The Abdus Salam
International Centre for Theoretical Physics

WEIZMANN INSTITUTE OF SCIENCE

2nd International Conference and Advanced School
"Turbulent Mixing and Beyond"

Velocity and energy profiles in two- vs. three-dimensional channels: Effects of an inverse *vs*. a direct energy cascade

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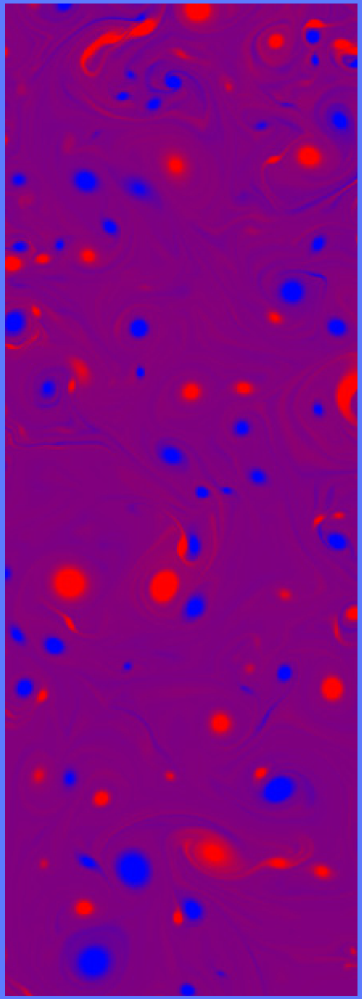
Overview



Formulation of the Problem

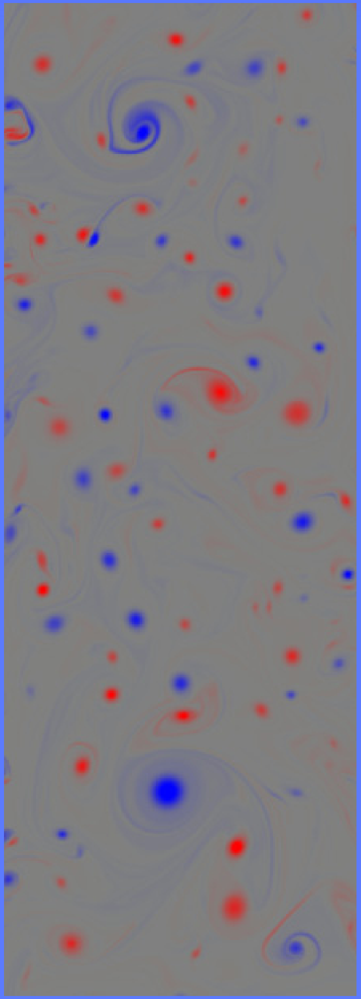
Milestones of the Model

- Mean Momentum Balance
- Kinetic Energy Balance
- Reynolds Stress Balance
- “Outer scale” of turbulence

Results

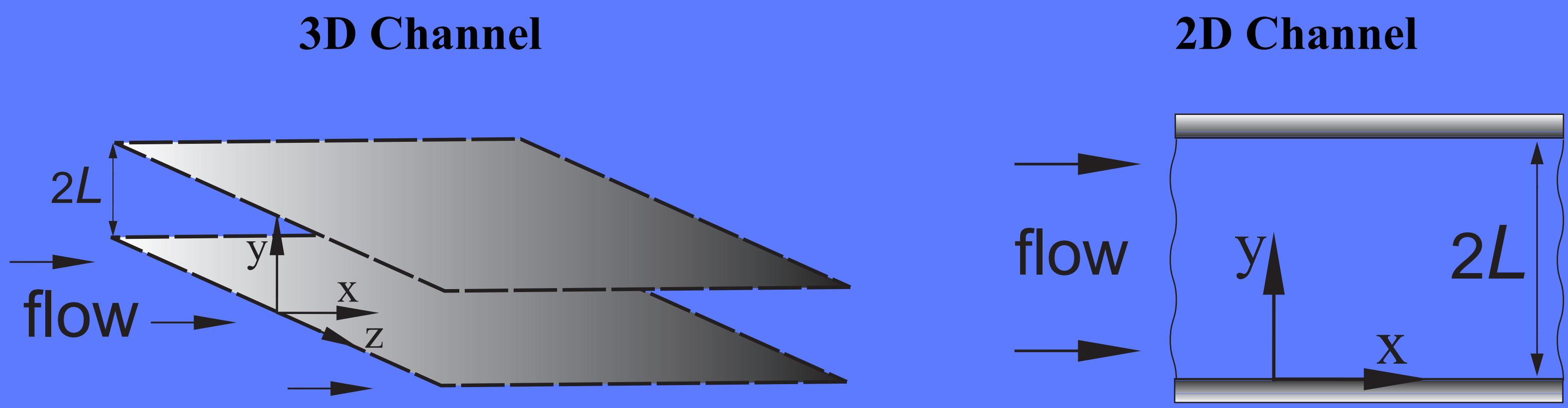
- Mean Velocity Profiles
- Kinetic Energy Profiles
- Reynolds Stress Profiles
- Kinetic Energy Balance

Conclusions



Problem Formulation

Stationary fully developed turbulent flow



For 3D:
z – spanwise direction,
$-\infty \le z \le \infty$.

x – streamwise direction,
$-\infty \le x \le \infty$,
y – wall-normal direction,
$0 \le y \le 2L$.

Driven force – constant pressure gradient: $p' \equiv -d\left\langle p(x)\right\rangle / dx > 0$.

Fluid velocity (Reynolds decomposition): $\mathbf{U}(\mathbf{r}) = V(y)\hat{\mathbf{x}} + \mathbf{u}(\mathbf{r}), \quad V(y) \equiv \left\langle \mathbf{U}(\mathbf{r})\right\rangle$.

Mean Momentum balance

$$\nu S(y) + W(y) = p'(L - y).$$

Mean Shear: $S(y) \equiv \frac{dV(y)}{dy},$

Reynolds Shear Stress: $W(y) \equiv -\langle u_x u_y \rangle,$

Turbulent Kinetic Energy: $K(y) \equiv \frac{1}{2}\langle \mathbf{u}^2 \rangle.$

Kinetic Energy Balance

$$P(y) = \varepsilon(y) + D(y),$$

Energy Production: $P(y) = W(y)S(y),$

Energy Dissipation: $\varepsilon(y) = \nu\left\langle\left(\partial_k u_i\right)^2\right\rangle,$

Energy Diffusion: $D(y) = \frac{d}{dy}\left[\frac{1}{2}\left\langle u_y\left(\mathbf{u}^2 + \tilde{p}\right)\right\rangle - \nu\frac{d}{dy}K(y)\right].$

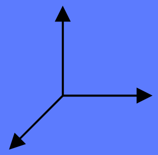
Model for Diffusion: $D(y) \approx \frac{d}{dy}\left\{\left[-\nu_T(y)\frac{d}{dy}K(y)\right] - \nu\frac{d}{dy}K(y)\right\},$

$$\nu_T(y) = a\,\ell(y)\sqrt{K(y)}\quad .$$

Dissipation in the bulk

Dissipation: $$\varepsilon(y) \approx \nu \int \mathrm{d}k\, k^2 \tilde{K}(k),$$

Kinetic energy: $$K(y) \approx \int_{1/\ell(y)}^{\infty} \mathrm{d}k\, \tilde{K}(k).$$



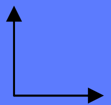
3D Channel

Direct *energy* cascade, Kolmogorov spectrum

$$\tilde{K}_{\mathrm{3D}} \sim \varepsilon^{2/3} k^{-5/3},$$

$$\varepsilon_{\mathrm{3D}} \sim \frac{K^{3/2}}{\ell},$$

2D Channel



Direct *enstropy* cascade, Kraichnan spectrum

$$\tilde{K}_{\mathrm{2D}} \sim \beta^{2/3} k^{-3} \ln^{-1/3}\left[k\ell(y)\right],$$

$$\varepsilon_{\mathrm{2D}} \sim \nu \frac{K}{\ell^2} \ln^{2/3}\left[\nu^{-1}\ell\sqrt{K} + \mathrm{const}\right],$$

Dissipation near walls

Near wall expansion: $$\varepsilon(y) \underset{y\to 0}{\to} 2\nu K(y)/y^2\,,$$

$$\varepsilon(y) \approx 2\nu \frac{K(y)}{\ell(y)^2}, \quad \ell(y) \underset{y\to 0}{\to} y\,.$$

<u>Model</u> <u>for Dissipation</u>:

$$\varepsilon(y) \approx \begin{cases} 2\nu \dfrac{K(y)}{\ell(y)^2} + b\dfrac{K^{3/2}(y)}{\ell(y)}, & \text{3D}\,, \\ 2\nu \dfrac{K(y)}{\ell^2(y)} \ln^{2/3}\left[\nu^{-1}\ell(y)\sqrt{K(y)} + e\right], & \text{2D}\,. \end{cases}$$

Reynolds Stress Balance

Boussinesq closure:

$$W(y) \approx \nu_{\mathrm{T}}(y) S(y),$$

$$\nu_{\mathrm{T}}(y) \sim \ell(y)\sqrt{K(y)}.$$

$$r_{\mathrm{W}}(y) W(y) \approx c\,\ell(y)\sqrt{K(y)}\, S(y),$$

$$r_{\mathrm{W}}(y) = \left[1 + \left(\ell_{\mathrm{buf}}/y\right)^6\right]^{1/6}.$$

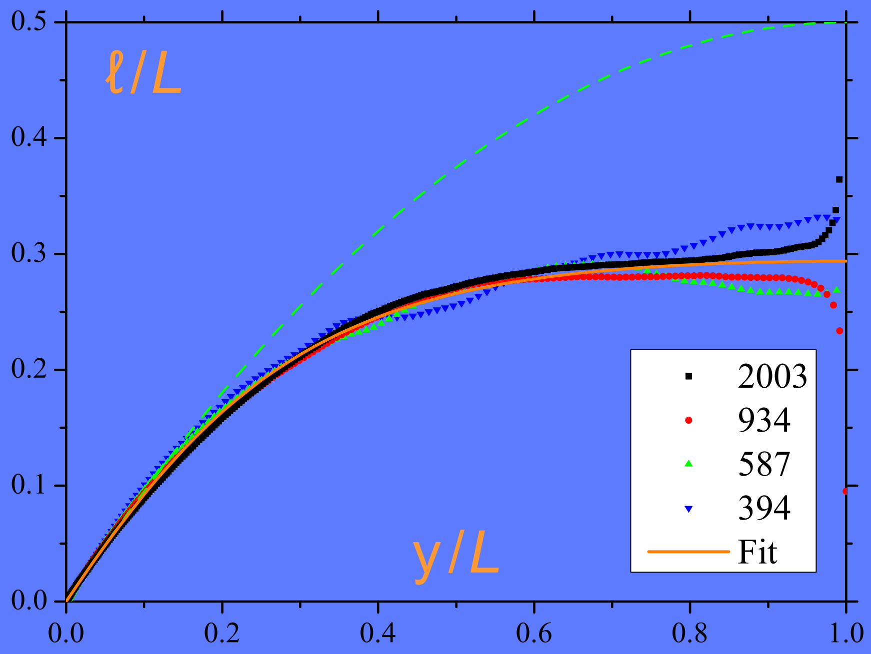
By fit to (3D) **D**irect **N**umerical **S**imulations (DNS) data:

$$\ell_{\mathrm{buf}}^{+} \equiv \ell_{\mathrm{buf}} \sqrt{p'L}\big/\nu \approx 43.$$

V. S. L'vov, I. Procaccia, and O. Rudenko, *Phys. Rev. Lett.* **100**, 054504 (2008).

"Outer scale" of turbulence

"Extracted" from 3D channel and adopted for 2D channel.



$$L_S = 0.311\,L\,,$$

$$\lambda(y) = y\left(1 - y/2L\right)/L_S\,,$$

$$\ell(y) = L_S\left\{1 - \exp\left[-\lambda\left(1 + \frac{\lambda}{2}\right)\right]\right\}.$$

Symbols: **D**irect **N**umerical **S**imulations (DNS) at four Reynolds numbers by

S. Hoyas and J. Jimenez, *Phys. Fluids* **18**, 011702 (2006);
R. G. Moser, J. Kim, and N. N. Mansour, *Phys. Fluids* **11**, 943 (1999);

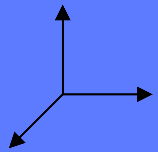
The fit (orange solid line) is proposed in

V. S. L'vov, I. Procaccia, and O. Rudenko, *Phys. Rev. Lett.* **100**, 054504 (2008).

Final Set of Equations

$$\nu S + W = p'(L-y)\,, \qquad \left[1+\left(\ell_{\mathrm{buf}}/y\right)^6\right]^{1/6} W \approx c\,\ell\sqrt{K}\,S\,,$$

$$WS + \frac{d}{dy}\left[\left(a\,\ell\sqrt{K}+\nu\right)\frac{d}{dy}K\right] \approx \begin{cases} 2\nu\dfrac{K}{\ell^2} + b\dfrac{K^{3/2}}{\ell}\,, & \mathrm{3D}\,,\\[2ex] 2\nu\dfrac{K}{\ell^2}\ln^{2/3}\left[\nu^{-1}\ell\sqrt{K}+e\right], & \mathrm{2D}\,. \end{cases}$$



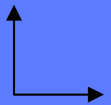
3D Channel

$$\kappa_{\mathrm{3D}} = \left(c^3/b\right)^{1/4} \approx 0.415\,.$$

By fit to (3D) DNS data:

$$a \approx 0.218\,,\quad b \approx 0.310,\quad (c \approx 0.386)\,.$$

2D Channel



$$\kappa_{\mathrm{2D}} \approx 0.2\,,\quad \text{at } \mathrm{Re}_\tau \sim 10^3.$$

N. Gutternberg and N. Goldenfeld,
Phys. Rev. E **79**, 065306(R) (2009)

$$c \approx 0.047\,.$$

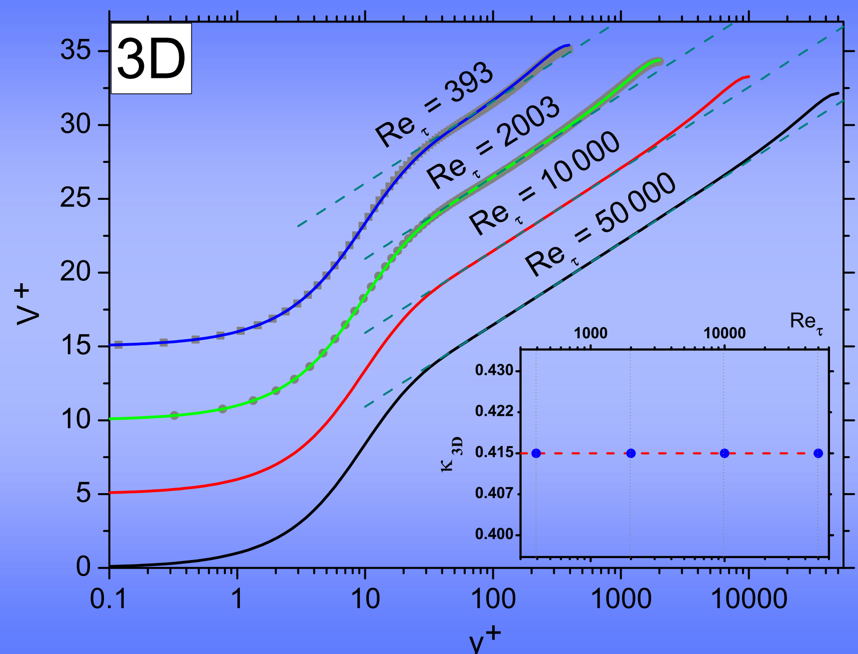
We <u>fix</u> the same values for 2D and 3D for $a,\ \ell_{\mathrm{buf}},\ L_s$.

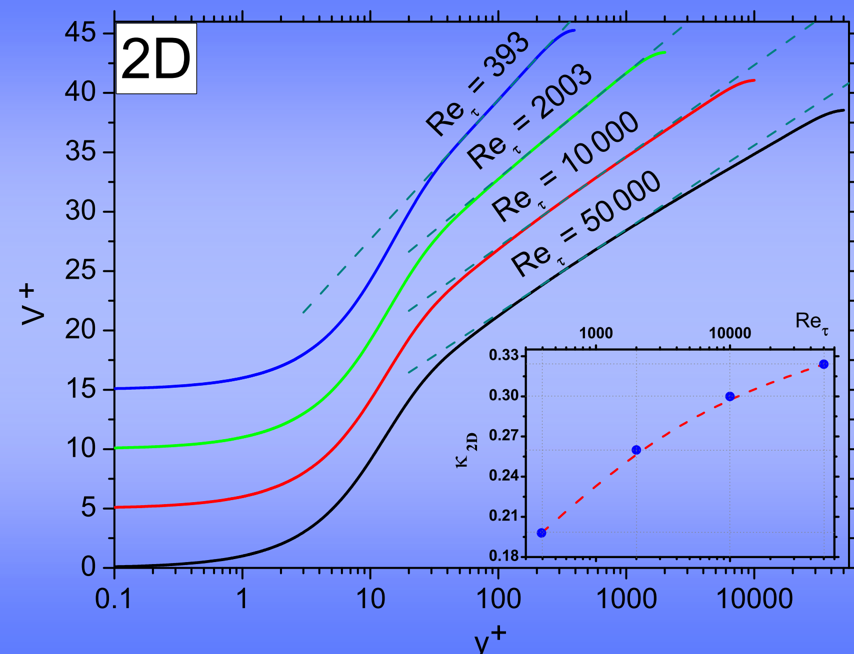
Mean Velocity Profiles

Wall Units: $u_\tau = \sqrt{p'L}, \quad \ell_\tau = \nu / u_\tau \,.$

Friction Reynolds Number: $\mathrm{Re}_\tau = L u_\tau / \nu = L^+ \,.$

$$V^+ = V/u_\tau \,, \quad y^+ = y/\ell_\tau = y\nu/u_\tau \,.$$

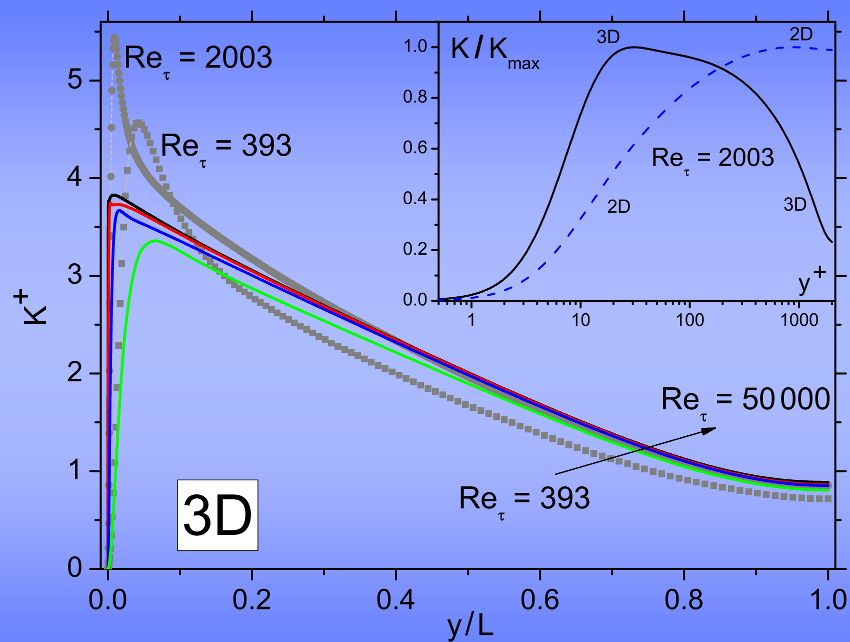


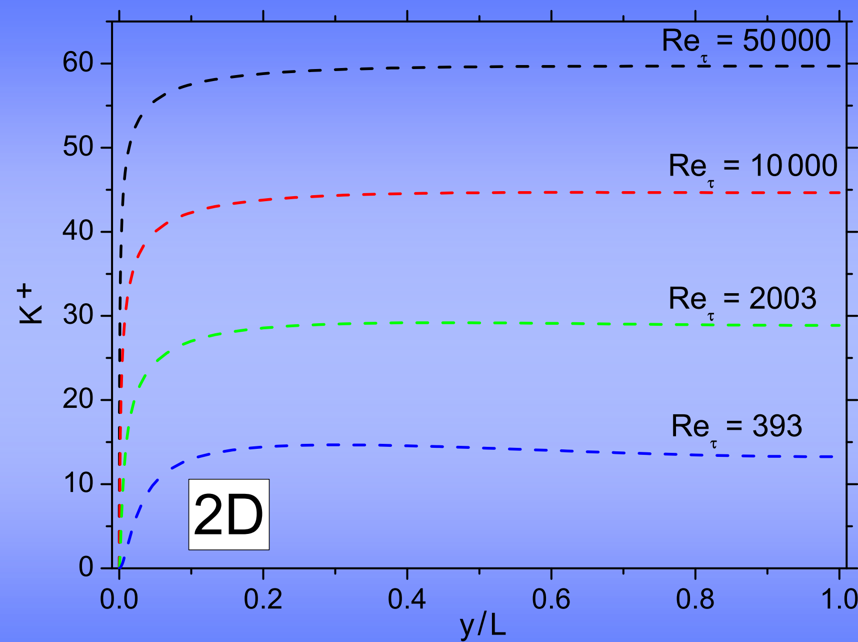


Symbols: Direct Numerical Simulations

Kinetic Energy Profiles

$$K^+ = K/u_\tau^2\,,\quad \mathrm{Re}_\tau = L u_\tau/\nu\,.$$





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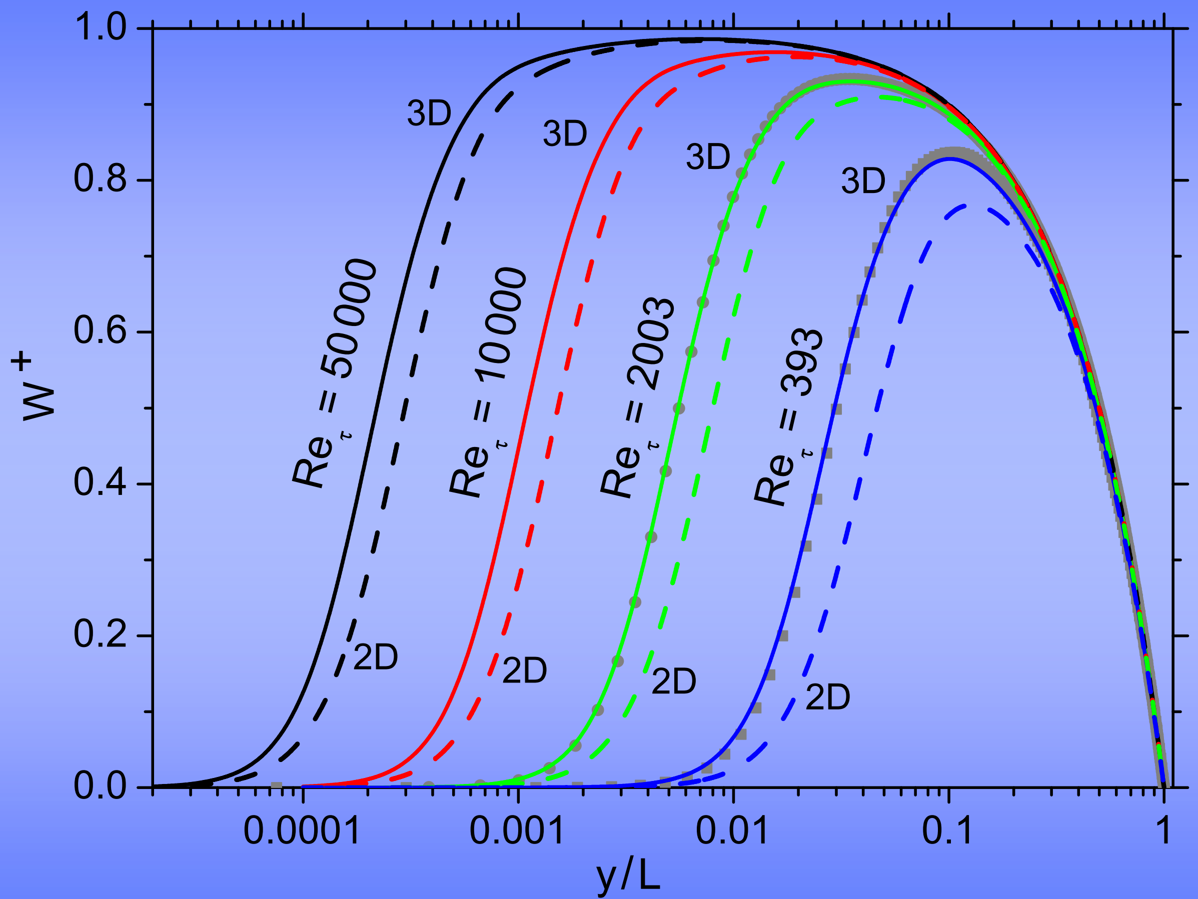
Reynolds Stress Profiles

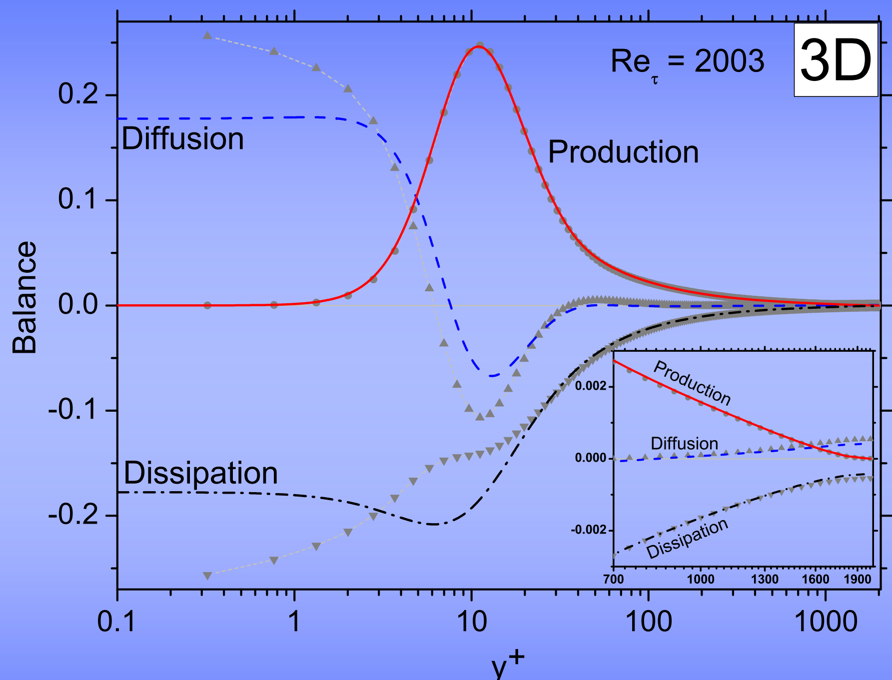
$$W^+ = W/u_\tau^2 , \quad \mathrm{Re}_\tau = L u_\tau / \nu .$$

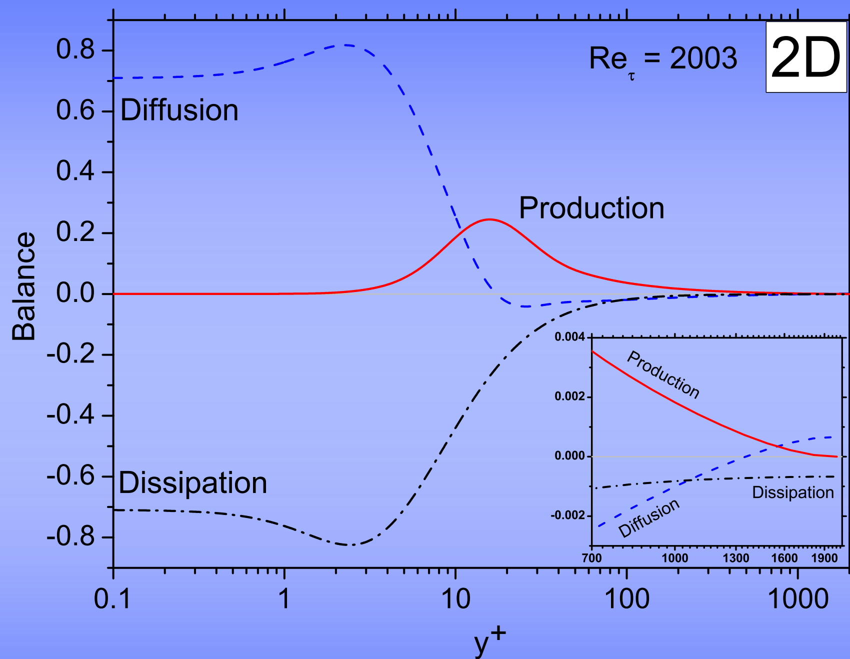


Symbols: Direct Numerical Simulations

Kinetic Energy Balance

$$P^+\left(y^+\right) = \varepsilon^+\left(y^+\right) + D^+\left(y^+\right), \quad y^+ = y/\ell_\tau = y\nu/u_\tau \,.$$





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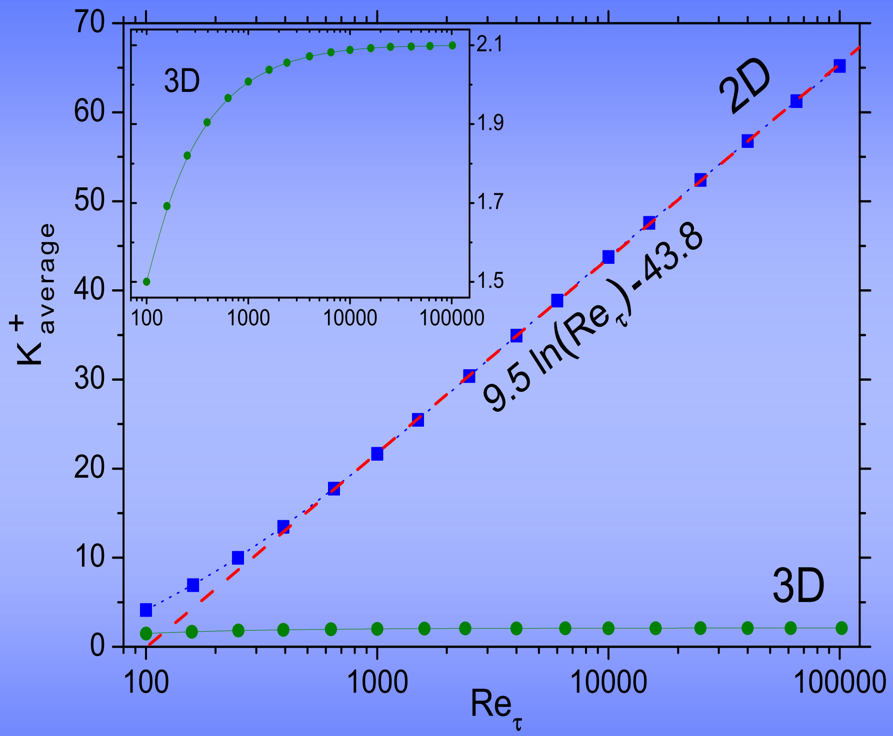
Conclusions

- Model for 2D and 3D turbulent channel flows: profiles of $V(y)$, $W(y)$, $K(y)$, as well as $S(y)$, $D(y)$, $\varepsilon(y)$, TKE balance, etc.;
- Direct *energy* cascade (3D) & direct *enstrophy* cascade (2D): difference in the dissipation of $K(y)$;
- Reynolds stress profiles, $W(y)$, in 2D and 3D look similar;
- Von-Kármán log-law: "exists" in 3D, just apparent in 2D;
- 2D channel is much energetic, with $K \sim \ln(Re_\tau)$.

Kinetic Energy *vs.* Re_τ

$$K^+ = K/u_\tau^2\,, \quad \mathrm{Re}_\tau = L u_\tau/\nu\,.$$



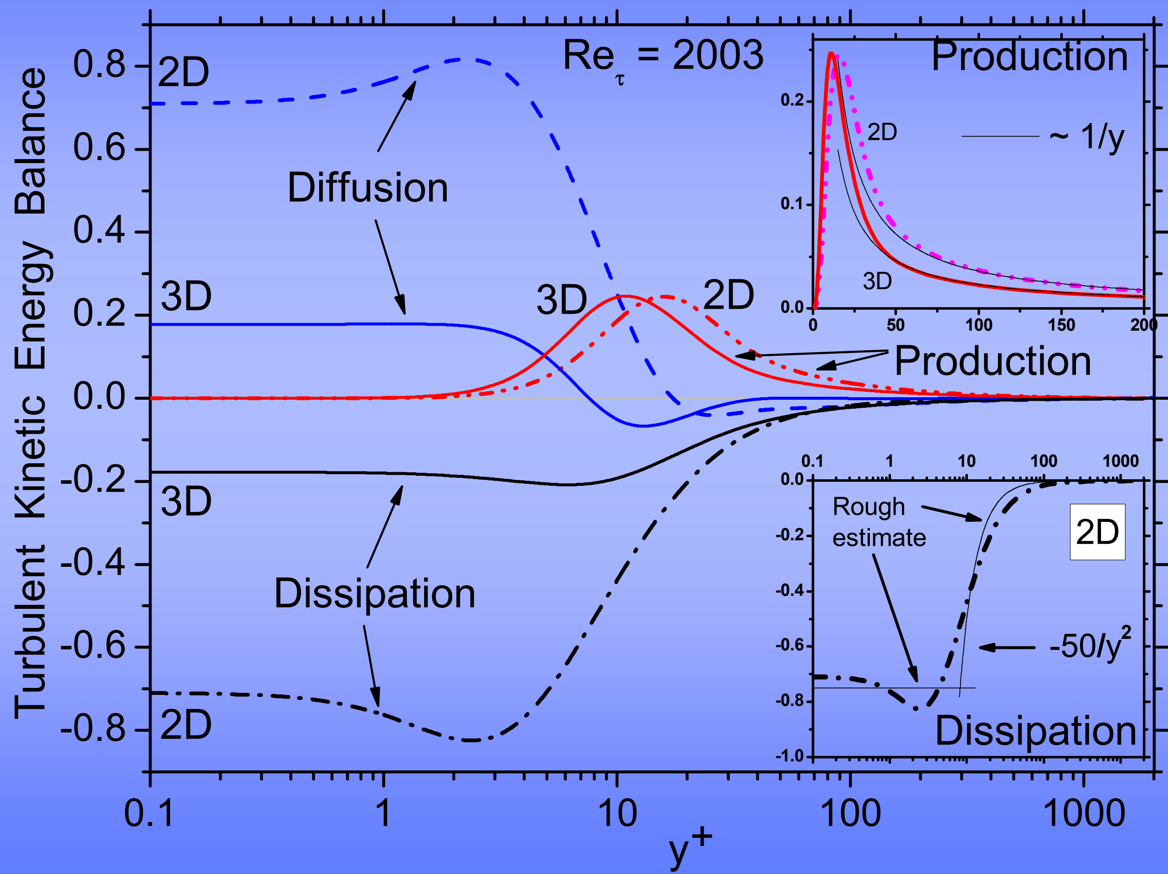
$$K_{\mathrm{average}} \equiv \frac{1}{L}\int_0^L K(y)\,\mathrm{d}y \sim u_\tau^2 \ln\left(\mathrm{Re}_\tau\right).$$

THANK YOU VERY MUCH!

THE END.

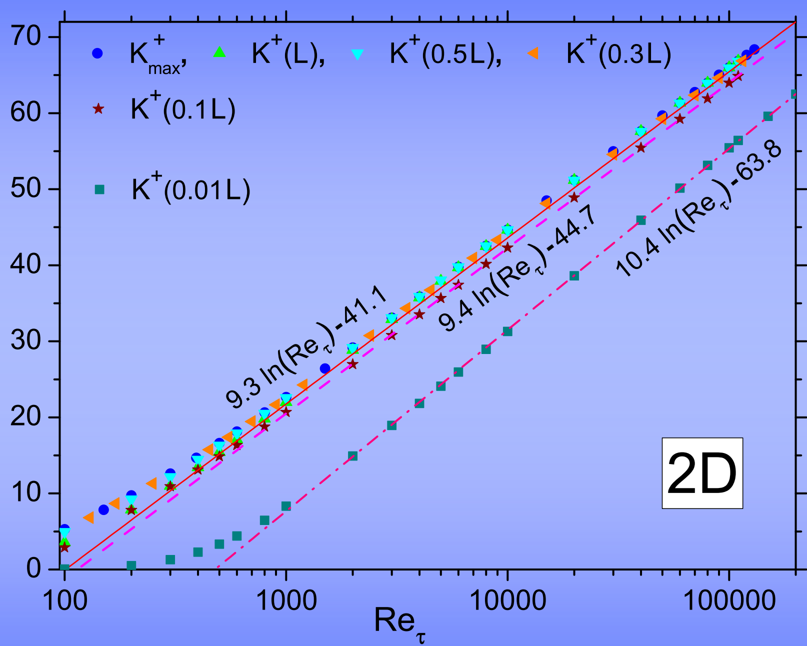
Appendix: TKE Balance

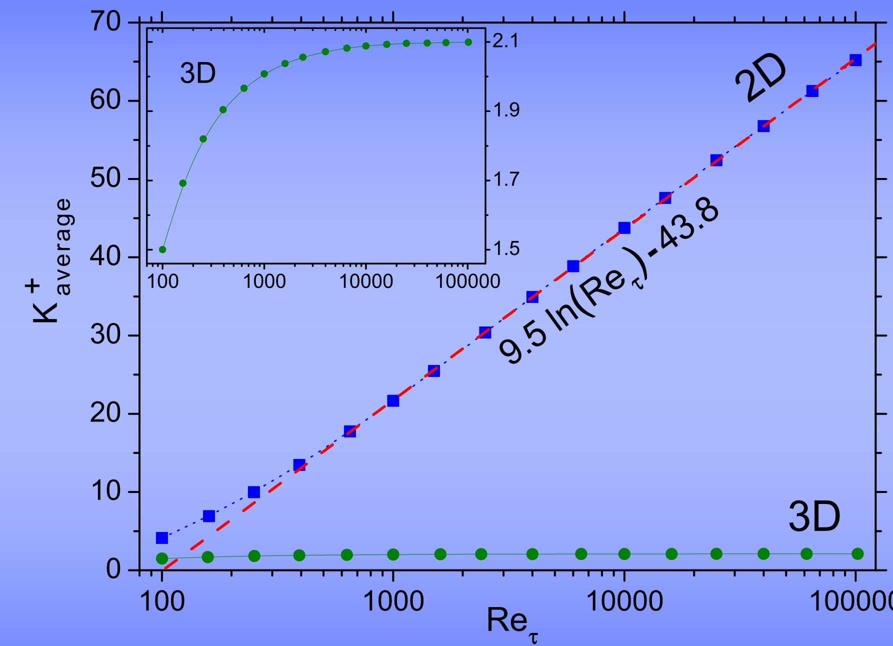
$$P^{+}\left(y^{+}\right)=\varepsilon^{+}\left(y^{+}\right)+D^{+}\left(y^{+}\right), \quad y^{+}=y/\ell_{\tau}=y\nu/u_{\tau} .$$



Kinetic Energy *vs.* Re_τ

$$K^+ = K/u_\tau^2\,,\quad \mathrm{Re}_\tau = L u_\tau/\nu\,.$$



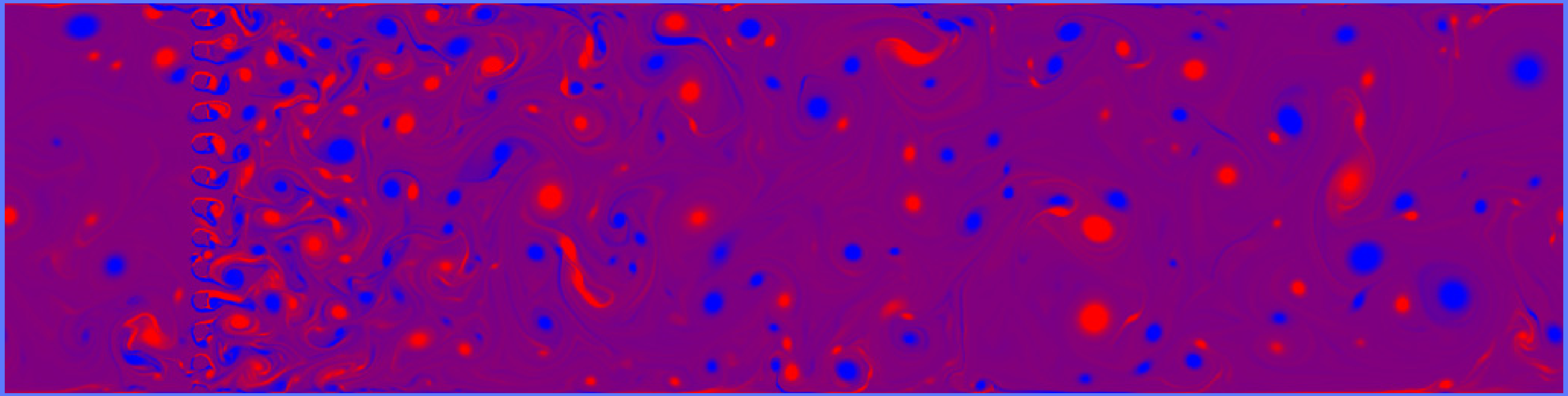


$$K_{\text{typical}} \sim \ln\left(\mathrm{Re}_\tau\right) + \text{const}\,.$$

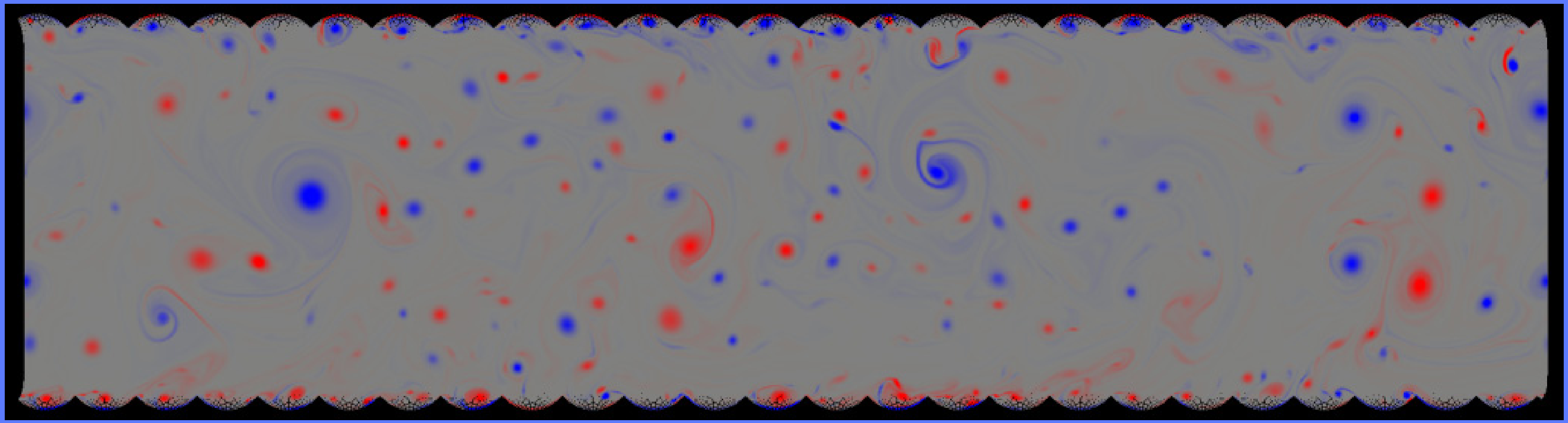
$$K_{\text{average}} \equiv \frac{1}{L}\int_0^L K\left(y\right)\mathrm{d}y\,.$$

DNS by Nicholas Guttenberg

Nicholas Gutternberg and Nigel Goldenfeld, *Phys. Rev. E* **79**, 065306(R) (2009):



Grid generated turbulence in a **smooth** pipe. The color scheme indicates positive (blue) and negative (red) vorticity.



Roughness generated turbulence in a **rough** pipe generated via conformal mapping.