Numerical Simulation of Turbulence Transition Regimes in Pipe Flow Using Solenoidal Bases

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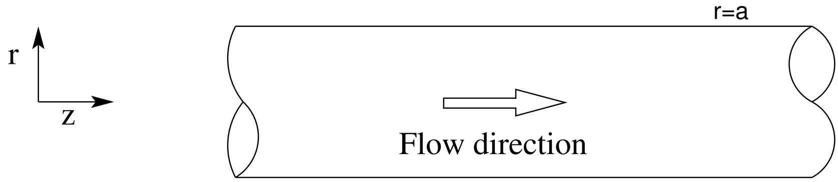
August 7, 2009

Flow geometry



Why study pipe flow?

Because, pipe flow is,

- Linearly stable even at very high Reynolds numbers,
- Technologically relevant,
- Easy to conduct experiments, wealth of experimental data.

Transition Regime

- Onset of turbulence,
- Spatially coherent, temporally chaotic,
- A case of deterministic chaos.

Why use Solenoidal Bases?

- Continuity equation is exactly satisfied,
- A dynamical system is obtained, suitable for bifurcation analysis,
- Construction of basis is independent of the Reynolds number,
- The pressure is eliminated.

N-S Equations

Navier-Stokes Equations In Polar coordinates, core scaling $(Re = \frac{u_c R}{\nu})$:

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= \mathrm{K}\ \mathbf{e_z} - \nabla p + \frac{1}{\mathrm{Re}}\ \Delta \mathbf{u} \\ \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{u}(1, \theta, z, t) &= 0 \\ \mathbf{u}(r, \theta, z, 0) &= \mathbf{u_0} \end{aligned} \tag{1}$$

Perturbation Formulation

Base, fluctuation expansion

$$\mathbf{U}(r,\theta,z,t) = \mathbf{u_B}(r) + \mathbf{u}(r,\theta,z,t) \qquad (2)$$

$$P(r,\theta,z,t) = P_B(r) + p(r,\theta,z,t)$$

Perturbation Formulation

Base, fluctuation expansion

$$\begin{aligned} \mathbf{U}(r,\theta,z,t) &= \mathbf{u_B}(r) + \mathbf{u}(r,\theta,z,t) \\ P(r,\theta,z,t) &= P_B(r) + p(r,\theta,z,t) \end{aligned} \tag{2}$$

The perturbation formulation of NS

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla p + \frac{1}{\mathrm{Re}}\,\Delta\mathbf{u} - (\mathbf{u_B}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u_B} \\ \nabla\cdot\mathbf{u} &= 0 \\ \mathbf{u}(1,\theta,z,t) &= 0 \\ \mathbf{u}(r,\theta,z,0) &= \mathbf{u_0} \end{aligned} \tag{3}$$

The bases

Basis function spaces, (regularity at $r = 0$ for physical basis, Priymak & Miyazaki)

$$
\begin{aligned}
\mathbf{\Phi}_{lnm}^{(1,2)}(1,\theta,z) &= 0\\
\mathbf{\Psi}_{lnm}^{(1,2)}(1,\theta,z)\cdot\mathbf{e_r} &= 0\\
\nabla\cdot\ \mathbf{\Psi}_{lnm}^{(1,2)} &= 0\\
\nabla\cdot\mathbf{\Phi}_{lnm}^{(1,2)} &= 0
\end{aligned}
$$

Representation using Fourier expansion along $\theta,\ z$

$$\mathbf{\Phi}^{(1,2)}_{lnm}(r,\theta,z) = e^{i(n\theta+2\pi lz/Q)}\ \mathbf{v}^{(1,2)}_{lnm}(r)$$

$$\mathbf{\Psi}^{(1,2)}_{lnm}(r,\theta,z) = e^{i(n\theta+2\pi lz/Q)}\ \tilde{\mathbf{v}}^{(1,2)}_{lnm}(r)$$

$$\mathbf{v}_{lnm} = \mathbf{v}^{(1)}_{lnm} + \mathbf{v}^{(2)}_{lnm}$$

$$\tilde{\mathbf{v}}_{lnm} = \tilde{\mathbf{v}}^{(1)}_{lnm} + \tilde{\mathbf{v}}^{(2)}_{lnm}$$

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Continuity equation in Fourier space

$$D_+ v_r + \frac{in}{r}\ v_\theta + ilv_z = 0$$
$$D_+ = D + \frac{1}{r}$$

Case I: l≠0 n=0

Physical "1-basis"

$$\mathbf{v}^{(1)} = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix}$$

$$v_\theta = r(1 - r^2)\ P_{2m}$$

Physical "2-basis"

$$\mathbf{v}^{(2)} = \begin{pmatrix} -il\ v_r \\ 0 \\ D_+ v_r \end{pmatrix}$$

$$v_r = r(1 - r^2)^2\ P_{2m}$$

Dual "1-basis"

$$\tilde{\mathbf{v}}^{(1)} = \begin{pmatrix} 0 \\ \tilde{v}_\theta \\ 0 \end{pmatrix}$$

$$\tilde{v}_\theta = P_{2m}$$

Dual "2-basis"

$$\tilde{\mathbf{v}}^{(1)} = \begin{pmatrix} -il\ \tilde{v}_r \\ 0 \\ D_+ \tilde{v}_r \end{pmatrix}$$

$$\tilde{v}_r = (1 - r^2)\ P_{2m}$$

Case II: l=0 n≠0

Physical "1-basis"

$$\mathbf{v}^{(1)} = \begin{pmatrix} -in\ v_r \\ D(r\ v_r) \\ 0 \end{pmatrix}$$

$$v_r = r^{|n|-1}(1-r^2)^2\ P_{2m}$$

Physical "2-basis"

$$\mathbf{v}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ in\ v_z \end{pmatrix}$$

$$v_z = r^{|n|}(1-r^2)\ P_{2m}$$

Dual "1-basis"

$$\tilde{\mathbf{v}}^{(1)} = \begin{pmatrix} -in\ \tilde{v}_r \\ D(r\ \tilde{v}_r) \\ 0 \end{pmatrix}$$

$$\tilde{v}_r = r^{(n \mod 2)}(1-r^2)\ P_{2m}$$

Dual "2-basis"

$$\tilde{\mathbf{v}}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ in\ \tilde{v}_z \end{pmatrix}$$

$$\tilde{v}_z = r^{(n+1 \mod 2)}\ P_{2m}$$

The inner product

$$(\Psi_{lnm}, \mathbf{f}) = \int_0^Q \int_0^{2\pi} \int_0^1 e^{-i(n\theta + 2\pi l z/Q)} \; \tilde{\mathbf{v}}_{lnm}(r)^* \cdot \mathbf{f}(r, \theta, z) \, r dr \, d\theta \, dz$$

The inner product

$$(\Psi_{lnm}, \mathbf{f}) = \int_0^Q \int_0^{2\pi} \int_0^1 e^{-i(n\theta + 2\pi lz/Q)} \; \tilde{\mathbf{v}}_{lnm}(r)^* \cdot \mathbf{f}(r, \theta, z) \, rdr \, d\theta \, dz$$

Discrete inner product

$$\hat{f}_{lnm} = \sum_{i=0}^{L_d-1} \sum_{j=0}^{N_d-1} \sum_{k=0}^{M_d-1} e^{-i(n\theta + 2\pi lz/Q)} \, r_k w_k \tilde{\mathbf{v}}_{ln1m}(r_k)^* \; \mathbf{f}_{\mathbf{ijk}} \tag{4}$$

Expansion for the velocity

$$\mathbf{u}(r,\theta,z) = \sum_{l=-L}^{L} \sum_{n=-N}^{N} \sum_{m=0}^{M} e^{i(n\theta + 2\pi l z/Q)} \, a_{lnm}(t) \;\; \mathbf{v}_{lnm}(r) \qquad (5)$$

$$a_{lnm} = \left[a_{lnm}^{(1)}, a_{lnm}^{(2)} \right]$$

The dynamical system

$$
\begin{aligned}
\mathbf{A}_{lnm}\,\dot{a}_{lnm} &= \mathbf{B}_{lnm}\,a_{lnm} - b_{lnm} \qquad (6)\\
\mathbf{A}_{lnm} &= (\boldsymbol{\Psi}_{lnm}, \boldsymbol{\Phi}_{lnm})\\
\mathbf{B}_{lnm} &= (\boldsymbol{\Psi}_{lnm}, \frac{1}{\mathrm{Re}}\Delta\boldsymbol{\Phi}_{lnm} - (\mathbf{v_B}\cdot\nabla)\boldsymbol{\Phi}_{lnm} - (\boldsymbol{\Phi}_{lnm}\cdot\nabla)\mathbf{v_B})\\
b_{lnm} &= (\boldsymbol{\Psi}_{lnm}, (\mathbf{u}\cdot\nabla)\mathbf{u})\\
a_{lnm} &= \left[a^{(1)}_{lnm}, a^{(2)}_{lnm}\right]
\end{aligned}
$$

Time Integration

- 4^{th} order
- Advective (non-linear) terms treated with backward difference,
- Dissipative (linear) terms with Adams-Bashford.

The semi implicit time stepping scheme

$$
\begin{aligned}
&(25\mathbf{A} - 12\Delta t\ \mathbf{B})\, a^{(k+1)} = \\
&\mathbf{A}(48a^{(k)} - 36a^{(k-1)} + 16a^{(k-2)} - 3a^{(k-3)}) \\
&- \Delta t(48b^{(k)} - 72b^{(k-1)} + 48b^{(k-2)} - 12b^{(k-3)})
\end{aligned}
\tag{7}
$$

Initial perturbation energies

2D

$$a^0_{lnm} = \begin{cases} \varepsilon_0 & \text{for} \quad l = 0,\ n = \pm 1,\ m = 0 \\ 0 & \quad\quad\ \text{otherwise} \end{cases}$$

3D

$$a^0_{lnm} = \begin{cases} \varepsilon_0 & \text{for} \quad l = 0,\ n = \pm 1,\ m = 0 \\ \varepsilon_1 & \text{for} \quad l = \pm 1,\ n = 0, \pm 1,\ m = 0 \\ 0 & \quad\quad\ \text{otherwise} \end{cases}$$

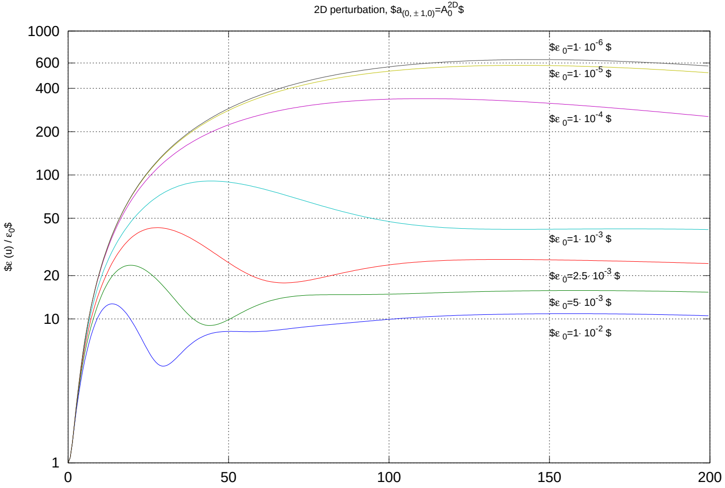

Figure: Evolution of various 2D perturbations, , Re=3000

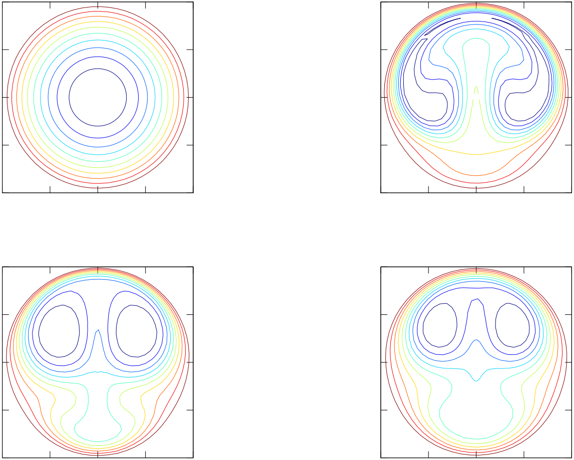

Figure: Streak formation, $\mathrm{Re} = 3000$, $\epsilon_0 = 1 \cdot 10^{-2}$, modulated axial flow at $0, 17, 75, 150$ seconds

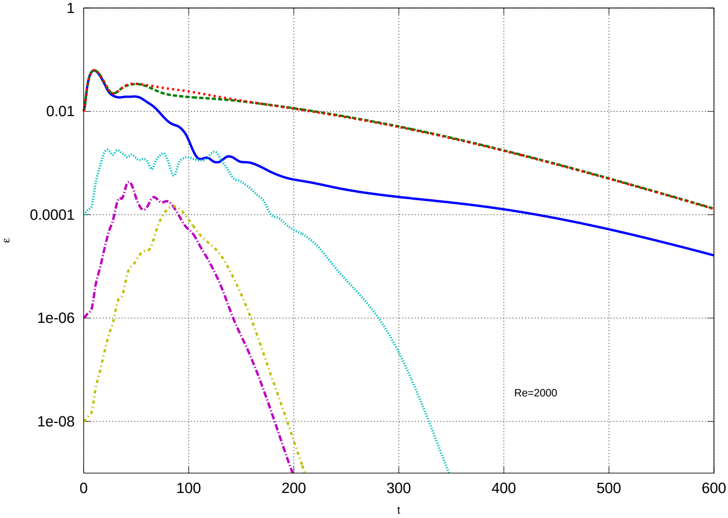

Figure: Evolution of 2D and 3D perturbations, $\mathrm{Re} = 2000$

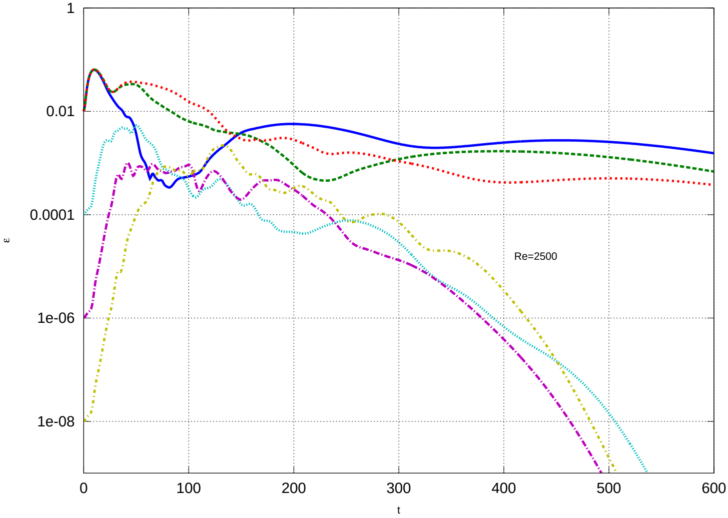

Figure: Evolution of 2D and 3D perturbations, $\mathrm{Re} = 2500$

Final Remarks

Advantages of using solenoidal bases:

- There is no separate simulation stage, unlike POD.
- Easy to code.
- In contrast to POD, bases are “parameter-independent”, independent of Re.
- Continuity equation is exactly satisfied.
- We do not need a pressure solver.

Final Remarks

Disadvantages of using solenoidal bases:

- In cases where Re is fixed, the bases are not optimal in energy sense, unlike POD bases.
- Restricted to simple geometries.
- Analytical derivation of bases is necessary.