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Rapid distortion of axisymmetric turbulence

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A generalization of the isotropic theory of Batchelor & Proudman (1954) is developed to estimate the effect of sudden but arbitrary three-dimensional distortion on homogeneous, initially axisymmetric turbulence. The energy changes due to distortion are expressed in terms of the Fourier coefficients of an expansion in zonal harmonics of the two independent scalar functions that describe the axisymmetric spectral tensor. However, for two special but non-trivial forms of this tensor, which represent possibly the simplest kinds of non-isotropic turbulence and specify the angular distribution but not the wavenumber dependence, the energy ratios have been determined in closed form. The deviation of the ratio from its isotropic value is the product of a factor containing R, the initial value of the ratio of the longitudinal to the transverse energy component, and another factor depending only on the geometry of the distortion. It is found that, in axisymmetric and large two-dimensional contractions, the isotropic theory gives nearly the correct longitudinal energy, but (when $R > 1$) over-estimates the increase in the transverse energy; the product of the two intensities varies little unless the distortion is very large, thus accounting for the stress-freezing observed in rapidly accelerated shear flows.

Comparisons with available experimental data for the spectra and for the energy ratios show reasonable agreement. The different ansatzes predict results in broad qualitative agreement with a simple strategem suggested by Reynolds & Tucker (1975), but the quantitative differences are not always negligible.

1. Introduction

The response of turbulence to a suddenly imposed strain has been a question of long-standing interest, because it appears in a variety of practical problems and seems to be amenable to analyses in which viscous and nonlinear inertial forces play a secondary role. The first investigations date back to Prandtl (1933) and Taylor (1935), both of whom were interested in studying the passage of turbulence through a wind-tunnel contraction. A formal analysis of the problem, recognizing the spatial and temporal randomness of turbulence (which had been ignored by both Prandtl and Taylor) but assuming it to be (initially) isotropic, was made by Ribner & Tucker (1953) for axisymmetric contractions and independently by Batchelor & Proudman (1954) for arbitrary distortions; the latter paper (to be referred to below as BP) gives a definitive and particularly complete account of the solution. The conditions under which the distortion can be considered sufficiently rapid for the nonlinear and viscous effects to be negligible have recently been closely re-examined by Hunt (1973), who finds that they are generally less stringent than BP suggested.

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The results of BP are however limited by the assumption of isotropy, which is rarely (if ever) found in practice. On the other hand, situations involving non-isotropic and even inhomogeneous turbulence in which rapid distortion appears to be the dominant mechanism have multiplied. In many rapidly accelerated shear flows, for example, it has been shown that the Reynolds shear stress freezes and the turbulent structure responds largely as in rapid distortion (Narasimha & Prabhu 1972; Narasimha & Sreenivasan 1973; Ramjee, Badri Narayanan & Narasimha 1972). Hunt (1973), Hunt & Mulhearn (1973) and Bearman (1972) have shown the importance of the rapid-distortion limit in wind flow past buildings and in pollutant dispersion. It therefore appears necessary and useful to analyse the effect of rapid distortion on more general types of turbulence. The present paper considers homogeneous but (initially) axisymmetric turbulence; the effects of inhomogeneity will be treated elsewhere (see Sreenivasan & Narasimha (1974) for a preliminary account). Axisymmetric turbulence is fairly common, particularly in wind-tunnel streams and in the atomsphere. Approximations to axisymmetry are also found (as we shall argue elsewhere) in some limited regions of free shear layers (wakes, jets) and in the outer part of a turbulent boundary layer.

In isotropic turbulence, the spectral tensor is determined completely in terms of a single scalar function of a single scalar variable; e.g. the energy spectrum function $E(k)$ of Batchelor (1953, p. 36). Furthermore (see BP), the ratio of turbulent energy after distortion to that before does not depend on the precise form of $E(k)$. In axisymmetric turbulence, on the other hand, kinematic arguments (e.g. Batchelor 1953, p. 43) show that the spectral tensor could in general involve two independent functions of two scalar variables. (These variables can be taken as k and $k_1/k \equiv \cos\theta$, where k is the magnitude of the wavenumber vector $\mathbf{k}$ and k_1 is the component of $\mathbf{k}$ along the axis of symmetry.) A complete analysis of the axisymmetric problem is impossible without some knowledge of the dependence of these functions on θ. Direct measurements of spectra are invariably one-dimensional, and provide information usually only on the planes k_1 = constant in $\mathbf{k}$ space; transverse correlations similarly describe (after taking an appropriate Fourier transform) the plane k_2 = constant. Experimental data on these quantities are not sufficiently detailed to allow us to infer the two functions referred to above, although partial checks are possible and will later be made.

We proceed here by representing the axisymmetric spectral functions as suitable expansions involving Legendre polynomials in $\cos\theta$. If such an expansion for the initial spectrum function is known, the energy ratios for arbitrary distortion can be completely determined; they can in particular be expressed as sums of the known isotropic results (from BP) and appropriate correction terms for departure from isotropy. However, in those cases where certain functions contained in the spectral tensor may be assumed to depend only on the wavenumber magnitude k, the correction terms no longer depend on the details of the spectral function, but only on the ratio R of the longitudinal energy component to either of the other two in the initial state. Two of the corresponding 'ansatzes', which arise as special truncations of the general expansion, are studied here in the same detail as isotropic turbulence; in particular the results for the energy ratios are expressible in closed form, in elliptic integrals. A subclass of the first of these ansatzes, involving only a single scalar function of k, represents what is possibly the simplest model for non-isotropic turbulence, and is pursued somewhat further. Comparison is made with the available experimental information in order to

examine the validity of the results based on these ansatzes and to assess the importance of the correction terms in common flow situations.

Earlier work on the problem has been incomplete and, in some places, even incorrect. Acharya (1956) and Swamy (1972) both considered what is here called ansatz II (see § 4.1). Acharya wrote down the expressions for the component energies for the special case of an axisymmetric contraction, in terms of integrals involving the appropriate scalar functions, but made no attempt to evaluate them. Swamy made further assumptions (which are here found to be unnecessary) about the explicit forms of the scalar functions and numerically integrated Acharya's expressions. However, an algebraic error in Acharya's work, carried over by Swamy, vitiates the latter's results for lateral energy at small distortions. The work reported here is more general, and penetrates further than these studies.

In a different approach to the problem, Reynolds & Tucker (1975) have proposed a 'simple stratagem' based on the realization that a given undistorted initially axisymmetric turbulence may be considered to have resulted from the application of an appropriate axisymmetric strain to a phyothetical field of initially isotropic turbulence. The effect of the actual distortion on the axisymmetric field is then considered to be the same as that of an equal incremental distortion on the hypothetical isotropic field. The results of this 'hypothetical strain' method are compared with the present results in § 5.4.

Incidentally, a particular case of axisymmetric turbulence, similar in spirit to those studied here but different in detail, has been examined by Herring (1974) in connexion with the approach of axisymmetric turbulence to isotropy. The relation between these various types of axisymmetry is examined in § 4.2.

2. The spectral tensor in axisymmetric turbulence

We call turbulence axisymmetric if the mean value of any product of fluctuating velocity components along given directions at a given set of points is invariant (i) to rigid-body rotations of the configurations of points and directions about a given unit vector $\mathbf{a}$ (along the axis of symmetry, say Ox_1) and (ii) to reflexion of the configuration in any point. Thus there is an axis of symmetry but no preferred direction, and the spectral tensor must be even in the wavenumber component along the axis (see figure 1).

The most general second-order spectral tensor for such axisymmetric turbulence is of the form (see, for example, Batchelor 1953, p. 43)

$$\phi_{ij}(\mathbf{k}) = A_1 k_i k_j + A_2 a_i a_j + A_3 \delta_{ij} + A_4 a_i k_j + A_5 a_j k_i \tag{2.1a}$$

in Cartesian-tensor notation, where the A_r ($r = 1, \ldots, 5$) are functions, not all independent, of k and $\mathbf{k}.\mathbf{a} = k_1$. Continuity in incompressible flow requires ϕ_{ij} to be orthogonal to $\mathbf{k}$, and symmetry in the indices i and j demands that A_4 and A_5 be equal. Equation (2.1a) can therefore be written as

$$\phi_{ij}(\mathbf{k}) = (k_i k_j - k^2\delta_{ij}) A_1 + [k_1^2 \delta_{ij} + k^2 a_i a_j - k_1(a_i k_j + a_j k_i)] A_2/k^2. \tag{2.1b}$$

Introducing

$$G = -A_2/k^2, \quad F = -A_1 - G, \tag{2.2}$$

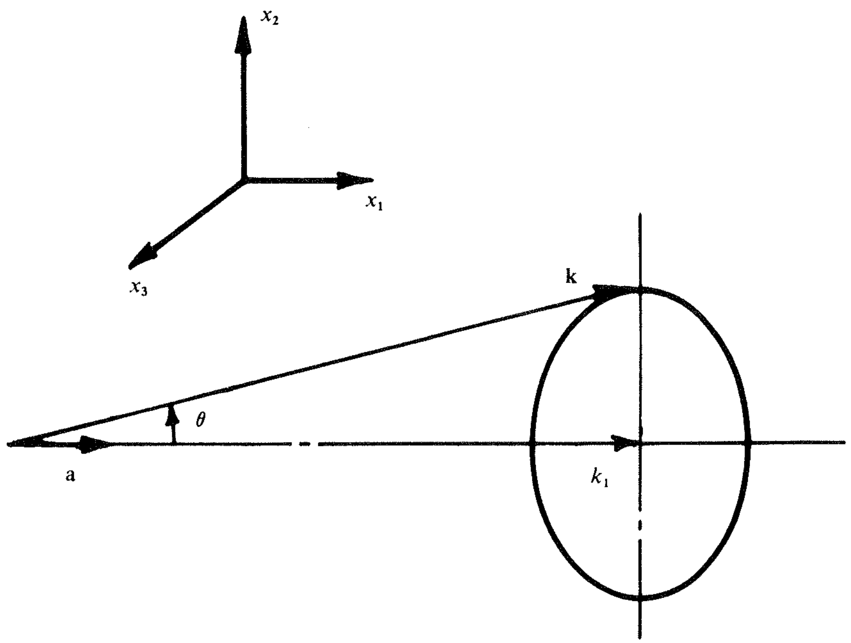

FIGURE 1. Sketch defining notation.

we can write the diagonal components of ϕ_{ij} as

$$\phi_{11}(\mathbf{k}) = (k^2-k_1^2)\,F(k,k_1),$$

$$\phi_{nn}(\mathbf{k}) = (k^2-k_n^2)\,F(k,k_1)+(k^2-k_1^2-k_n^2)\,G(k,k_1), \quad n = 2,3,$$

and

$$\sum_i \phi_{ii}(\mathbf{k}) = 2k^2F+(k^2-k_1^2)\,G. \tag{2.3}$$

Here and in the following, we do *not* follow the summation convention; the indices i and j can take the values 1, 2 and 3 but n takes only the values 2 and 3.

It is convenient to change the arguments of F and G from (k, k_1) to $(k, \theta = \cos^{-1}k_1/k)$, and expand in zonal harmonics with the Legendre polynomials $P_{2m}(\cos\theta)$ as basis:

$$\left.\begin{aligned} F(k,\theta) &= \sum_{m=0}^{\infty} F_{2m}(k)\,P_{2m}(\cos\theta),\\ G(k,\theta) &= \sum_{m=0}^{\infty} G_{2m}(k)\,P_{2m}(\cos\theta). \end{aligned}\right\} \tag{2.4}$$

Because of the absence of a preferred direction as noted earlier, only even-order polynomials appear in the expansions. If (2.3) is integrated over wavenumber space in a spherical polar co-ordinate system, the dependence of the rest of the integrand (i.e. apart from F and G) on the polar angle involves no powers higher than $\sin^3\theta$. It follows, from the orthogonality of the Legendre polynomials, that only the first two terms in the expansion (2.4) can contribute to the energy; thus we get for the mean-square velocity components

$$\langle u_1^2\rangle = \tfrac{8}{15}\pi(5\bar{F}_0-\bar{F}_2),$$

$$\langle u_n^2\rangle = \tfrac{4}{15}\pi(10\bar{F}_0+\bar{F}_2+5\bar{G}_0-\bar{G}_2), \tag{2.5}$$

where

$$\bar{F}_{2m} = \int_0^\infty F_{2m}(k)\,k^4dk, \quad \bar{G}_{2m} = \int_0^\infty G_{2m}(k)\,k^4dk. \tag{2.6}$$

In isotropic turbulence $F = F_0$ and $G = 0$.

3. The general solution

3.1. *Post-distortion spectral tensor*

By following the procedure used by BP, it is easy to show that in the present problem the diagonal components of the post-distortion spectral tensor (indicated by a prime) are given by

$$\phi'_{11}(\boldsymbol{\chi}) = \frac{Fk^2}{e_1^2\chi^4}\left[k_1^2\left\{\frac{k_2^2}{e_2^4}+\frac{k_3^2}{e_3^4}\right\}+(\chi^2-\chi_1^2)^2\right]+\frac{Gk_1^2k_2^2k_3^2}{\chi^4}\left(\frac{e_2^2}{e_3^2}+\frac{e_3^2}{e_2^2}-2\right), \tag{3.1}$$

$$\begin{aligned}\phi'_{nn}(\boldsymbol{\chi}) = {} & \frac{Fk^2}{e_n^2\chi^4}\left[k_n^2\left\{\frac{k_1^2}{e_1^4}+e_1^4e_n^4(k^2-k_1^2-k_n^2)\right\}+(\chi^2-\chi_n^2)^2\right]\\ & +\frac{G(k^2-k_1^2-k_n^2)}{\chi^4}\left[e_1^4e_n^2(k^2-k_1^2)^2+k_1^2\left\{\frac{k_1^2}{e_1^4e_n^2}+2(k^2-k_1^2)\right\}\right],\end{aligned} \tag{3.2}$$

where $\boldsymbol{\chi}$ is the post-distortion wavenumber vector, with components $\chi_i = k_i/e_i$, and e_i is the extension ratio, obeying the mass conservation law $e_1e_2e_3 = 1$ in incompressible flow. In flow through a duct with mean velocity U_j along the axis x_j, the distortion experienced by a fluid particle in a time t is

$$e_j = \exp\int_0^t \frac{\partial U_j}{\partial x_j}\,dt \tag{3.3}$$

(no sum over j!).

Turbulence which is initially axisymmetric will remain so when subjected to axisymmetric distortion, and hence only the first two terms in the harmonic expansion of ϕ'_{ii} will contribute to post-distortion energy. However each of these terms is itself in general a linear combination of all the initial coefficients F_{2m} and G_{2m}, so that a finite number of them is not sufficient to determine the energy ratio.

3.2. *The energy ratios*

The ratio of each component energy after distortion to its value before is

$$\mu_i = \int\phi'_{ii}(\boldsymbol{\chi})\,D\boldsymbol{\chi}\Big/\int\phi_{ii}(\mathbf{k})\,D\mathbf{k}.$$

Integrating (3.1), and noting that in isotropic turbulence $F = F_0$ and $G = 0$, we can separate the isotropic part μ_1^0 of μ_i and write

$$\mu_i = \mu_1^0-\frac{R-1}{R}\Delta\mu_1, \quad \mu_2 = \mu_2^0-(R-1)\,\Delta\mu_2, \tag{3.4}$$

where

$$\mu_1^0 = 3J_0/4e_1^2, \quad \mu_2^0 = 3J_0^*/4e_2^2, \tag{3.5}$$

$$-\frac{(R-1)}{R}\Delta\mu_1 = \left(1-\frac{1}{5}\frac{\bar{F}_2}{\bar{F}_0}\right)^{-1}\left[\tfrac{1}{5}\mu_1^0\frac{\bar{F}_2}{\bar{F}_0}+\frac{3}{4e_1^2}\sum_{m=1}^{\infty}J_{2m}\frac{\bar{F}_{2m}}{\bar{F}_0}+\frac{3}{4}\left(\frac{e_2^2}{e_3^2}+\frac{e_3^2}{e_2^2}-2\right)\sum_{m=0}^{\infty}j_{2m}\frac{\bar{G}_{2m}}{\bar{F}_0}\right], \tag{3.6}$$

$$-(R-1)\,\Delta\mu_2 = (R-1)\,\mu_2^0+\left(1-\frac{1}{5}\frac{\bar{F}_2}{\bar{F}_0}\right)^{-1}\left[\tfrac{1}{5}R\mu_2^0\frac{\bar{F}_2}{\bar{F}_0}+\frac{3R}{4e_2^2}\sum_{m=1}^{\infty}J_{2m}^*\frac{\bar{F}_{2m}}{\bar{F}_0}+\tfrac{3}{4}Re_1\sum_{m=1}^{\infty}j_{2m}^*\frac{\bar{G}_{2m}}{\bar{F}_0}\right]. \tag{3.7}$$

The quantities J and j are functions of (e_1, e_2, e_3) defined by the following integrals:

$$J_{2m}(e_1, e_2, e_3) = \int_0^1 [2(B_2 B_3)^{\frac{3}{2}} - B_2 C_3 - B_3 C_2]\,(B_2 B_3)^{-\frac{3}{2}} P_{2m}(\xi)\,d\xi,$$

$$e_2^{-2} J^*_{2m}(e_1, e_2, e_3) \equiv e_1^{-2} J_{2m}(e_3, e_1, e_2),$$

$$j_{2m} = \int_0^1 (B_2 B_3)^{-\frac{1}{2}}\,(B_2^{\frac{1}{2}} + B_3^{\frac{1}{2}})^{-2}\,\xi^2 (1-\xi^2)^2 P_{2m}(\xi)\,d\xi,$$

$$j^*_{2m} = \int_0^1 [e_2^2 e_1^2 B_2 + (e_3^2 - e_2^2)\,\xi^2]\,(e_1^2 e_2^2 B_2)^{-1}(1-\xi^2)\,P_{2m}(\xi)\,d\xi,$$

where $$B_n = \xi^2/e_1^2 + (1-\xi^2)/e_n^2, \quad C_n = (\xi^2/e_1^2 e_n^4)\,(2e_n^2 - e_1^2 + e_n^2\,\xi^2).$$

It is possible to evaluate the J_{2m}, J^*_{2m}, j_{2m} and j^*_{2m} in terms of elliptic integrals (Sreenivasan 1973, 1974), but the resulting expressions are not very useful, particularly for computation, and so will not be quoted here. We note however that each integral has the form

$$\int_0^1 Q(\xi)\,P_{2m}(\xi)\,d\xi,$$

where Q consists of terms even in ξ. As m becomes large $P_{2m}(\xi)$ oscillates rapidly, so we may expect that the integrals for large m are small in comparison with J_0. In fact, the j_{2m} are generally very small, so that the contribution of G_{2m} to the longitudinal energy ratio μ_1 as given by (3.6) will be correspondingly small.

4. The simpler ansatzes

Useful results about the behaviour of axisymmetric turbulence can be obtained only if some reasonable hypotheses about the coefficients F_{2m} and G_{2m} in (2.4) can be made. One way would be to truncate the series (2.4) at some suitable value of m. A more fruitful approach is to postulate that some pair of the five functions A_r depend only on k and not on k_1 (recall that any three of them can be eliminated by symmetry and continuity). However, the requirement that ϕ_{ij} should be even in k_1 rules out such a postulate for $A_4 = A_5$. Of the remaining three functions we can choose two in three possible ways: each combination defines a possible simple ansatz for the spectral tensor. We examine these in turn.

4.1. *Ansatzes* I *and* IA

Ansatz I is defined by taking A_1 and A_2 to depend only on k; from (2.2) it follows (a superscript I indicating the ansatz) that

$$F^{\mathrm{I}}_{2m} = 0 = G^{\mathrm{I}}_{2m} \quad \text{for} \quad m = 1, 2, \ldots, \tag{4.1}$$

and from (2.3) and (2.4) that

$$\left.\begin{aligned} \phi^{\mathrm{I}}_{11}(\mathbf{k}) &= (k^2 - k_1^2)\,F_0(k), \\ \phi^{\mathrm{I}}_{nn}(\mathbf{k}) &= (k^2 - k_n^2)\,F_0(k) + (k^2 - k_1^2 - k_n^2)\,G_0(k). \end{aligned}\right\} \tag{4.2}$$

Thus the angular dependence of ϕ_{11} is like $\sin^2\theta$, exactly as in isotropic turbulence.

If the one-dimensional spectra

$$\phi_i(k_1) = \iint_{-\infty}^{+\infty} \phi_{ii}(k, k_1)\,dk_2\,dk_3 = 2\pi \int_{k_1}^{\infty} \phi_{ii}(k, k_1)\,dk \tag{4.3}$$

were known, say from measurement, it would in principle be possible to infer F_0 and G_0; for, by using (4.2) in (4.3) it can be shown that

$$\left.\begin{aligned} F_0(k) &= \frac{1}{4\pi k}\frac{d}{dk}\left(\frac{1}{k}\frac{d\phi_1(k)}{dk}\right), \\ G_0(k) &= \frac{1}{2\pi k}\frac{d}{dk}\left(\frac{1}{k}\frac{d\phi_n(k)}{dk} + \frac{1}{2}\frac{d^3\phi_1(k)}{dk^3}\right). \end{aligned}\right\} \tag{4.4}$$

Unfortunately use of these formulae calls for the very difficult third-order differentiation of numerical data.

From (2.6) it follows that

$$\bar{G}^{\mathrm{I}}_0/\bar{F}^{\mathrm{I}}_0 = -2(R-1)/R. \tag{4.5}$$

This suggests consideration of an even simpler ansatz, which we shall call IA, in which

$$G^{\mathrm{I}A}_0(k) = (\bar{G}^{\mathrm{I}}_0/\bar{F}^{\mathrm{I}}_0)\,F^{\mathrm{I}A}_0(k) = -2(R-1)\,F^{\mathrm{I}A}_0(k)/R; \tag{4.6}$$

the spectral tensor in this case is completely determined by the function $F_0(k)$ and the parameter R.

Ansatz IA is perhaps the simplest possible form of non-isotropic turbulence. It has many interesting properties: the spectral tensor in the final period of decay of axisymmetric turbulence (studied by Chandrasekhar 1950) conforms to IA (see appendix). The one-dimensional spectra ϕ_n and ϕ_1 are related to each other by the equation

$$2\phi^{\mathrm{I}A}_n(k_1) = (2-R)\,\phi^{\mathrm{I}A}_1 - Rk_1\,\partial\phi^{\mathrm{I}A}_1/\partial k_1, \tag{4.7}$$

which can be derived using (4.3) and (4.6). If we put $R = 1$ we recover the well-known relation for isotropic turbulence (Batchelor 1953, p. 50).

There is some experimental evidence to suggest that ansatz IA is quite realistic: among the many sets of data we have analysed, we shall cite three here to demonstrate the kind of success achieved. Turbulence at the centre-line of a pipe should be axisymmetric (although not homogeneous); from measured $\phi_1(k_1)$ one can therefore predict $\phi_n(k_1)$ using (4.7)† and compare it with experiment. For the data of Lawn (1971), whose values for R range from 1·6 to 2·0, the computed $\phi^{\mathrm{I}A}_2$ exhibits the kind of low wavenumber bump that is observed; but while it is certainly much closer to the measured spectrum than that based on isotropic theory (figure 2), there is some quantitative discrepancy at low wavenumbers. At higher wavenumbers, where turbulence is more nearly isotropic, there is no great difference between isotropic and axisymmetric theories.

The measurements of Laufer (1954) show excellent agreement with the predicted $\phi^{\mathrm{I}A}_2$ *if* R is taken as 1·6; but Laufer's own value of R is about 1·2, which is unusually low compared with the results of recent measurements like those of Lawn cited above and those of Perry & Abell (1975), who find $R \simeq 1{\cdot}6$.

The axisymmetric theory appears to work well in some applications where it would at first seem less justifiable. For example, the predicted spectrum of u_2 shows excellent agreement with the measurements of Comte-Bellot (1963) at the centre-line of a

† The differentiation involved in computing ϕ_n was carried out here by fitting a four-point Lagrangian formula to a smooth curve through the experimental points for the u_1 spectra.

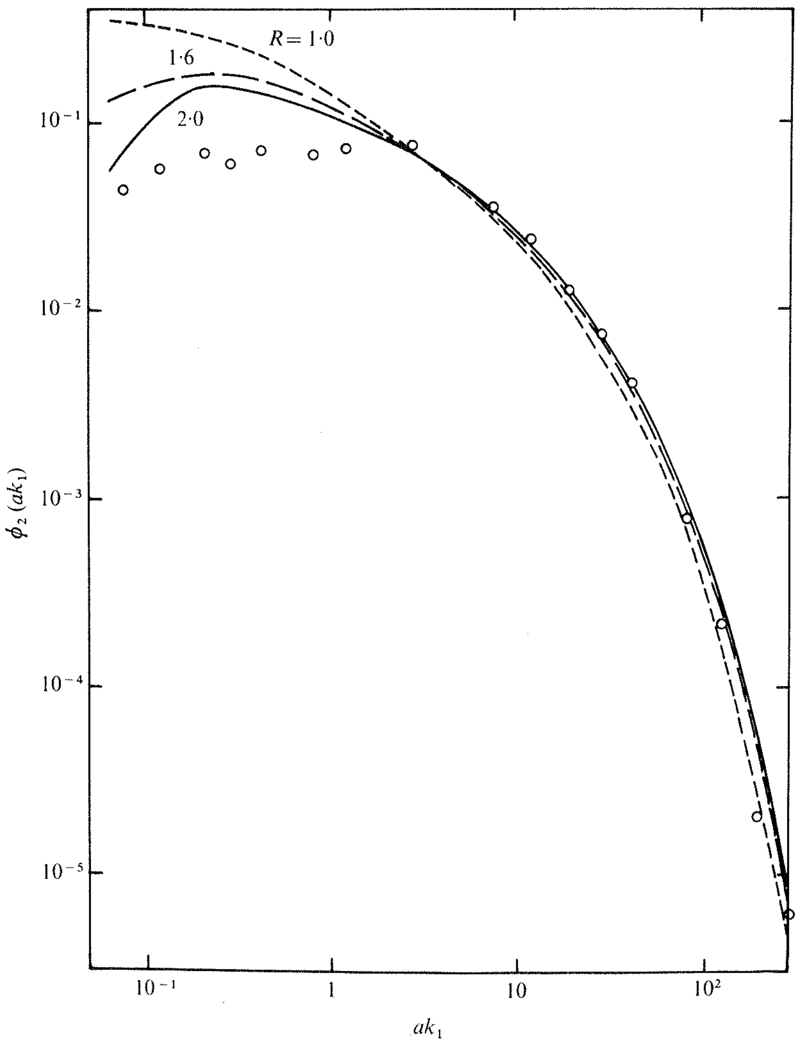

FIGURE 2. Power spectral density of radial velocity component at centre-line of pipe of radius a. Experimental data from Lawn (1971) at a pipe Reynolds number of 9×10^4. Theory: ---, isotropic; — —, ansatz IA, R = 1·6; ——, ansatz IA, R = 2·0.

channel and those of Klebanoff (1954) in a constant-pressure boundary layer (see figure 3). In these calculations we have taken for R the average value

$$R = \tfrac{1}{2}(\langle u_1^2\rangle/\langle u_2^2\rangle + \langle u_1^2\rangle/\langle u_3^2\rangle),$$

where u_1 denotes the streamwise component, u_2 is normal to the surface and u_3 is transverse. Similarly good agreement is also found with the data of Champagne, Harris & Corrsin (1970) for nearly homogeneous shear flow.

4.2. *Ansatz* II

We put $A_2 = A_2(k \text{ only})$ and $A_3 = A_3(k \text{ only})$. Examination of the requirement of continuity on (2.1) and comparison with (2.4) show that this implies

$$G_0 = \tfrac{3}{2}F_2, \quad F_{2m+2} = 0 = G_{2m}, \quad m = 1, 2, \ldots .$$

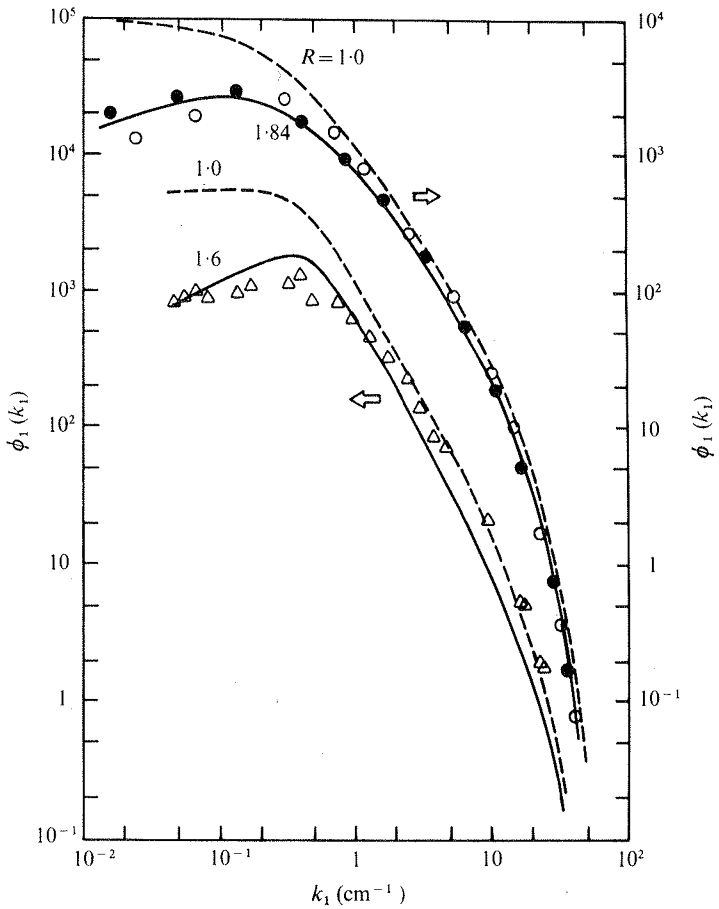

FIGURE 3. u_2 spectrum calculated from measured u_1 spectrum, according to ansatz IA (solid curve) and isotropic relation (dashed curve). Measurements on the centre-line of channel flow (average $R \doteqdot 1{\cdot}84$), Comte-Bellot (1963): ○, u_2 spectrum; ●, u_3 spectrum. Measurements at 0·8 boundary-layer thickness from wall in constant-pressure turbulent boundary layer ($R \doteqdot 1{\cdot}6$), Klebanoff (1954): △, u_2 spectrum.

Thus ansatz II contains one more harmonic than ansatz I, and is a special case of the two-term expansion of (2.4). For this ansatz

$$\phi_{11}^{\mathrm{II}}(\mathbf{k}) = (k^2-k_1^2)^2A_2 + (k-k_1^2)\,A_3, \tag{4.8a}$$

$$\phi_{nn}^{\mathrm{II}}(\mathbf{k}) = k^2k_n^2A_2 + (k^2-k_n^2)\,A_3, \tag{4.8b}$$

with $\quad A_2 = 3F_2/2k_1^2, \quad A_3 = (F_0 - \tfrac{1}{2}F_2) + (3k_1^2/2k^2)\,F_2.$

Corresponding to (4.4) we now have the more complicated inversion relations

$$A_3 + \tfrac{3}{2}k\frac{dA_3}{dk} = -\frac{k}{\pi}\frac{d}{dk}\frac{1}{k}\left[\frac{d}{dk}\left(\phi_n + \frac{k}{8}\frac{d\phi_1(k)}{dk}\right)\right],$$

$$2A_1 + 4A_2 - k\frac{dA_3}{dk} = -\frac{k^3}{4\pi}\frac{d}{dk}\left[\frac{1}{k}\frac{d}{dk}\left(\frac{1}{k}\frac{d\phi_1}{dk}\right)\right]$$

and

$$\frac{\overline{G}_0}{\overline{F}_0} = \frac{3}{2}\frac{\overline{F}_2}{\overline{F}_0} = -\frac{30(R-1)}{17R+4}.$$

4.3. *Ansatz* III

Here $A_1 = A_1(k \text{ only})$ and $A_3 = A_3(k \text{ only})$. This implies

$$\phi_{11}^{\mathrm{III}}(\mathbf{k}) = \frac{A_1}{k_1^2}(k^2-k_1^2)^2 + \frac{A_3}{k_1^2}(k^2-k_1^2), \tag{4.9a}$$

$$\phi_{nn}^{\mathrm{III}}(\mathbf{k}) = A_1 k_n^2 + A_3. \tag{4.9b}$$

If $A_1 k^2 = -A_3$, these equations reduce exactly to isotropic forms. If $A_1 k^2 \neq -A_3$, $\phi_{11}^{\mathrm{III}}(\mathbf{k})$ has an algebraic singularity at $k_1 = 0$, which is physically undesirable as it implies that the longitudinal correlation does not die away at infinity. One can of course avoid the singularity by choosing $A_1 k_1^{-2}$ and $A_2 k_1^{-2}$ as functions only of k, but then it turns out that isotropic results are not recoverable merely by putting $R = 1$, as an additional dependence of A_1 and A_2 on k_1^2 then still remains in the spectral tensor.

For these reasons, we pay greater attention to ansatzes I and II, but some incidental remarks will be made on ansatz III in connexion with the energy ratios in the normal direction.

We may finally note that the present ansatzes are quite different from the proposals made recently by Herring (1974) in a somewhat different context. Herring used an expansion in spherical harmonics for functions ϕ^1 and ϕ^2 related to the functions F and G of this paper as follows:

$$\phi^1 = k^2F + (k^2-k_1^2)\,G, \quad \phi^2 = k^2F. \tag{4.10}$$

He then considered the special case in which ϕ^1 and ϕ^2 are independent of k_1. Equations (4.10) show that this cannot be our ansatz I; from the easily derived relations

$$A_2 = (\phi^2-\phi^1)\,k^2/(k^2-k_1^2), \quad A_3 = \phi^1,$$

obtained from the relations among the A_r (implied by continuity), it follows that Herring's proposal is not our ansatz II either.

5. Results for the energy ratios

5.1. *Ansatz* I

From (3.6) and (4.1),

$$\Delta\mu_1^{\mathrm{I}} = \frac{3}{2}\left(\frac{e_2^2}{e_3^2}+\frac{e_3^2}{e_2^2}-2\right)j_0, \quad \mu_2^{\mathrm{I}} = \tfrac{3}{2}e_1 j_1^* - \mu_2^0, \tag{5.1}$$

where $\Delta\mu_1^{\mathrm{I}}$ and $\Delta\mu_2^{\mathrm{I}}$ depend only on the geometry of the distortion and are, in particular, independent of R. It is therefore sufficient to present information on $\Delta\mu_i^{\mathrm{I}}$ and μ_1^0 to enable calculation of energy ratios according to this ansatz. For completeness, we first give, in figure 4, μ_1^0 *vs.* e_1 and e_2. Figures 5 and 6 give $\Delta\mu_1^{\mathrm{I}}$ and $\Delta\mu_2^{\mathrm{I}}$, respectively, plotted against e_1 and e_2 in a convenient manner. (By symmetry, figure 6 also represents the variation of $\Delta\mu_3^{\mathrm{I}}$ with e_1 and e_3.)

From figure 5 it is clear that the correction term for μ_1 due to departure from isotropy is negligible for distortions that are two-dimensional or nearly so (i.e. e_2 or $e_3 \doteq 1$). Similarly, figure 6 shows that for a constant-area distortion of the kind studied by Townsend (1954), where $e_1 = 1$, the correction term is nearly independent of e_2 when $e_2 > 1$. A more general result is that for $e_1 \geqslant 1$ (in practice, contractions or constant-

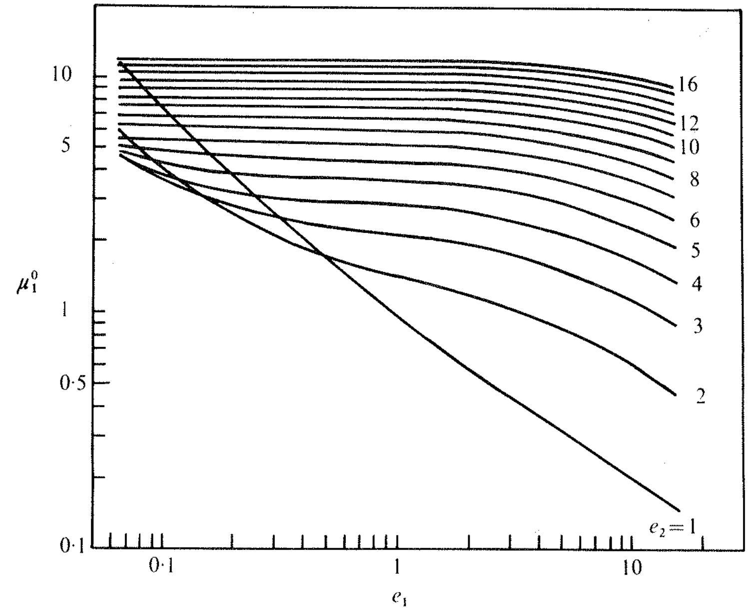

FIGURE 4. Longitudinal energy ratio calculated from the Batchelor–Proudman theory for initially isotropic turbulence, as a function of the extension ratios. Other component energy ratios can be obtained by symmetry.

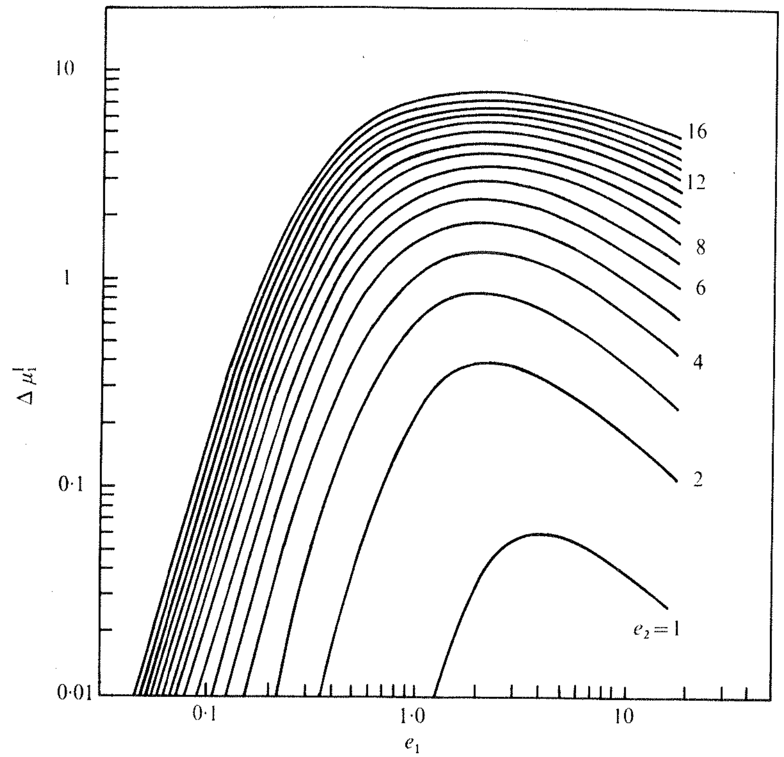

FIGURE 5. Correction term for the longitudinal energy ratio according to anzatz I.

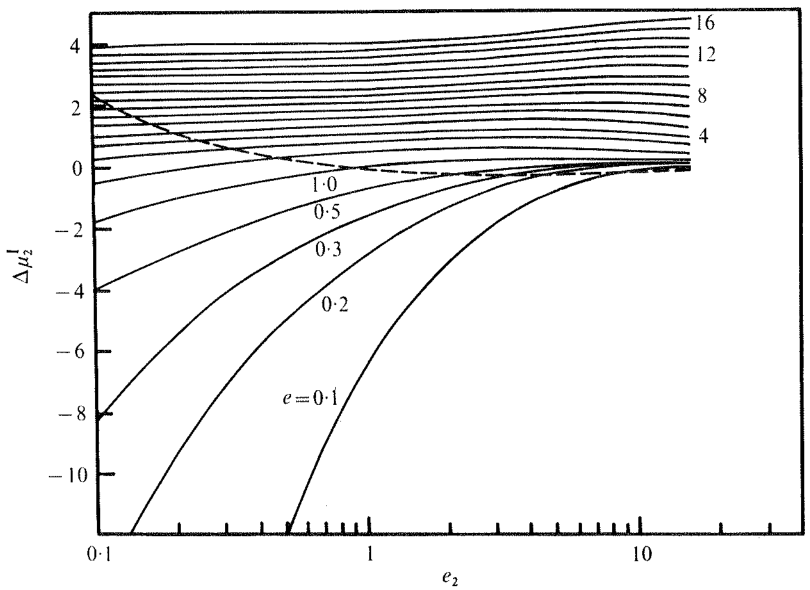

FIGURE 6. Correction term for the transverse energy ratio according to ansatz I. ---, two-dimensional distortion ($e_1e_2 = 1 = e_3$).

area distortions) $\Delta\mu_2^{\rm I}$ is approximately equal to $\frac{1}{4}e_1$ and is almost independent of e_2; for $e_1 < 1$ (diffusers) $\mu_2^{\rm I}$ becomes increasingly sensitive to squashing with increasing e_1: thus $\mu_2^{\rm I}$ differs considerably from μ_2^0 for large e_3. Similar considerations hold for $\mu_3^{\rm I}$ and $\Delta\mu_3^{\rm I}$. Further, for a two-dimensional distortion ($e_3 = 1$), figure 6 shows that the correction term is negligible, particularly if $e_2 > 1$.

For an axisymmetric strain field $e_2 = e_3 = e_1^{-\frac{1}{2}}$, we see from (5.1) that $\Delta\mu_1^{\rm I} = 0$; we have in fact

$$\mu_1^{\rm I} = \mu_1^0, \quad \mu_n^{\rm I} = \mu_n^0-(R-1)(e_1-\mu_n^0). \tag{5.2}$$

For large e_1, these reduce to

$$\mu_1^{\rm I} = \mu_1^0 = (3/4e_1^3)(\ln 4e_1^3-1), \quad \mu_n^{\rm I} = e_1(1-\tfrac{1}{4}R). \tag{5.3}$$

For large two-dimensional contractions with $e_2 = 1$ and $e_1 = e_3^{-1} \gg 1$,

$$\begin{aligned}\mu_1^{\rm I} &= \mu_1^0-\frac{R-1}{R}(e_1-e_1^{-2})\left[\frac{1}{2e_1(e_1-1)^2}+O(e_1^{-5})\right]\\ &= \mu_1^0+O(e_1^{-2})\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}\mu_n^{\rm I} &= R\mu_n^0-e_1(R-1)\left[1-\tfrac{1}{4}(-1)^n(e_1^2-1)\,e_1^{-4}+O(e_1^{-6})\right]\\ &= e_1(1-\tfrac{1}{4}R)+O(e_1^{-1}).\end{aligned} \tag{5.5}$$

These results indicate that for large contractions, as far as the correction terms are concerned,

(i) there is no difference between axisymmetric and plane distortion;

(ii) there is no difference between axisymmetric and isotropic turbulence as far as the longitudinal energy component is concerned;

(iii) if the initial transverse component energy is less than the longitudinal, its increase during distortion is over-estimated by isotropic theory.

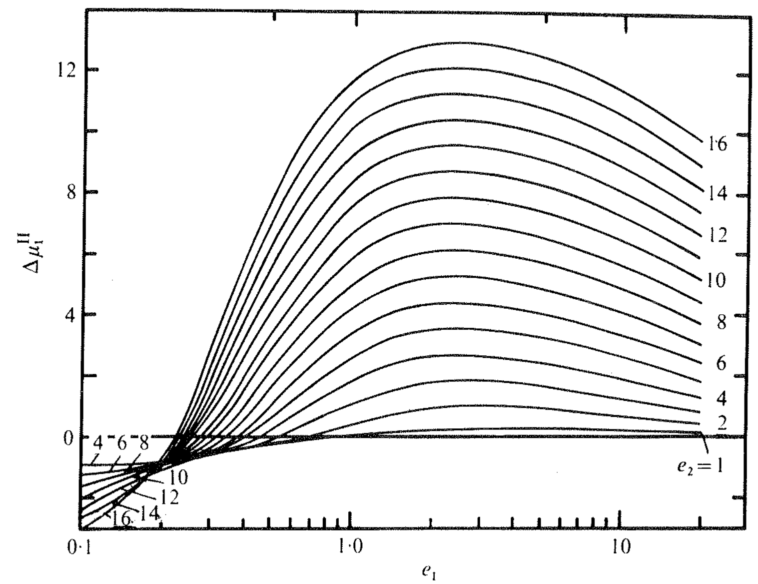

FIGURE 7. Correction term for the longitudinal energy ratio according to ansatz II.

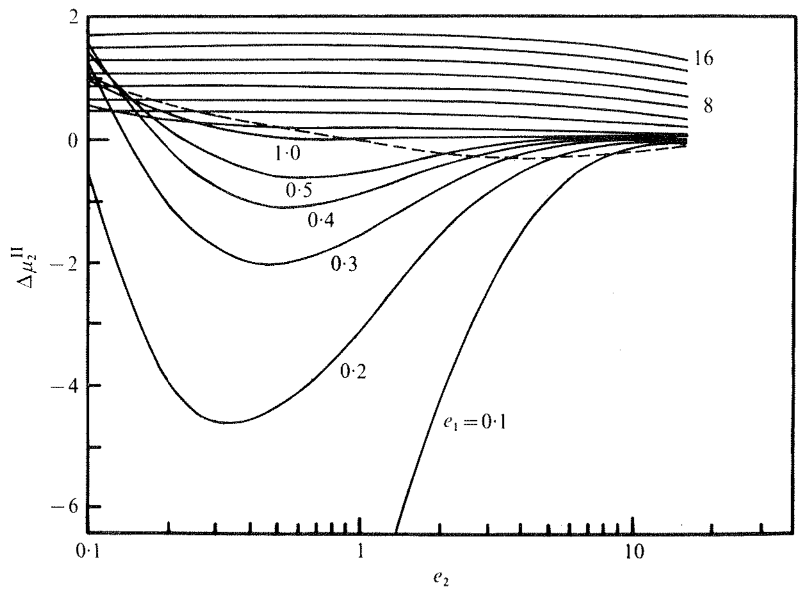

FIGURE 8. Correction term for the transverse energy ratio according to ansatz II. ---, two-dimensional distortion ($e_1e_2 = 1 = e_3$)

5.2. *Ansatz* II

The results here are

$$\Delta\mu_1^{\rm II} = \frac{8\mu_1^0}{7}+\frac{5}{7}\frac{J_2J_0}{e_1^2}+\frac{15}{14}\left(\frac{e_2^2}{e_3^2}+\frac{e_3^2}{e_2^2}-2\right)j_0, \tag{5.6a}$$

$$\Delta\mu_2^{\rm II} = \frac{1}{7}\mu_2^0+\frac{5}{7}\frac{J_2^*-J_0^*}{e_2^2}+\frac{15}{14}e_1j_0^*; \tag{5.6b}$$

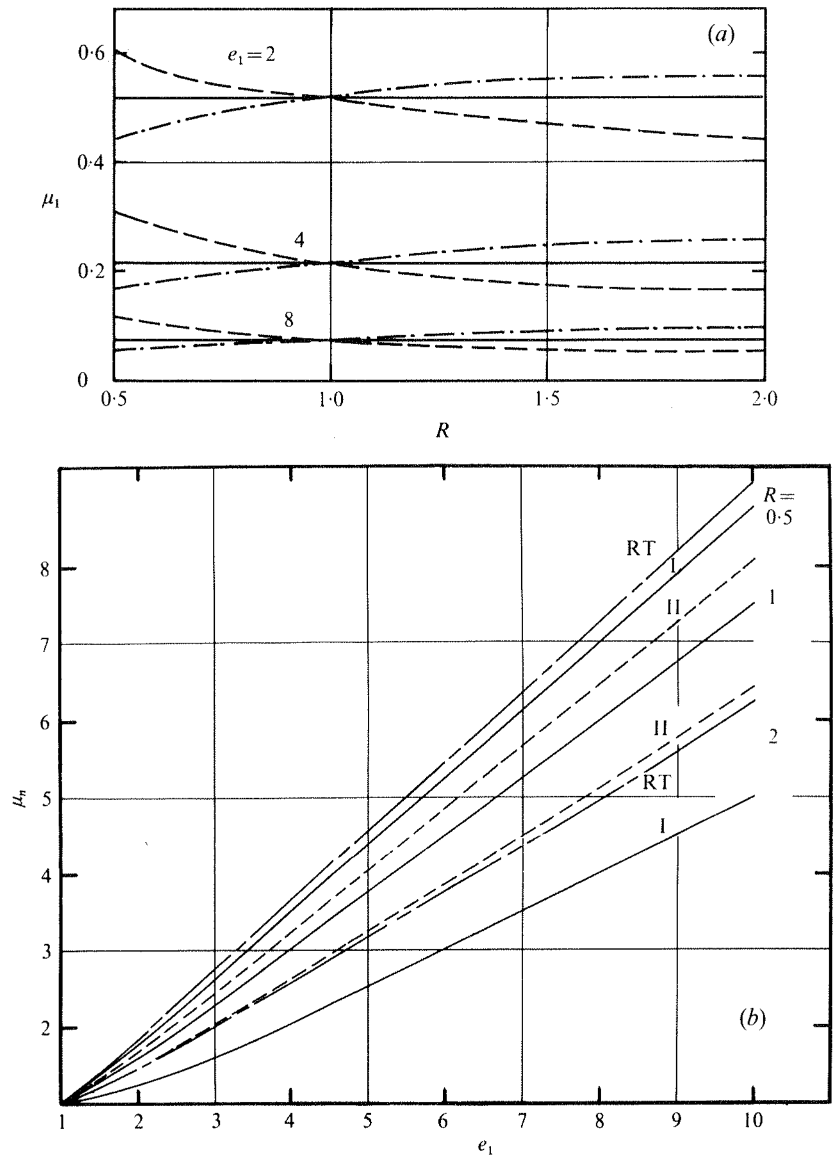

FIGURE 9. Comparison of (a) predicted longitudinal energy ratios and (b) predicted lateral energy ratios in axisymmetric distortion. ——, ansatz I; ---, ansatz II; —·— Reynolds & Tucker's simple stratagem.

a similar result holds for $\Delta\mu_3^{II}$. Expressions (5.6) are plotted in figures 7 and 8. Comparing these with figures 5 and 6 we see that the two ansatzes give qualitatively similar results for either the longitudinal or the transverse energy component when the turbulence is being stretched in the corresponding direction. In the limiting case of a large contraction, we get by straightforward algebra

$$\mu_1^{II} = \frac{3}{4e_1^2}\left(1-\frac{8}{7}\frac{R-1}{R}\right)\ln 4e_1^3-\frac{3}{4e_1^2}\left(1-4\frac{R-1}{R}\right),$$
$$\mu_n^{II} = \mu_n^0[1-\tfrac{1}{7}(R-1)] = \tfrac{3}{28}e_1(8-R).$$

If $R > 1$, these results show that the reduction in longitudinal energy is greater, and the increase in transverse energy less, than in isotropic turbulence. Once again, there is no difference between large axisymmetric and large plane contractions.

5.3. *Ansatz* III

As noted in §4.3, this ansatz has a singularity at $k_1 = 0$ except in a special isotropic case. It may be shown (Sreenivasan 1974) that as a consequence the energy ratio exhibits singular behaviour at $R = 1$. Thus, while the correct isotropic value is recovered at $R = 1$, the longitudinal energy ratio μ_1 is e_1^{-2}, independently of R, for all $R \neq 1$. Although this simple result for μ_1 is reminiscent of that obtained by Prandtl, the singularity at $R = 1$ does not inspire confidence and hence this ansatz will not be considered further here.

5.4. *Comparison with the Reynolds–Tucker method*

The basis of this method has already been touched upon in §1. Reasonable results may be expected if, as Reynolds & Tucker have speculated, the component energy ratios are not sensitive to the details of the initial spectra. The results of §§5.1 and 5.2 show that this cannot in fact be true. Although qualitatively similar, the results of ansatzes I and II show a dependence on the initial spectral form.

Figure 9(*a*) shows a comparison of the longitudinal energy ratios μ_1 calculated for $e_1 = 2$, 3 and 8. This illustrates that while departures from isotropic results are small, according to any of these methods, in the vicinity of $R = 1$ ansatz II and the Reynolds–Tucker method indeed show opposing trends! Ansatz I lies almost midway between the two. For the lateral energy ratios μ_n (figure 9*b*), the two present ansatzes and the Reynolds–Tucker method depart from isotropic results in the same direction. Here again, the Reynolds–Tucker method is closer to ansatz I than to ansatz II. This suggests a somewhat systematic dependence of the component energy ratio on the form of the initial spectra.

5.5. *Changes in velocity products*

It has been observed (e.g. Launder 1964) that, in the outer region of a highly accelerated turbulent boundary layer, the Reynolds shear stress is nearly frozen along a given streamline. Narasimha & Sreenivasan (1973) suggested that the major mechanism responsible for the phenomenon could be the rapid distortion of turbulence caused by sudden mean flow acceleration. A preliminary account given by Sreenivasan & Narasimha (1974) supports this conclusion. Narasimha & Prabhu (1972), who observed similar features in a distorted wake, also attributed this to the same phenomenon. It is therefore of some interest to examine the changes in the product $\langle u_1^2\rangle^{\frac{1}{2}}\langle u_2^2\rangle^{\frac{1}{2}}$, which is proportional, through a correlation coefficient, to the Reynolds shear stress. It is of particular significance because, in a sufficiently accelerated shear flow, the correlation coefficient itself does not vary greatly along a given streamline not too close to the wall. For example, data of Blackwelder & Kovasznay (1972) show that, on a streamline roughly half-way across the boundary layer, the correlation coefficient increases by no more than about 30% over a distance of about 25 boundary-layer thicknesses. Similar corroborating results were obtained by Ramjee *et al.* (1972) in an accelerated two-dimensional channel flow.

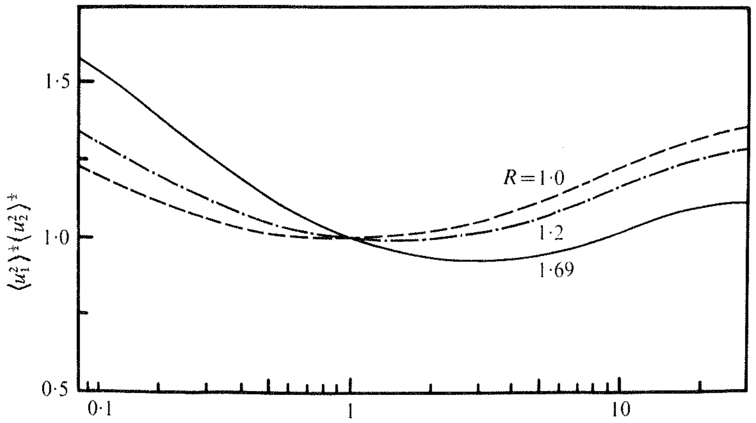

FIGURE 10. Variation with extension ratio e_1 of the product $\langle u_1^2\rangle^{\frac{1}{2}}\langle u_2^2\rangle^{\frac{1}{2}}$ in homogeneous turbulence subjected to a two-dimensional distortion at different values of the initial anisotropy factor R.

Figure 10 shows the results for a two-dimensional distortion using $R = 1{\cdot}0$, $1{\cdot}2$ and $1{\cdot}69$ in ansatz I. It is seen that over the range $0{\cdot}5 \lesssim e_1 \lesssim 5$ the velocity product $\langle u_1^2\rangle^{\frac{1}{2}}\langle u_2^2\rangle^{\frac{1}{2}}$ changes little, lending support to the suggested explanation of stress-freezing. Note that, in applying the result to a shear layer with the mainstream $U_1(x)$ along the x axis, we get from (3.3)

$$e_1 = \exp\int\frac{\partial U_1}{\partial t}\,dt = \exp\int_{x_0}^{x}\frac{\partial U_1}{\partial x}\frac{dx}{U_1} = \frac{U_1(x)}{U_1(x_0)}.$$

6. Comparison with experiment

The main conclusion from the above calculations is that, if departure from isotropy is not large, the component energy ratios can be expressed as the sum of their value given by isotropic theory and certain corrections. In particular, in a large contraction the longitudinal energy $\langle u_1^2\rangle$ decreases appreciably downstream whereas the transverse component $\langle u_2^2\rangle$ increases, although (in the case where $\langle u_2^2\rangle < \langle u_1^2\rangle$ initially) this increase is less rapid than if the turbulence had been isotropic.

There is some experimental evidence to support these conclusions. In their experiments on axisymmetric contraction of initially axisymmetric turbulence ($R \simeq 3$), Klein & Ramjee (1973) observed first a sharp drop in the longitudinal energy followed by a considerable increase; they attributed this behaviour to the strong initial departure from isotropy. This conclusion is however not supported by the present results, which show no strong change in qualitative behaviour even for $R = 3$; also, similar non-monotonic behaviour of $\langle u_1^2\rangle$ had been observed earlier by Uberoi (1956) at lower values of R and was attributed by him to the intercomponent exchange of energy promoted by pressure fluctuations – an explanation that seems more plausible. In many of Klein & Ramjee's experiments, $\langle u_1^2\rangle$ drops to half its initial value in a very short distance while $\langle u_2^2\rangle$ has yet hardly changed: this is likely to bring intercomponent energy transfer rapidly into play. The measured transverse energy ratios are compared in figure 11 with the predictions of ansatzes I and II. For both the Thwaites and the

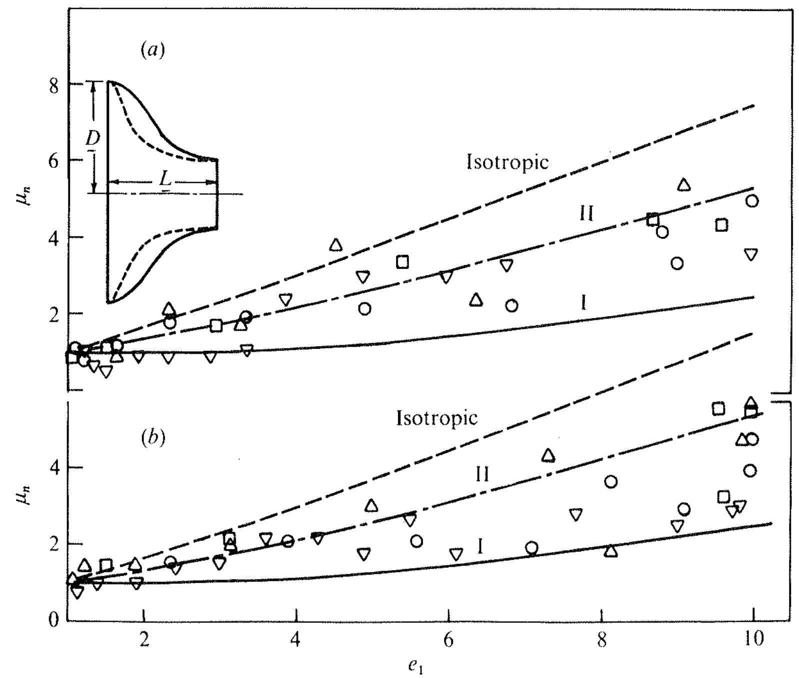

FIGURE 11. Comparison of measured energy ratios in strongly non-isotropic axisymmetric turbulence ($R = 3$) with theory. Experimental data from Klein & Ramjee (1973) for contracting ducts with (*a*) a Thwaites contour and (*b*) a Batchelor–Shaw contour. □, $L/D = 1$; ▽, 1·5; △, 2·0; ○, 3·0. Theory: ---, isotropic; ——, ansatz I; -·-·-, ansatz II.

Batchelor–Shaw contours of contraction used by Klein & Ramjee, the experimental results lie between the predictions of the two ansatzes, but seem somewhat closer to those of ansatz II.

In the experiments of Tucker & Reynolds (1968) the distortion was not sufficiently rapid for a direct comparison with the theory; a correction for viscous decay is necessary. Applying this following Tucker & Reynolds, we get for ansatz I the results shown in figure 12; the axisymmetric theory is clearly closer to the measurements. Results for ansatz II show no significant differences.

In the experiments reported by Uberoi (1956), Uberoi & Wallis (1966) and Comte-Bellot & Corrsin (1966), isotropic theory applied to each component shows reasonable agreement; the present theory improves the agreement but the differences are relatively small.

7. Conclusions

A complete specification of the spectral tensor in axisymmetric turbulence needs in general two scalar functions of two scalar variables. However, there are certain special cases in which it is sufficient to give two functions of a single variable (ansatzes I and II) or even one function of a single variable with an additional parameter (ansatz I*A*). In all these special cases, the turbulent energy after an arbitrary three-dimensional distortion can be worked out in closed form, just as in isotropic turbulence. The results

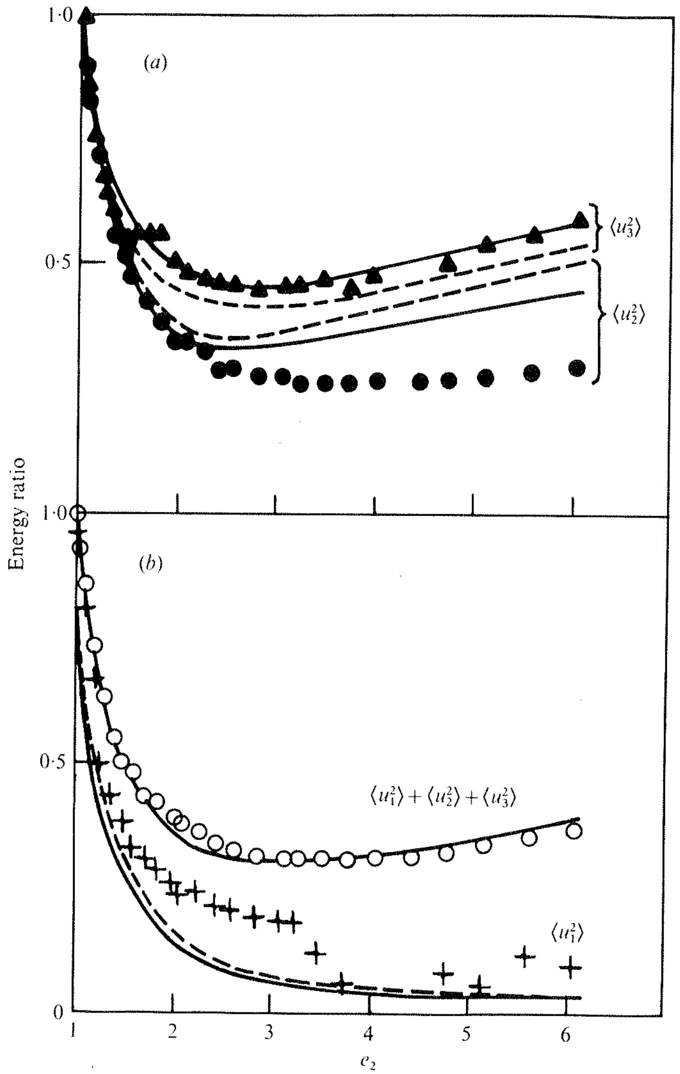

FIGURE 12. Comparison of the calculated energy ratios with measurements (points, data of Tucker & Reynolds 1968) for constant-area distortion of axisymmetric turbulence. Theory: ——, ansatz I; – – –, isotropic. Note the small difference between the isotropic and axisymmetric theories for the variation of the total energy.

from the different ansatzes are in broad qualitative agreement with each other, as well as with those of Reynolds & Tucker's hypothetical-strain method: thus in large plane or axisymmetric contractions the decrease in longitudinal energy is nearly the same as in isotropic turbulence, but the transverse energy, if less than the longitudinal in the initial state, does not increase by as much as the isotropic theory predicts. Nevertheless, there are significant qualitative differences; e.g. the predictions of ansatz II and the hypothetical-strain method differ with regard to the sign of the departure of μ_1 from isotropic theory. The transverse energy ratio predicted by the hypothetical-strain method is about 25 % more than that predicted by ansatz II, but agrees closely with ansatz I. In any case, severe distortions can cause energy changes very different from

those predicted by isotropic theory; but for small departures from isotropy and moderate distortions, an *ad hoc* application of isotropic theory to each component should not cause serious errors.

Unfortunately, it is not possible to make a categorical recommendation about which of the different ansatzes proposed here is the most appropriate; indeed it is likely that each of them has its special place. Nevertheless, ansatz I*A* is the simplest possible form of non-isotropic turbulence, and ought therefore to be chosen if possible. On the other hand, ansatz II appears to be rather more successful, especially in cases where the departure from isotropy is large.

Appendix

It was shown by Chandrasekhar (1950) that, for turbulence with weak inertia effects, the functions (say Q_1 and Q_2) which determine the second-order tensor of the type (2.1*a*) for correlations approach, in the limit $t \to \infty$, a state independent of the direction cosine, say r_1, of the separation-distance vector $\mathbf{r}$. He showed then that the correlation tensor has components given by

$$R_{11}(r) = \left(\frac{r_1^2}{r}\frac{\partial}{\partial r} - r\frac{\partial}{\partial r} - 2\right) Q_1 \exp(-\alpha r^2), \qquad \text{(A 1)}$$

$$R_{22}(r) + R_{33}(r) = -\left(\frac{r_1^2}{r}\frac{\partial}{\partial r} + r\frac{\partial}{\partial r} + 4\right) Q_1 \exp(-\alpha r^2) + \left(\frac{r_1^2}{r}\frac{\partial}{\partial r} - r\frac{\partial}{\partial r} - 2\right) Q_2 \exp(-\alpha r^2), \qquad \text{(A 2)}$$

where Q_1, Q_2 and α are functions only of time.

Taking Fourier transforms of (A 1) and (A 2), one gets

$$\phi_{11}(\mathbf{k}) = -(Q_1/2\alpha)(\pi/\alpha)^{\frac{3}{2}}(k^2 - k_1^2)\exp(-k^2/4\alpha) \qquad \text{(A 3)}$$

and

$$\phi_{22}(\mathbf{k}) + \phi_{33}(\mathbf{k}) = Q_1(/2\alpha)(\pi/\alpha)^{\frac{3}{2}}(k^2 + k_1^2)\exp(-k^2/4\alpha) - (Q_2/2\alpha)(\pi/\alpha)^{\frac{3}{2}}(k^2 - k_1^2)\exp(-k^2/4\alpha). \qquad \text{(A 4)}$$

On comparing (A 3) and (A 4) with (4.2) it is clear that this form is consistent with ansatz I; it further follows that

$$Q_1/Q_2 = F_0/G_0 = -R/2(R-1). \qquad \text{(A 5)}$$

This case in which F_0/G_0 is a constant independent of k is precisely what we have called ansatz I*A* [see (4.6)].

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