Combinatorics and Geometry

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The goal of this talk was to illustrate how the two concepts of the title, perhaps seemingly unrelated at first sight, are really tied together quite closely. Combinatorics, the art of counting, naturally deals with discrete objects. Geometry on the other hand mostly evokes the opposite, the continuous, dealing with shapes in space and the like. But there is a tight connection between the two. We may even go further and speculate whether our universe is continuous or discrete. Arguably, the natural answer is the former but it increasingly it looks like it might just be the latter.

§1. What is a projective plane? We can abstract axiomatically the basic properties of what a projective should be: a set $\mathcal{A}$ whose elements we call *points* and a collection of subsets of $\mathcal{A}$ which we call *lines* satisfying a few simple axioms:

- Two distinct points lie in a unique line.
- Two distinct lines meet in a unique point.
- There exist four points not all in a line

The first two axioms are the key features of a projective plane, the third avoids dealing with some trivial cases.

The main point we would like to emphasize is that *no finiteness is required*: $\mathcal{A}$ could be a finite set. For example, Fano in 1892 produced a projective plane consisting of seven points and seven lines (see Fig.1). In general, in any finite projective plane all lines have the same number of points, say $q + 1$ points for some q. The total number of points of the plane is then $q^2 + q + 1$, with $q = 2$ in Fano's case (the smallest possible).

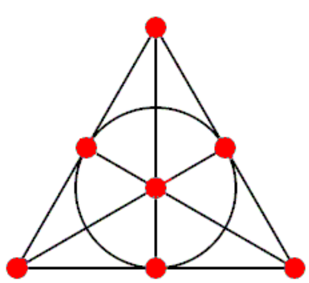

Figure 1: Fano plane

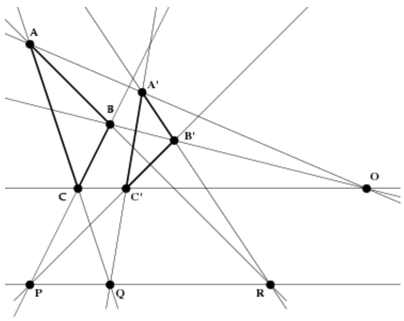

Figure 2: Desargues theorem

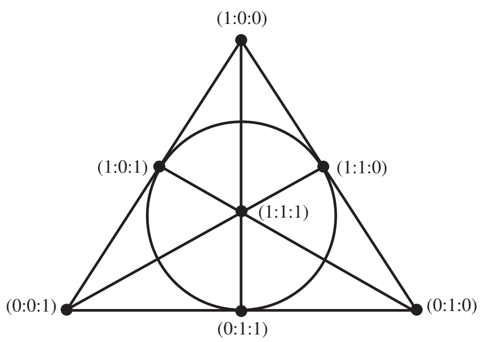

Figure 3: Fano plane coordinates

The familiar theorem of Desargues of usual projective geometry (see Fig.2, the theorem says that the lines A, A', B, B' and C, C' necessarily meet at a point) does not follow from the above axioms. In fact, it holds if and only if we can give coordinates to our plane. If Desargues theorem holds we can use it to define all the usual operations with coordinates: sums, multiplication by scalars, etc. The scalars obtained inherit the algebraic structure of a field. For a finite plane we get a finite field $\mathbb{F}_q$, necessarily of size $q = p^n$ elements for some prime number p. Up to isomorphism this field is uniquely determined by its size q. You can see in Fig. 3 the coordinates system in the Fano plane where the scalars consist of the field of two elements $\mathbb{F}_2 = \{0, 1\}$.

§2. In algebraic geometry we study the zero locus $X(\mathbb{C})$ of polynomials, say, $F_1, \ldots, F_m$ with complex coefficients in variables $x_1, \ldots, x_n$. If the coefficients of F_i are actually integers we may also consider their solutions $X(\mathbb{F}_q)$ with coordinates in the finite field $\mathbb{F}_q$. What, if any, is the relation between the complex points $X(\mathbb{C})$ and the finite field points $X(\mathbb{F}_q)$?

Consider for example,

$$X : \quad y^2 = f(x)$$

with $f \in \mathbb{Z}[x]$ square-free of degree 8. This is an algebraic curve of genus $g = 3$. Its complex points $X(\mathbb{C})$ look like Figure 4. Topologically $X(\mathbb{C})$ is a three-holed doughnut.

What can we say about $X(\mathbb{F}_q)$? Pictures of this finite set are not particularly useful. Passing from $\mathbb{C}$ to $\mathbb{F}_q$ we go from the continuous to the discrete and loose our ordinary spatial intuition. But we gain something else: we can count.

Thanks to the work of Weil we know that for a smooth projective curve X of genus g we have for all n

$$\#X(\mathbb{F}_{p^n}) = p^n + 1 - \sum_{i=1}^{2g} \alpha_i^n, \qquad |\alpha_i| = p^{\frac{1}{2}}$$

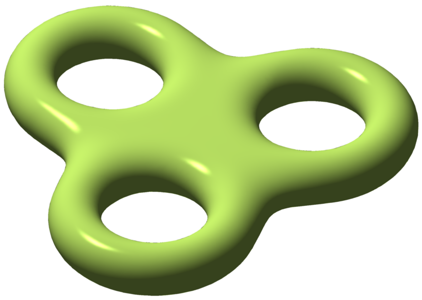

Figure 4: $X(\mathbb{C})$ genus 3 curve

Hence with $q = p^n$

$$|\#X(\mathbb{F}_q) - q - 1| \leq 2g\sqrt{q}. \tag{0.1}$$

In particular, if $g = 0$ then $\#X(\mathbb{F}_q) = q + 1$. This is as it should be! Indeed, X is isomorphic to a projective line, which as we discussed has $q + 1$ points if the field of scalars is $\mathbb{F}_q$.

We can interpret the inequality (0.1) as saying that a general algebraic curve of genus g has roughly $q+1$ points over $\mathbb{F}_q$, the points in a line, with an error bounded by $2g\sqrt{q}$. This error does indeed range over the whole interval $[-2g\sqrt{q}, 2g\sqrt{q}]$ as X and q vary. Hence, on one hand g determines the topological shape of $X(\mathbb{C})$ and on the other it controls the rough behaviour of $\#X(\mathbb{F}_q)$.

In general, what precisely can we recover of $X(\mathbb{C})$ from the $\#X(\mathbb{F}_q)$ data? In a loose analogy the situation is similar to that of tomography. We may think of passing to a given finite field as analogous to taking the sectional image of an object in space along a given plane. In tomography one reconstructs the shape of object from the sectional images along all planes. One may hope that data on $X(\mathbb{F}_q)$ would allow the recovery of the shape of $X(\mathbb{C})$. To some extent this is the case (see [4] and [2] for two well know and important examples) thanks to the Weil conjecures proved by Deligne.

A particular, simple situation is the following. Suppose $X(\mathbb{C})$ is smooth, compact and $\#X(\mathbb{F}_q) = C(q)$ for a certain polynomial C. Let $b_j(X) := \dim H^j(X, \mathbb{C})$, the Betti numbers of X. (These are topological invariants of the space $X(\mathbb{C})$; for example, for an a algebraic curve of genus g we have $b_0 = b_2 = 1, b_1 = 2g$ and all others are zero.) Then $b_{2i+1}(X) = 0$ and

$$C(q) = \sum_{i=0}^{\dim X} b_{2i}(X)\, q^i \tag{0.2}$$

For example, if $X = \mathbb{P}^1$, the projective line, then

$$C(q) = q + 1$$

and indeed we have $b_0 = b_2 = 1$, $b_1 = 0$ since $g = 0$. Similarly, if $X = \mathbb{P}^2$, the projective plane, then as we mentioned in §1

$$C(q) = q^2 + q + 1$$

and indeed $b_0 = b_2 = b_4 = 1, b_1 = b_3 = 0$.

We must point out a technical but crucial point for what follows. The equality (0.2) holds in a bit more generality. It is enough to know that the natural mixed Hodge structure in the cohomology of X is pure and $\#X(\mathbb{F}_q) = C(q)$ or all finite fields with $q = p^n$ for al

Note that such counting polynomials $C(q)$ must have non-negative integer coefficients (these coefficients being dimensions of vector spaces). The relation (0.2) is actually a two-way street. We may use it to compute

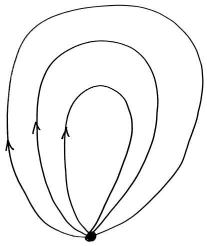

Figure 5: S_g quiver

Betti numbers by counting or to prove that certain polynomials have non-negative coefficients (because they happen to equal $C(q)$ for an appropriate X). We discuss an example along the lines of the latter situation in §3 below.

Consider for example the Grassmanian $X = G(k, n)$ of all dimension k subspaces in a fixed n dimensional space. Its number of points is given by the q-binomial coefficient

$$\#G(k,n)(\mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}, \qquad [n]! := (1-q)\cdots(1-q^n),$$

which is a polynomial in q. E.g.,

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$$

and therefore for $X = G(2,5)$ we have $b_0 = b_2 = b_{10} = b_{12} = 1$, $b_4 = b_6 = b_8 = 2$ and all other Betti numbers are zero.

§3. A quiver Q is a directed graph. A representation of Q is an assignment:

$$\begin{array}{ccc} \text{vertex} & \mapsto & \text{vector space} \\ \text{arrow} & \mapsto & \text{linear map} \end{array}$$

We are interested in representations up to isomorphism.

For example, if Q is the quiver S_g (see Fig. 5) consisting of one vertex with g loops attached then a representation is a tuple $(A_1, \ldots, A_g)$ of $n \times n$ matrices for some n. Two representations $(A_1, \ldots, A_g)$ and $(A'_1, \ldots, A'_g)$ are isomorphic if the tuples of matrices are simultaneously conjugate. I.e.

$$(A'_1, \ldots, A'_g) = U(A_1, \ldots, A_g)U^{-1}$$

for some invertible matrix U. For $g = 1$ this is Jordan's problem: to classify matrices up to conjugation; it has a beautiful solution that we learn in a linear algebra course.

Can we classify in some form representations up to isomorphim in general? For example, can we classify $g > 1$ tuples of matrices up to simultaneous conjugation? Mostly we cannot; these are typically difficult linear algebra problems.

Kac (in the early 80's) thought of passing to finite fields and counting representations up to isomorphim. Fix Q and a dimension vector α (recording the dimension of the vector spaces attached to the vertices of Q). Consider absolutely indecomposable representations of dimension α. These are representations that do not decompose in a non-trivial way as a direct sum even after extending scalars to a larger field. In Jordan's case an absolutely indecomposable representation of dimension n is given by an $n \times n$ matrix with exactly one Jordan block. To illustrate the issue of extending scalars, note that for example the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is not absolutely indecomposable. It represents an indecomposable representation over $\mathbb{R}$ but not over $\mathbb{C}$ since its Jordan decomposition is diagonal with eigenvalues i and $-i$.

Kac showed that up to isomorphism the number of absolutely indecomposable representations of a quiver Q of dimension α equals $A_\alpha(q)$ a polynomial in q with integer coefficients. It is not a particularly easy polynomial to compute though there is a rather daunting formula for a certain generating function of these [6]. For example, for the S_g quiver we have

$\alpha\backslash g$	1	2	3	4
1	q	q^2	q^3	q^4
2	q	q^5+q^3	$q^9+q^7+q^5$	$q^{13}+q^{11}+q^9+q^7$
3	q	$q^{10}+q^8+q^7+\cdots$	$q^{19}+q^{17}+q^{16}+\cdots$	$q^{28}+q^{26}+q^{25}+\cdots$
4	q	$q^{17}+q^{15}+q^{14}+\cdots$	$q^{33}+q^{31}+q^{30}+\cdots$	$q^{49}+q^{47}+q^{46}+\cdots$
5	q	$q^{26}+q^{24}+q^{23}+\cdots$	$q^{51}+q^{49}+q^{48}+\cdots$	$q^{76}+q^{74}+q^{73}+\cdots$
6	q	$q^{37}+q^{35}+q^{34}+\cdots$	$q^{73}+q^{71}+q^{70}+\cdots$	$q^{109}+q^{107}+q^{106}+\cdots$

Note that for $g = 1$ we have $A_\alpha(q) = q$ for all α since, as mentioned, an absolutely indecomposable representation of dimension α consists of one Jordan block of size α. This Jordan block is uniquley determined up to isomorphism by its eigenvalue for which there are $q = |\mathbb{F}_q|$ possibilities.

Kac conjectured that the coefficients of $A_\alpha(q)$ are in fact non-negative. Crawley-Boevey and van der Bergh proved the conjecture when α indivisible (not a proper multiple of another integral vector). For the quivers S_g, for example, it only applies to the case of dimension 1. With Hausel and Letellier we extended the proof to the general case, see §4.

The argument of Crawley-Boevey and van der Bergh shows that, in fact,

$$A_\alpha(q) = \sum_i \dim\left(H_c^{2i}(\mathcal{Q}_\alpha;\mathbb{C})\right) q^{i-d_\alpha/2}, \tag{0.3}$$

where $\mathcal{Q}_\alpha$ is an associated smooth Nakajima quiver variety of dimension d_α. The hypothesis on α being indivisible is crucial for the existence of $\mathcal{Q}_\alpha$. This variety is an appropriate replacement for a naive "space of isomorphism classes" of representations of the type we want to count and the identity (0.3) is far from obvious.

The proof of (0.3) is along the lines of our discussion in §2. Namely,

$$\#\mathcal{Q}_\alpha(\mathbb{F}_q) = q^{d_\alpha/2} A_\alpha(q).$$

However, this is not quite enough to deduce (0.3) because though $\mathcal{Q}_\alpha$ is smooth it is not compact. A further argument is needed to show that (0.2) still holds for $\mathcal{Q}_\alpha$ because the natural mixed Hodge structure on its cohomology is pure.

§4. Given a quiver Q and arbitrary dimension vector α with Hausel and Letellier (see [5] and the references given therein) we consider an extended quiver $\tilde{Q}$ adding legs to every vertex. On $\tilde{Q}$ we take the dimension vector $\tilde{\alpha}$ where if α_i is the dimension at the i-th vertex of Q then we put dimensions $\alpha_i - 1, \alpha_i - 2, \ldots$

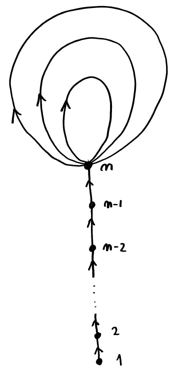

Figure 6: Extended S_g quiver

along the vertices of the attached leg. For example, if $Q = S_g$ and $\alpha = n$ then $\tilde{Q}$ is as in Fig.6 with a leg of length n and $\tilde{\alpha} = n, n-1, \ldots, 2, 1$.

The vector $\tilde{\alpha}$ is indivisible and we may hence consider the associated Nakajima quiver variety $\tilde{\mathcal{Q}}_{\tilde{\alpha}}$ of dimension $d_{\tilde{\alpha}}$. Each vertex of the quiver gives rise to a reflection that acts on the cohomology of $\tilde{\mathcal{Q}}_{\tilde{\alpha}}$. The reflections along the i-th leg generate a copy of the symmetric group S_{α_i} and we therefore obtain an action of $S_{\alpha_1} \times S_{\alpha_2} \cdots$. Let $\epsilon = \epsilon_1 \times \epsilon_2 \cdots$ where ϵ_i is the sign character of S_{α_i}. Our main result is that

$$A_\alpha(q) = \sum_i \dim \left(H_c^{2i}(\mathcal{Q}_{\tilde{\alpha}}; \mathbb{C})_\epsilon \right) q^{i - d_{\tilde{\alpha}}/2},$$

where the subscript indicates we take the ϵ isotypical component of the corresponding space. This again is proved by counting points over finite fields using the whole machinery for counting points on character and quiver varieties developed in our previous papers.

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