small nor very large in comparison with most existing facilities. The spectral density of the streamwise velocity fluctuation u in the free-stream showed no discrete peaks. Three wake generators were used. Two of them were nylon threads stretched across the width of the wind tunnel, 0.024 cm and 0.036 cm in diameter, giving aspect ratios of about 2000 and 1500 respectively; the third was an aluminum tube 4 mm in diameter (aspect ratio = 175).

A large part of the data to be presented below is in the form of power spectral densities of u. For nearly all the signals, digitization was done at sufficiently high frequency (60 kHz or more) to ensure that whenever the signal was periodic, at least 30 digitized points were contained in one period of the basic frequency (so that it was a good representation of the

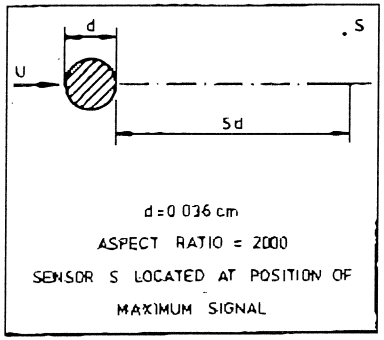


Fig. 1

Schematic of experimental conditions.

analog signal). Further, the entire length of the signal, which contained at least 100 cycles of the basic frequency, was Fourier transformed at once, using the Cooley-Tukey FFT algorithm. The overriding criterion was that the spectral resolution should be as good as possible (here, about 0.9 Hz compared with shedding frequencies of the order of 2000 Hz or more) and must not miss any low frequency modulations.

All velocity signals were obtained with a hot-wire operated on a DISA 55M01 constant temperature anemometer. The speed of the tunnel was monitored with a Pitot tube connected to a calibrated MKS Baratron with adequate resolution (and an averager). The hot-wire and the Pitot tube were mounted on a specially-made slim holder. Most measurements were made approximately 5 diameters downstream from the cylinder and about a diameter or so off the centerplane where the signal was the largest (see Figure 1).

3. RESULTS

Figure 2 shows the logarithm (to base 10) of the normalized power spectral density of the hot-wire signal at a Reynolds number Re (based on the free stream velocity and the diameter of the vortex shedding cylinder) of about 36, which is just about the onset value for vortex shedding. Notice that the general noise level is around 10^{-8}, while the peak of the spectrum (marked f_1), corresponding to the basic vortex shedding frequency behind the cylinder cylinder, is at around $10^{-0.5}$, about 7 1/2 orders of magnitude higher than the noise level! The sharpness is excellent, which also holds for the other peaks to the right of f_1, which are the harmonics of f_1.

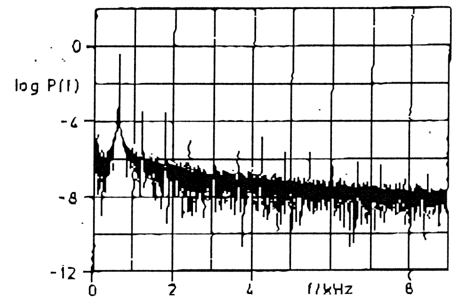


Fig. 2 Frequency spectrum of streamwise velocity fluctuations at Re = 36. Note that the power is plotted on a logarithmic scale. The peak at f_1 = 590 Hz corresponding to vortex shedding, and the subsequent strong peaks above the noise level are simply harmonics of f_1. Notice that the background noise is not white.

At a somewhat higher Reynolds number of 54, there appear a number of peaks in the spectrum (Figure 3a). As shown in the expanded version (Figure 3b), all the peaks can be identified precisely in terms of the interaction of the two frequencies - the basic vortex shedding frequency f_1 and another incommensurate frequency f_2. That it contains only two frequencies can be seen also from a combination of the phase plot and its Poincaré section (Figures 4 and 5). Figure 4, which is a computer plot of the time derivative $\dot{u}$ of the signal against the signal u itself, is seen to be a complicated structure; the Poincaré section (Figure 5), which is simply $\dot{u}$ vs u sampled at the frequency f_2, is essentially a circle - as it ought to be if the signal contained only frequencies f_1 and f_2. At a slightly higher Reynolds number of 62, the second frequency becomes much weaker (Figure 6); that it has not disappeared can be seen clearly from the corresponding phase plot (Figure 7). At Re = 76, several peaks can be seen in the spectral density (Figure 8) and, as shown in detail in Figure 9, these peaks can all be identified with great precision (actually 5 decimal places) as arising from the interaction of three irrational frequencies. After a finite (though small) increase in Reynolds number, one can see about an order of magnitude increase in broad-band frequency content to the left of f_1 (Figure 10). In the language of dynamical systems, we may now consider chaos to have set in.

This progression towards chaos - underlying the possible presence of a strange attractor - proceeds more or less according to the prescription given by NEWHOUSE, RUELLE & TAKENS [2]. There are deviations! These include the conspicuous weakening at Re = 62 of the second frequency after its strong appearance at Re = 54, as well as the moderate and finite increase in Reynolds number that is required between the appearance of the third frequency (Re = 76) and the onset of chaos (Re = 93). It is still

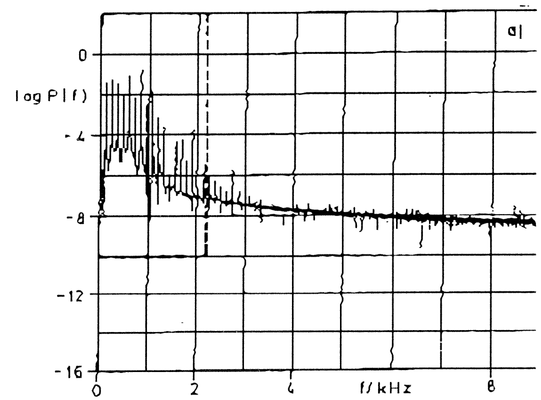


Fig. 3a Frequency spectrum at Re = 54. f_1 = 835 Hz.

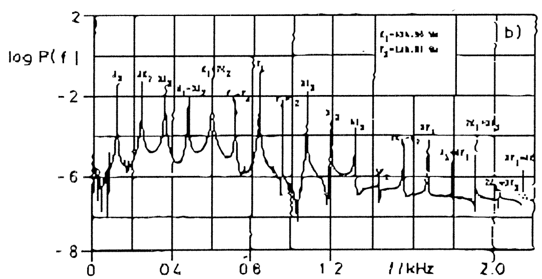


Fig. 3b Expanded version of Figure 3a in the frequency range 0 - 2200 Hz. Note that all significant peaks in Figure 3a are simply linear combinations of f_1 and another incommensurate frequency f_2 = 119 Hz.

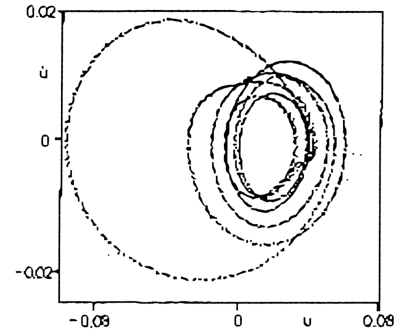


Fig. 4

A phase plot from the velocity signal u at Re = 54. The ordinate is simply the time derivative u̇ of the abscissa u. Number of data points = 3000.

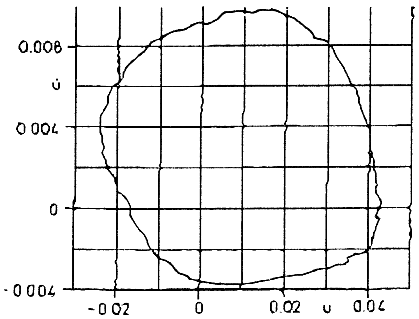


Fig. 5

Poincaré section for the phase plot of Figure 4. This is simply a plot of u̇ vs u sampled exactly at the frequency f_2.

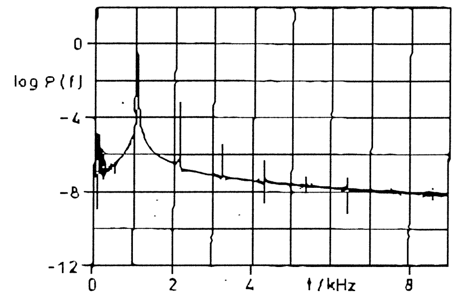


Fig. 6 Frequency spectrum at Re = 62. Notice that the second frequency has diminished in importance.

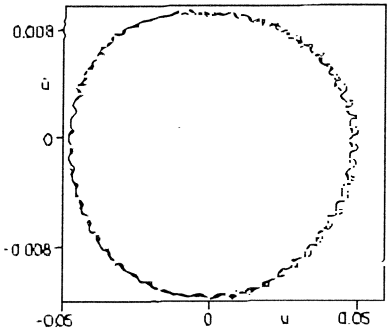


Fig. 7

Phase plot for Re = 62; all details as in Figure 4.

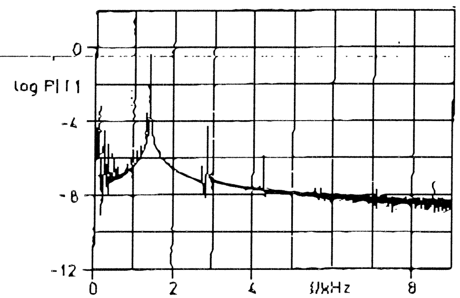
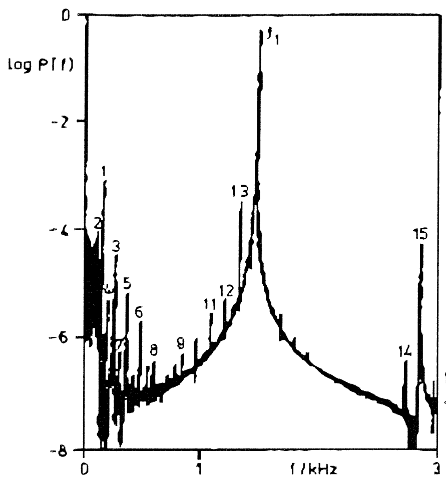


Fig. 8

Frequency spectrum at Re = 76.



f_1 = 1422.73 Hz f_2 = 119.02 Hz

f_3 = 29.30 Hz

1 = f_2	2 = $f_2 - f_3$
3 = $2f_2$	4 = $2f_2 - 2f_3$
5 = $f_1/3 - f_2$	6 = $f_1/3$
7 = $f_1/3 - 10f_3$	8 = $f_1/3 + f_2$
9 = $\frac{2}{3}f_1 - f_2$	10 = $\frac{2}{3}f_1$
11 = $f_1 - 3f_2$	12 = $f_1 - 2f_2$
13 = $f_1 - f_2$	14 = $2f_1 - f_2$
15 = $2f_1$	

Fig. 9 Expanded version of Figure 8 in the frequency range 0 - 3 kHz. All significant peaks are combinations of three frequencies f_1, f_2, and f_3.

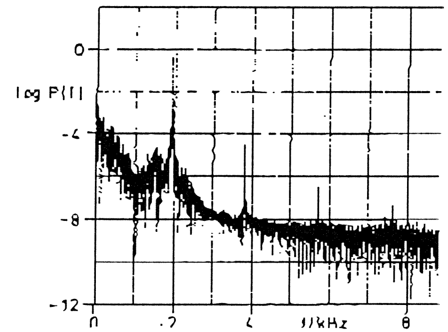


Fig. 10 First appearance of chaos at Re = 93. Notice that in comparison with Figure 8, the broadband "noise" level in the 0 - 2000 Hz range has gone up by an order of magnitude or so.

extraordinary that the "typical" behaviour indicated by RUELLE & TAKENS for a highly idealized mathematical system should have a nontrivial bearing on a rather complicated system.

It should be emphasized that the state we have recognized as chaotic is still far away from being turbulent. In fact, *most* of the energy is still contained in the discrete shedding frequency. Thus further increase in Reynolds number is in order.

With further increase in Reynolds number, the flow evolves into a much better organized state (Figure 11) at Re = 102, and the signal itself looks more periodic. We believe that there are *at least* two basic frequencies present in the system, although, because of the their low amplitude, we have been unable to recognize them precisely or to establish their connection to the frequencies occurring before the onset of chaos at Re = 93.

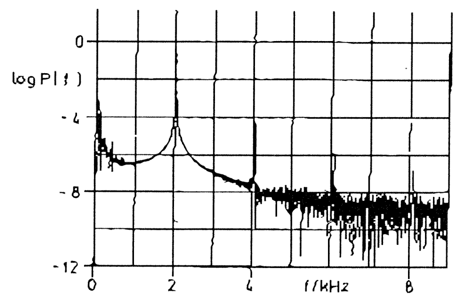


Fig. 11 "Reordering" at Re = 102.

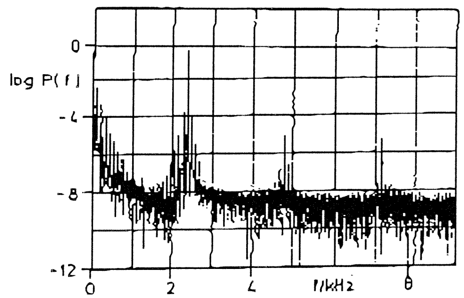


Fig. 12a Discrete frequencies at Re = 115.

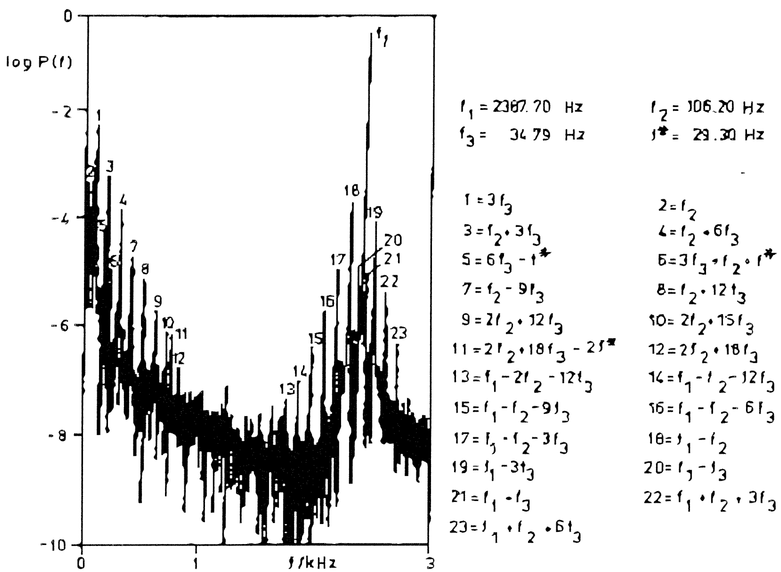


Fig. 12b Expanded version of Figure 12a in the range 0 - 3 kHz. Notice that four frequencies are required to account for all dominant peaks above the noise level.

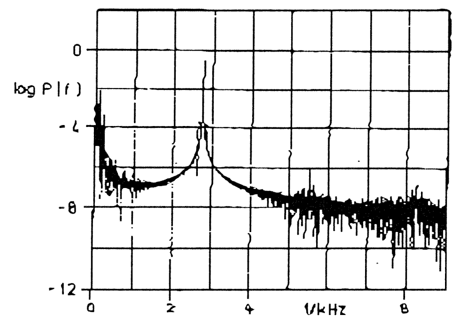


Fig. 13 Frequency spectrum at Re = 127. Notice the reduced relative dominance of the discrete frequencies f , f , and f* in comparison with f .

With further increase in Re, there is a reappearance (Re = 115) of relatively strong discrete components (Figure 12a); and, as outlined in detail in Figure 12b, there is a definite need for four frequencies and their various combinations to identify all the dominant peaks. Further increase in Re results in the weakening (but not the disappearance) of the discrete components at Re = 127 (Figure 13) and a reappearance of chaos at Re = 135 (Figure 14), as indicated by the broadband component in the power spectral density. This second appearance of chaos is marked by a larger fraction of energy content in the broadband component than was the case when chaos set in the first time at Re = 93.
Further increase in Re results in a return to a more ordered state (Figure 15), but this return to "order" is somewhat less convincing than the previous instance at Re = 102. As Re increases further, one sees the reappearance of chaos (Figure 16) at Re = 168; presumably, greater resolution in our measurements would reveal steps similar to those preceeding the appearance of chaotic states at Re = 93 and 135. This reappearance of chaos is also marked by a much higher fraction of energy in the broadband component of motion (or "background noise", as it is often labeled).

Two remarks should be made: First, we note that there is a well-defined state with four discrete frequencies (and their linear combinations) without the presence of a strange attractor - a statement we shall justify later (section 4). This is in contradiction to the NEWHOUSE-RUELLE-TAKENS projection, and to the popular - and as far as we know unproven - statement that "period three means chaos". Second, the complicated appearance and reappearance of ordering and chaos is not unusual in other dynamical systems, either.

The sequence of events described above is summarized in Figure 17. Even though these precise details have not been noted before, we believe that

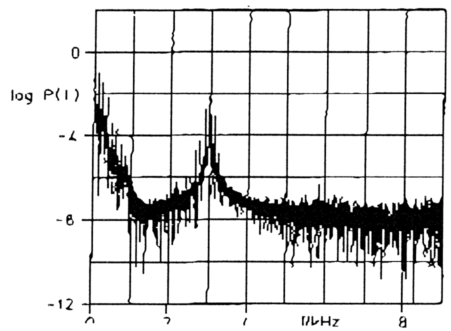


Fig. 14 Second appearance of chaos at Re = 135. Notice that the general noise level in the 0 - 1000 Hz range has gone up significantly.

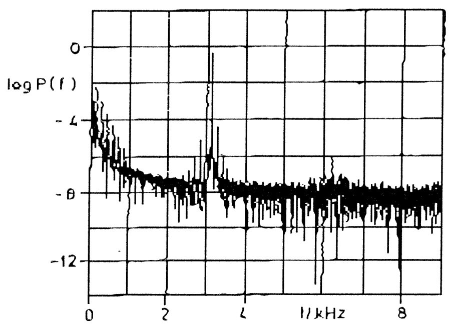


Fig. 15 "Reordering" at Re = 144.

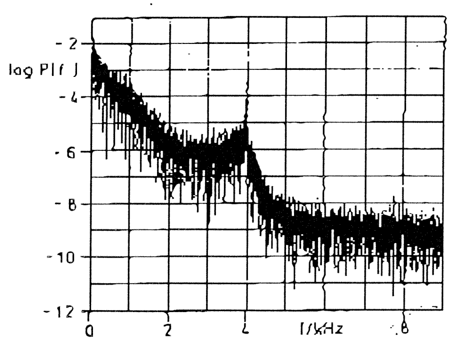


Fig. 16 Chaos at Re = 168. Notice the persistence of the votex shedding frequency.

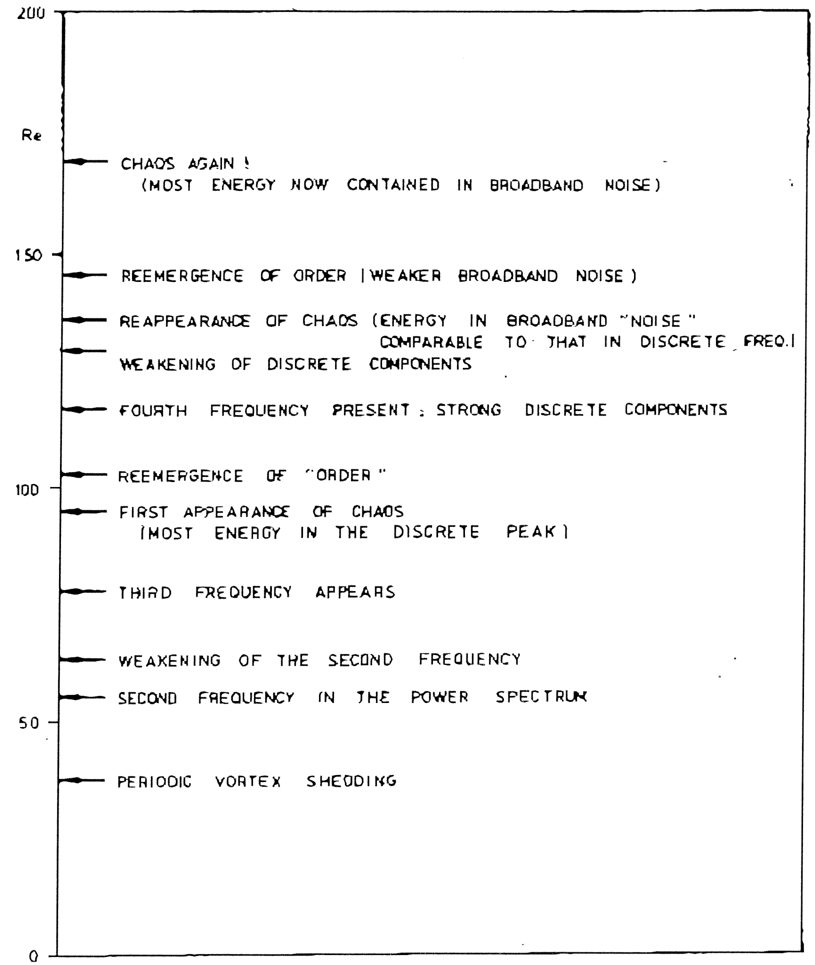


Fig. 17 Summary of the events.

related observations in wakes have in fact been made by previous researchers. For example, consider the windows of chaos and order alluded to above. Although we have not made detailed spectral measurements at higher Reynolds numbers, it is our contention that the succession of order and chaos in a wake continues indefinitely even at very high Reynolds numbers. This was noted several years ago by ROSHKO [8], who showed that order reappears in the Reynolds number range of 10^6. More recently, the fluctuating lift force measurements of SCHEWE [9] on a circular cylinder showed that the spectral density of the lift coefficient was broad at Re = 3.7×10^6 (upper end of transition) and became increasingly narrow until, at Re = 7.1×10^6, it was quite sharp, rather like a narrow-band-pass filtered signal. Although the fluctuating lift force can at best be related to the squared fluctuating velocity filtered via the transfer function corresponding to the response of the circular cylinder, its behaviour is nevertheless indicative of the flow itself in the vicinity of the cylinder.

As another example, consider the variation of the vortex shedding frequency with respect to Reynolds number (Figure 18). It can immediately be seen that the frequency does not monotonically increase with Re, but shows (at least) two distinct breaks. These breaks appear whenever there is a transition to chaos and reordering. Such breaks have been noted before [10,11,12], and perhaps most convincingly demonstrated in a beautiful experiment by FRIEHE [13]. Although the appearance of these breaks has been disputed, GASTER [14], our own data tend to support the conclusion that they do indeed appear. In TRITTON's first observations of the phenomena [10], a discontinuity in f_1 vs Re curve was observed in the range $80 <= Re <= 90$, while in his later experiments, TRITTON [12], it appeared at around Re = 110. Our conclusion is that they both appear, in agreement with FRIEHE's observation. Friehe varied the Reynolds number continuously at a low rate and obtained on an x-y plotter the frequency-Re variation directly.

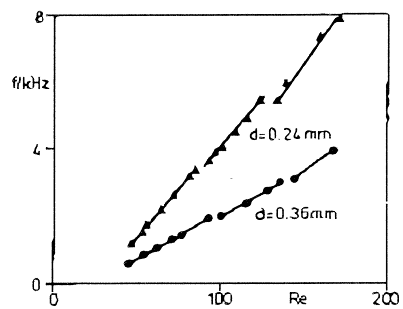


Fig. 18

Variation of the vortex shedding frequency with respect to Reynolds number.

4. DIMENSION OF THE STRANGE ATTRACTOR

It is clearly worth enquiring whether or not there is one single quantity that can successfully describe the many subtle changes that occur in the frequency spectra. There is indeed such a quantity, namely the dimension of the turbulence attractor. The concept of the dimension of the attractor is highlighted in studies of dynamical systems, and we may briefly digress here to discuss its meaning before presenting results of our measurements.

Let us consider that the attractor for turbulent signals is embedded in a (large) d-dimensional phase space. Let $N(\varepsilon)$ be the number of d-dimensional cubes of linear dimension ε required to cover the attractor to an accuracy ε. Obviously, making ε smaller renders N larger, but if the limiting quantity

$$D = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log (\frac{1}{\varepsilon})} \tag{4.1}$$

exists, it will be called the dimension of the attractor. An important characteristic of a strange attractor is that D is small even though d is large. We would be interested to see if turbulence has this property.

To see what the dimension means, let us write (4.1) as

$$N(\varepsilon) \sim \varepsilon^{-D}; \tag{4.2}$$

that is, if one specifies D and the accuracy ε to which we need to determine the attractor, we automatically know the number of cubes required to cover the attractor. The only missing information will now be the position of the cubes in the phase space. Thus, D can be considered as a measure of how much more information is required in order to specify the attractor completely; the larger the value of D, the larger is this missing information.

The dimension D, as defined in (4.1), has been called the fractal dimension by MANDELBROT [15], who has contributed a lot to our understanding of the quantity. As defined in (4.1), D is a geometric property of the attractor, and does not take into account the fact that a typical trajectory may visit some region of the phase space more frequently than others. Several measures taking this probability into account have been defined, and are believed to be closely related to the dynamical properties of the attractor. The most well-known among them are:

(a) the pointwise dimension
(b) the Grassberger-Procaccia dimension.

If the attractor is uniform, i.e., if each region in the phase space is as likely to be visited by the trajectory as every other, then the above two measures equal D defined by (4.1). Otherwise, they are generally smaller than D.

Let $S_\varepsilon(x)$ be a sphere of radius ε centered about a point x on the attractor, and let μ be the probability measure on the attractor. Then, the pointwise dimension is defined , FARMER, OTT & YORKE [16], as

$$d_p(x) = \lim_{\varepsilon\to 0} \frac{\log\mu[S_\varepsilon(x)]}{\log\varepsilon} \tag{4.3}$$

or

$$\mu[S_\varepsilon(x)] \sim \varepsilon^{d_p} \tag{4.4}$$

GRASSBERGER & PROCACCIA [17] have defined another measure ν that is related to the dimension of the attractor, as well as the information-theoretic entropy. The procedure for computing ν is as follows:

(i) Obtain the correlating sum $C(\varepsilon)$ from:

$$C(\varepsilon) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{\substack{i=j=1\\ i\neq j}}^{N} H(\varepsilon - |\underset{\sim}{x}_i - \underset{\sim}{x}_j|)$$

where H is the Heaviside step function and $\underset{\sim}{x}_i-\underset{\sim}{x}_j$ is difference in the two vector positions $\underset{\sim}{x}_i$ and $\underset{\sim}{x}_j$ on the phase space. Basically, what C does is to consider a window of size ε, and start a clock that ticks each time the difference $|\underset{\sim}{x}_i-\underset{\sim}{x}_j|$ lies within the box of size ε. Thus, one essentially has

$$C(\varepsilon) = \lim_{N\to\infty} \frac{1}{N^2} \left[\text{number of pairs of points } (i,j) \text{ with } |\underset{\sim}{x}_i-\underset{\sim}{x}_j| < \varepsilon\right]$$

(ii) Obtain ν from the relation, GRASSBERGER & PROCACCIA [17],

$$C(\varepsilon) \sim \varepsilon^{-\nu} \quad \text{as } \varepsilon \to 0 \tag{4.5}$$

In practice, not all components of x are known for constructing the phase space, but perhaps only one component, say x_m. One then constructs a d-dimensional "phase space" using delay coordinates

$$\{x_m, x_{m+\tau}, x_{m+2\tau}, \ldots\ldots, x_{m+(d-1)\tau}\}$$

where τ is some interval which is neither too small nor too large. If d is substantially larger than ν itself, reasonable results can be obtained.

Since one does not *a priori* know ν, one constructs several phase spaces of increasingly large values of d and evaluates ν for each of them; ν will first increase with d and eventually asymptote to a constant independent of d. This asymptotic value of ν is of interest to us as a measure of the dimension of the strange attractor.

We have computed both d_p and ν for turbulent velocity signals as described above, using the streamwise velocity fluctuations u and the delay coordinates described above to construct the phase space. Our confidence in the numerical values of these measures of dimension is very good when they are less than about 5 or 6, but becomes increasingly shaky at higher values. However, we do believe that they are reasonable, judging from their repeatability and the several precautions we have taken (such as taking the proper limit as $\varepsilon \to 0$ and using, in a couple of cases, double precision arithmetic in our computations). Figure 19 gives the data on ν and d_p as a function of the Reynolds numbers.

Several observations must be made. Concentrating first on the data at Reynolds numbers with discrete spectral peaks, we may conclude the following. At Re = 36, where there is only one independent degree of freedom (corresponding to the vortex shedding) (see Figure 2), the dimension of the attractor does indeed turn out to be about 1. When only two frequencies are present (Figures 3 and 4, Re = 54 and 62 respectively), the dimension is about 2, independent of the relative magnitude of the second frequency. One must note that at Re = 62, where the second frequency is of smaller amplitude, it is necessary to take the computations of the dimension to fairly small values of ε. At Re = 76, where there are three dominant frequencies, the dimension is about 2.7, not very different from the number of independent frequencies present. Lastly, at Re = 115, where there are four independent frequencies, the calculated ν is not very different from 4. Thus, making some small allowances for the computational uncertainties in calculating the dimension, it is seen to be a reasonable representation of the number of degrees of freedom in the system.

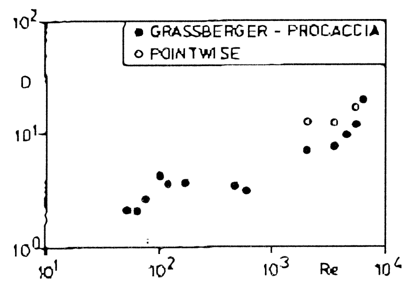


Fig. 19

Variation of the dimension of the attractor with respect to Reynolds numbers. Note that the dimension is about 1 when there is only vortex shedding (Re = 36), about 2 when there are only 2 frequencies (Re = 54), about 3 when there are 3 frequencies, etc..

Now getting back to other Reynolds numbers, it is clear that the first appearance of chaos at Re = 93 is characterized by a jump in the dimension (to about 4.3), whereas a return to "order" at Re = 102 is characterized by a dip in the dimension. We have not computed the dimensions in the Reynolds number range 200 - 500, but calculations in higher Reynolds number range up to about 7000 indicate that it does go up with Re, although not rapidly. In fact, it is about 20 at Re = 7000.

Keeping the above interpretation of the dimension in mind - namely, that it is indicative of the degrees of freedom of the system - it follows that the number of degrees of freedom even at Re = 7000 is of the order of 20. If this is true, it is clear that this information must be used to the best advantage.

It is pertinent to point out that, apart from our own earlier measurements of the dimension of the turbulence attractor, Reference 4 gives such measurements for a TAYLOR-COUETTE flow.

5. DISCUSSION

We have shown that many of the qualitative features of transition to turbulence behind circular cylinders are in essential agreement with the behaviour of dynamical systems. There are some deviations, but it is surprising that the dynamics of fluid motion which we believe to be particularly governed by the Navier-Stokes equations should be represented by extremely simple systems at all. We have shown that during early stages of transition, a strong connection (speculated previously, but never conclusively shown to be true) exists between the dimension of the attractor and the degrees of freedom of the fluid system. Provided that this interpretation is true at higher Reynolds numbers also, our results suggest that the degrees of freedom are not too many, even up to Reynolds number of the order of 10^4. We have several reasons to believe that the dimension of the attractor, as computed according to (4.4) and (4.5), is not very high, even at much higher Reynolds number corresponding to the fully turbulent state (Re = 10^6). Most of our data has been at 5 diameters downstream from the cylinder. At least at low enough Reynolds numbers, much the same happens at $x/d = 50$, for example.

Do we then conclude that the key to understanding transition and turbulence lies completely in the study of dynamical systems? We think that such statements are optimistic at best and misguided at worst, even though there is much that we can learn from dynamical systems. Consider our own experiments. Dynamical systems theory can correctly tell us that chaos occurs after 3 bifurcations, but does not at all tell us what those bifur-

cations are! The origin and physical structure of these bifurcations can be discovered only by looking at the particular form of F in (1.5). For the same reason, it will perhaps never be possible to predict C_D vs Re curve for the circular cylinder without worrying about the special form of F in (1.5). Furthermore, the spatial structure of the wake is an important element completely outside the scope of dynamical systems theory - at least as as it stands today.

What do we make of the fact that the dimensions of the attractor is not too high even at high Re? If the attractor is sufficiently low-dimensional, a clever projection of it can perhaps be used to our advantage. If the attractor dimension is even as high as 20, no matter what projection one devises, it will look uniformly dark. So it is unclear at this stage how this information could be used, except in the hope that it lends credence to concepts embodied in renormalization group theory or slaving principle, etc.. Perhaps it is appropriate to remind ourselves that some excellent fluid dynamicists have for many years toyed with the idea of a relatively small number of degrees of freedom in turbulence, and have not gone very far!

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