

Numerical Simulation of Turbulence Transition Regimes in Pipe Flow Using Solenoidal Bases

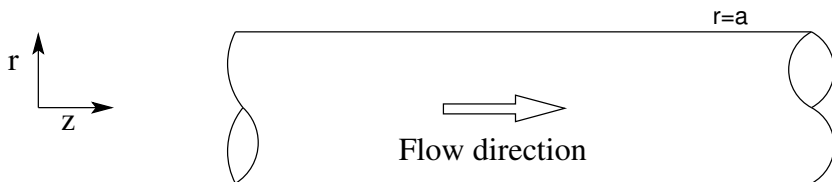
Ozan Tuğluk Hakan Tarman

Dept. of Engineering Sciences, METU/Ankara

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Flow geometry



Why study pipe flow?

Because, pipe flow is,

- **Linearly stable** even at very high Reynolds numbers,
- Technologically relevant,
- Easy to conduct experiments, wealth of experimental data.

Transition Regime

- Onset of turbulence,
- Spatially coherent, temporally chaotic,
- A case of deterministic chaos.

Why use Solenoidal Bases?

- Continuity equation is exactly satisfied,
- A dynamical system is obtained, suitable for bifurcation analysis,
- Construction of basis is independent of the Reynolds number,
- The pressure is eliminated.

N-S Equations

Navier-Stokes Equations In Polar coordinates, core scaling
($Re = \frac{u_c R}{\nu}$) :

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = K \mathbf{e}_z - \nabla p + \frac{1}{Re} \Delta \mathbf{u} \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u}(1, \theta, z, t) = 0$$

$$\mathbf{u}(r, \theta, z, 0) = \mathbf{u}_0$$

Perturbation Formulation

Base, fluctuation expansion

$$\mathbf{U}(r, \theta, z, t) = \mathbf{u}_B(r) + \mathbf{u}(r, \theta, z, t) \quad (2)$$

$$P(r, \theta, z, t) = P_B(r) + p(r, \theta, z, t)$$

Perturbation Formulation

Base, fluctuation expansion

$$\begin{aligned}\mathbf{U}(r, \theta, z, t) &= \mathbf{u}_B(r) + \mathbf{u}(r, \theta, z, t) \\ P(r, \theta, z, t) &= P_B(r) + p(r, \theta, z, t)\end{aligned}\quad (2)$$

The perturbation formulation of NS

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{\text{Re}} \Delta \mathbf{u} - (\mathbf{u}_B \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u}_B \quad (3)$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u}(1, \theta, z, t) = 0$$

$$\mathbf{u}(r, \theta, z, 0) = \mathbf{u}_0$$

The bases

Basis function spaces, (regularity at $r = 0$ for physical basis, Priymak & Miyazaki)

$$\Phi_{lm}^{(1,2)}(1, \theta, z) = 0$$

$$\Psi_{lm}^{(1,2)}(1, \theta, z) \cdot \mathbf{e}_r = 0$$

$$\nabla \cdot \Psi_{lm}^{(1,2)} = 0$$

$$\nabla \cdot \Phi_{lm}^{(1,2)} = 0$$

Representation using Fourier expansion along θ , z

$$\Phi_{lnm}^{(1,2)}(r, \theta, z) = e^{i(n\theta + 2\pi lz/Q)} \mathbf{v}_{lnm}^{(1,2)}(r)$$

$$\Psi_{lnm}^{(1,2)}(r, \theta, z) = e^{i(n\theta + 2\pi lz/Q)} \tilde{\mathbf{v}}_{lnm}^{(1,2)}(r)$$

$$\mathbf{v}_{lnm} = \left\{ \mathbf{v}_{lnm}^{(1)}, \mathbf{v}_{lnm}^{(2)} \right\}$$

$$\tilde{\mathbf{v}}_{lnm} = \left\{ \tilde{\mathbf{v}}_{lnm}^{(1)}, \tilde{\mathbf{v}}_{lnm}^{(2)} \right\}$$

Representation using Fourier expansion along θ , z

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Continuity equation in Fourier space

$$D_+ v_r + \frac{in}{r} v_\theta + i l v_z = 0$$

$$D_+ = D + \frac{1}{r}$$

Case I: $l \neq 0$ $n=0$

Physical "1-basis"

$$\mathbf{v}^{(1)} = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix}$$

$$v_\theta = r(1 - r^2) P_{2m}$$

Physical "2-basis"

$$\mathbf{v}^{(2)} = \begin{pmatrix} -il v_r \\ 0 \\ D_+ v_r \end{pmatrix}$$

$$v_r = r(1 - r^2)^2 P_{2m}$$

Dual "1-basis"

$$\tilde{\mathbf{v}}^{(1)} = \begin{pmatrix} 0 \\ \tilde{v}_\theta \\ 0 \end{pmatrix}$$

$$\tilde{v}_\theta = P_{2m}$$

Dual "2-basis"

$$\tilde{\mathbf{v}}^{(1)} = \begin{pmatrix} -il \tilde{v}_r \\ 0 \\ D_+ \tilde{v}_r \end{pmatrix}$$

$$\tilde{v}_r = (1 - r^2) P_{2m}$$

Case II: $l=0$ $n \neq 0$

Physical “1-basis”

$$\mathbf{v}^{(1)} = \begin{pmatrix} -in v_r \\ D(r v_r) \\ 0 \end{pmatrix}$$

$$v_r = r^{|n|-1} (1 - r^2)^2 P_{2m}$$

Physical “2-basis”

$$\mathbf{v}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ in v_z \end{pmatrix}$$

$$v_z = r^{|n|} (1 - r^2) P_{2m}$$

Dual “1-basis”

$$\tilde{\mathbf{v}}^{(1)} = \begin{pmatrix} -in \tilde{v}_r \\ D(r \tilde{v}_r) \\ 0 \end{pmatrix}$$

$$\tilde{v}_r = r^{(n \bmod 2)} (1 - r^2) P_{2m}$$

Dual “2-basis”

$$\tilde{\mathbf{v}}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ in \tilde{v}_z \end{pmatrix}$$

$$\tilde{v}_z = r^{(n+1 \bmod 2)} P_{2m}$$

The inner product

$$(\psi_{lnm}, \mathbf{f}) = \int_0^Q \int_0^{2\pi} \int_0^1 e^{-i(n\theta + 2\pi lz/Q)} \tilde{\mathbf{v}}_{lnm}(r)^* \cdot \mathbf{f}(r, \theta, z) r dr d\theta dz$$

The inner product

$$(\psi_{lnm}, \mathbf{f}) = \int_0^Q \int_0^{2\pi} \int_0^1 e^{-i(n\theta + 2\pi lz/Q)} \tilde{\mathbf{v}}_{lnm}(r)^* \cdot \mathbf{f}(r, \theta, z) r dr d\theta dz$$

Discrete inner product

$$\hat{f}_{lnm} = \sum_{i=0}^{L_d-1} \sum_{j=0}^{N_d-1} \sum_{k=0}^{M_d-1} e^{-i(n\theta + 2\pi lz/Q)} r_k w_k \tilde{\mathbf{v}}_{ln1m}(r_k)^* \mathbf{f}_{ijk} \quad (4)$$

Expansion for the velocity

$$\mathbf{u}(r, \theta, z) = \sum_{l=-L}^L \sum_{n=-N}^N \sum_{m=0}^M e^{i(n\theta + 2\pi lz/Q)} a_{lnm}(t) \mathbf{v}_{lnm}(r) \quad (5)$$

$$a_{lnm} = \begin{bmatrix} a_{lnm}^{(1)} \\ a_{lnm}^{(2)} \end{bmatrix}$$

Elimination of ∇p

$$\int_{\Omega} \Psi \nabla p \cdot \mathbf{r} \, d\Omega = \int_{\partial\Omega} p \underbrace{\Psi \cdot \boldsymbol{\nu}}_{=0} \, ds + \int_{\Omega} \underbrace{\nabla \cdot \Psi}_{=0} p \, d\Omega$$

The dynamical system

$$\mathbf{A}_{lnm} \dot{a}_{lnm} = \mathbf{B}_{lnm} a_{lnm} - b_{lnm} \quad (6)$$

$$\mathbf{A}_{lnm} = (\Psi_{lnm}, \Phi_{lnm})$$

$$\mathbf{B}_{lnm} = (\Psi_{lnm}, \frac{1}{\text{Re}} \Delta \Phi_{lnm} - (\mathbf{v}_B \cdot \nabla) \Phi_{lnm} - (\Phi_{lnm} \cdot \nabla) \mathbf{v}_B)$$

$$b_{lnm} = (\Psi_{lnm}, (\mathbf{u} \cdot \nabla) \mathbf{u})$$

$$a_{lnm} = \begin{bmatrix} a_{lnm}^{(1)} \\ a_{lnm}^{(2)} \end{bmatrix}$$

Time Integration

- 4th order
- Advective (non-linear) terms treated explicitly,
- Dissipative (linear) terms implicitly.

The semi implicit time stepping scheme, Adams-Bashford and backward difference combination

$$(25\mathbf{A} - 12\Delta t \mathbf{B}) a^{(k+1)} = \mathbf{A}(48a^{(k)} - 36a^{(k-1)} + 16a^{(k-2)} - 3a^{(k-3)}) - \Delta t(48b^{(k)} - 72b^{(k-1)} + 48b^{(k-2)} - 12b^{(k-3)}) \quad (7)$$

Initial perturbation energies

2D

$$a_{lnm}^0 = \begin{cases} \varepsilon_0 & \text{for } l = 0, n = \pm 1, m = 0 \\ 0 & \text{otherwise} \end{cases}$$

3D

$$a_{lnm}^0 = \begin{cases} \varepsilon_0 & \text{for } l = 0, n = \pm 1, m = 0 \\ \varepsilon_1 & \text{for } l = \pm 1, n = 0, \pm 1, m = 0 \\ 0 & \text{otherwise} \end{cases}$$

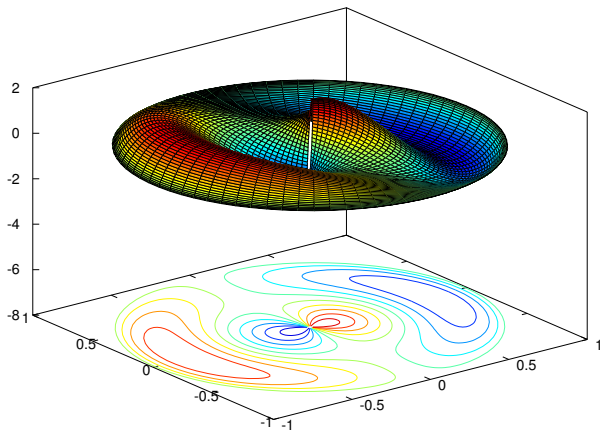


Figure: mode (0,1,0) in real space, v_θ contours

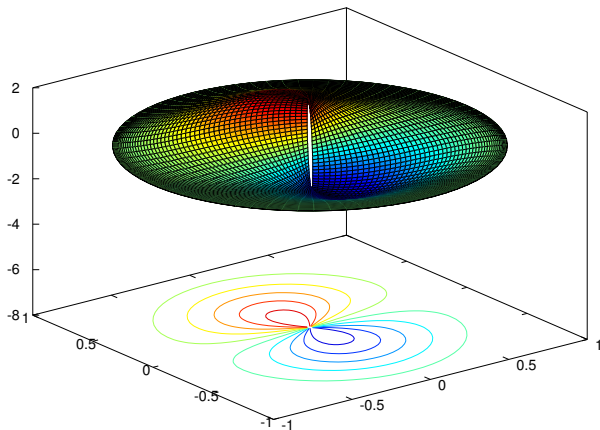


Figure: mode (0,1,0) in real space, v_r contours

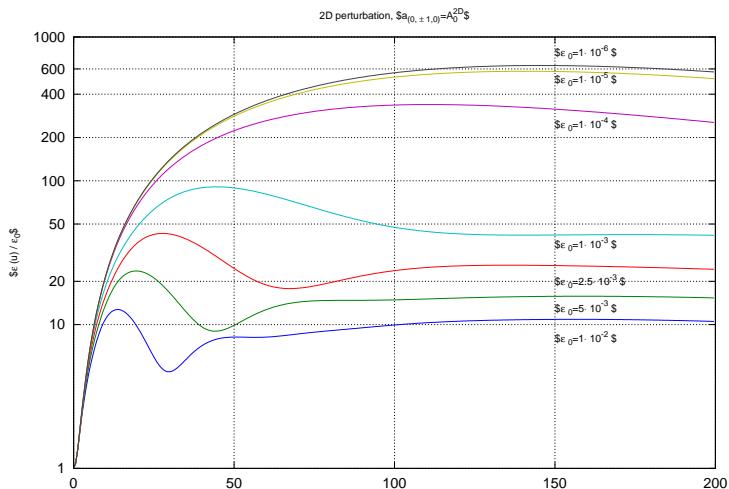


Figure: Evolution of various 2D perturbations, , Re=3000

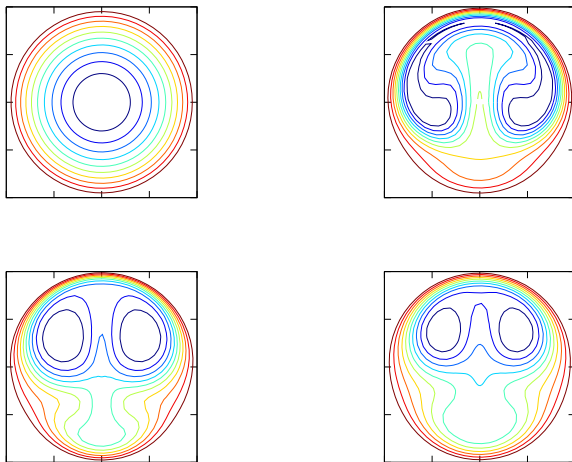


Figure: Streak formation, $\text{Re} = 3000$, $\epsilon_0 = 1 \cdot 10^{-2}$, modulated axial flow at 0, 17, 75, 150 seconds

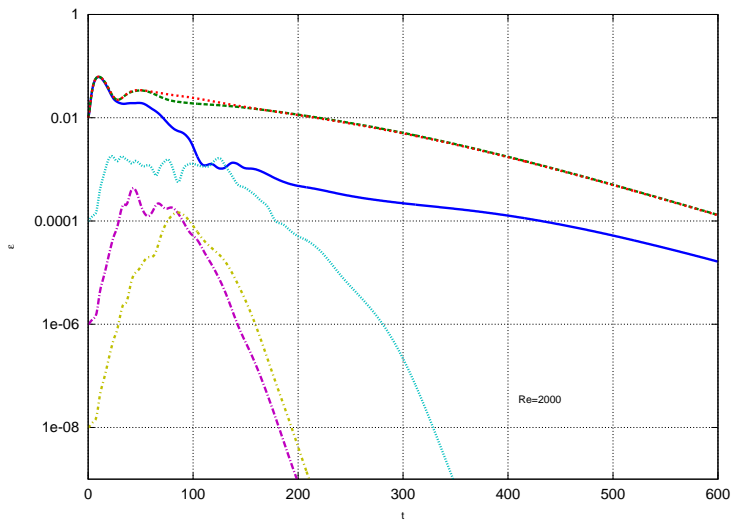


Figure: Evolution of 2D and 3D perturbations, $Re = 2000$

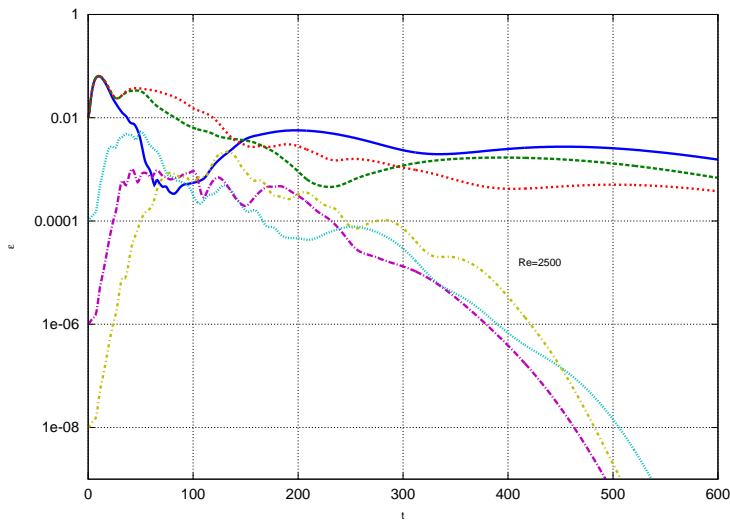


Figure: Evolution of 2D and 3D perturbations, $Re = 2500$

Final Remarks

Advantages of using solenoidal bases:

- There is no separate simulation stage, unlike POD.
- Easy to code.
- In contrast to POD, bases are “parameter-independent”, independent of Re .
- Continuity equation is exactly satisfied.
- We do not need a pressure solver.

Final Remarks

Disadvantages of using solenoidal bases:

- Not suitable for high Re , the matrices are badly-conditioned.
- In cases where Re is fixed, the bases are not optimal in energy sense, unlike POD bases.
- Restricted to simple geometries.
- Analytical derivation of bases is necessary.