

# Hybrid Stochastic – Statistical Strategies in Climate Science

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# Distinct Metastable Atmospheric Regimes Despite Nearly Gaussian Statistics: A Paradigm Model

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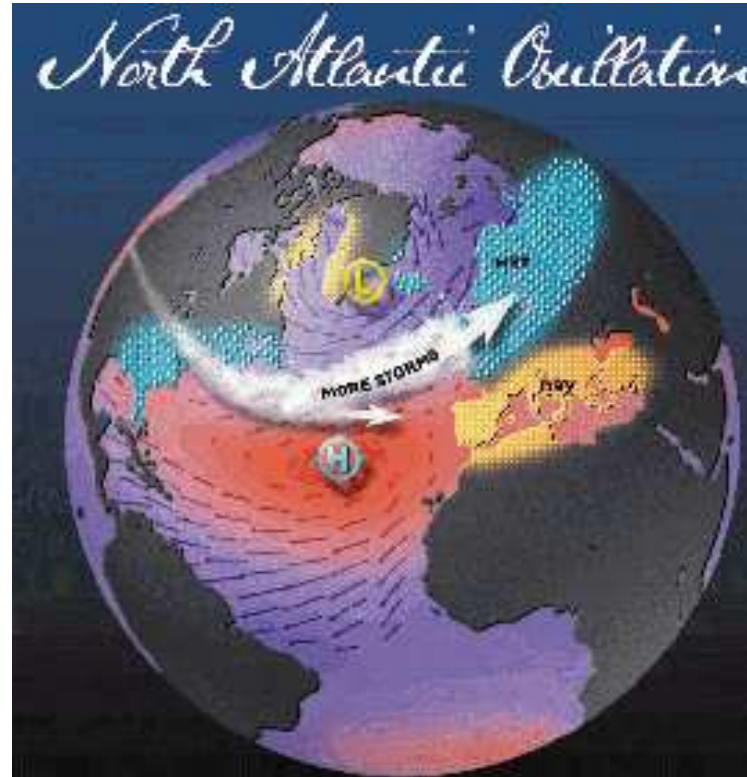


## Outline

- What are Metastable Atmospheric Regimes?
- Paradigm model: Barotropic flow over topography
- Objective regime identification through Hidden Markov Models (HMM)



# Why are Atmospheric Regimes important?



Source: [www.ldeo.columbia.edu/NAO](http://www.ldeo.columbia.edu/NAO)

- The structure of the low-frequency regime transitions among persistent teleconnection patterns (e.g. NAO and PNA) is of central importance for both long-range weather prediction and climate change projection



# What are Metastable Atmospheric Regimes?

- Possible Definitions of Metastable Atmospheric Regimes
  - Atmospheric regimes are slowly evolving or quasi-steady flow fields
  - Atmospheric regimes are recurring flow patterns (This does not necessarily mean a slow down of planetary waves)
  - Atmospheric regimes are regions in phase space where the trajectories slow down
  - Atmospheric regimes are regions in phase space with different dynamics



# What are Metastable Atmospheric Regimes?

- Previous Approaches
  - Fixed points of highly truncated models (Charney and DeVore 1979, JAS)
  - Multiple Attractors (Itoh and Kimoto 1997, JAS)
  - Recurrent Patterns: Identified by Cluster analysis (Cheng and Wallace 1993, JAS; Mo and Ghil 1988, JAS)
  - Multiple Extrema in PDF's (Corti et al. 1999, Nature)
  - Gaussian Mixtures (Smyth et al. 1999, JAS)
- Problem
  - Long integrations of GCM's show nearly Gaussian statistics
- Are there distinct atmospheric regimes despite nearly Gaussian statistics?



# Metastable Atmospheric Regimes: A Paradigm Model

- Paradigm model:
  - Barotropic flow over topography
  - Low-frequency waves: Blocked and Zonal states
  - Nearly Gaussian behavior
  - Truncated low-order model is Charney-DeVore model (1979; JAS)



## Metastable Atmospheric Regimes: A Paradigm Model

The barotropic quasi-geostrophic equations with a large scale zonal mean flow  $U$  on a  $2\pi \times 2\pi$  periodic domain are given by

$$\frac{\partial q}{\partial t} + \nabla^\perp \psi \cdot \nabla q + U \frac{\partial q}{\partial x} + \beta \frac{\partial \psi}{\partial x} = 0$$

$$q = \Delta \psi + h$$

$$\frac{dU}{dt} = \frac{1}{4\pi^2} \int h \frac{\partial \psi}{\partial x} dx dy$$

The model is truncated at  $|\mathbf{k}|^2 \leq 17$  (57 degrees of freedom).

Majda, A. J., I. Timofeyev and E. Vanden-Eijnden, 2003: Systematic Strategies for Stochastic Mode Reduction in Climate, J. Atmos. Sci.

Majda, A. J., C. L. Franzke, A. Fischer, and D. T. Crommelin, 2006: Distinct Metastable Atmospheric Regimes Despite Nearly Gaussian Statistics: A Paradigm Model, PNAS.





# Hidden Markov Model (HMM)

The conditional independence relations between  $X$  and  $Y$  are defined by the factorization

$$P(X_1, \dots, X_T, Y_1, \dots, Y_T) = P(X_1)P(Y_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(Y_t|X_t)$$

A HMM is defined by the following components:

- $N$  hidden States  $S = s_1, s_2, \dots, s_N$
- the observation space  $V \subset \mathbb{R}^d$
- a  $(N \times N)$  stochastic transition matrix  $A = (a_{ij})$
- a stochastic vector  $\pi = (\pi_1, \dots, \pi_N)$
- probability distributions  $B_n, n = 1, \dots, N$  on  $V$

Parameter estimation by EM and Viterbi algorithms

References: Rabiner (1989), Ghahramani (2001), Fischer et al. (2006)



## HMM Analysis for Metastable Regimes

$$A^{0.2} = \begin{pmatrix} 0.985 & 0.015 \\ 0.016 & 0.984 \end{pmatrix},$$

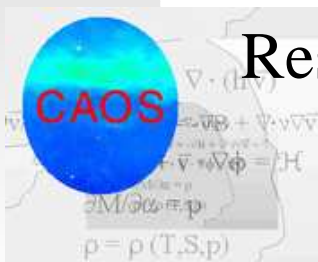
$$B_1 = \mathcal{N}(-0.035, 0.304), B_2 = \mathcal{N}(-0.789, 0.119)$$

$$\text{eigenvalues of } A^{0.2}: \lambda_1(A^{0.2}) = 1, \lambda_2(A^{0.2}) = 0.969$$

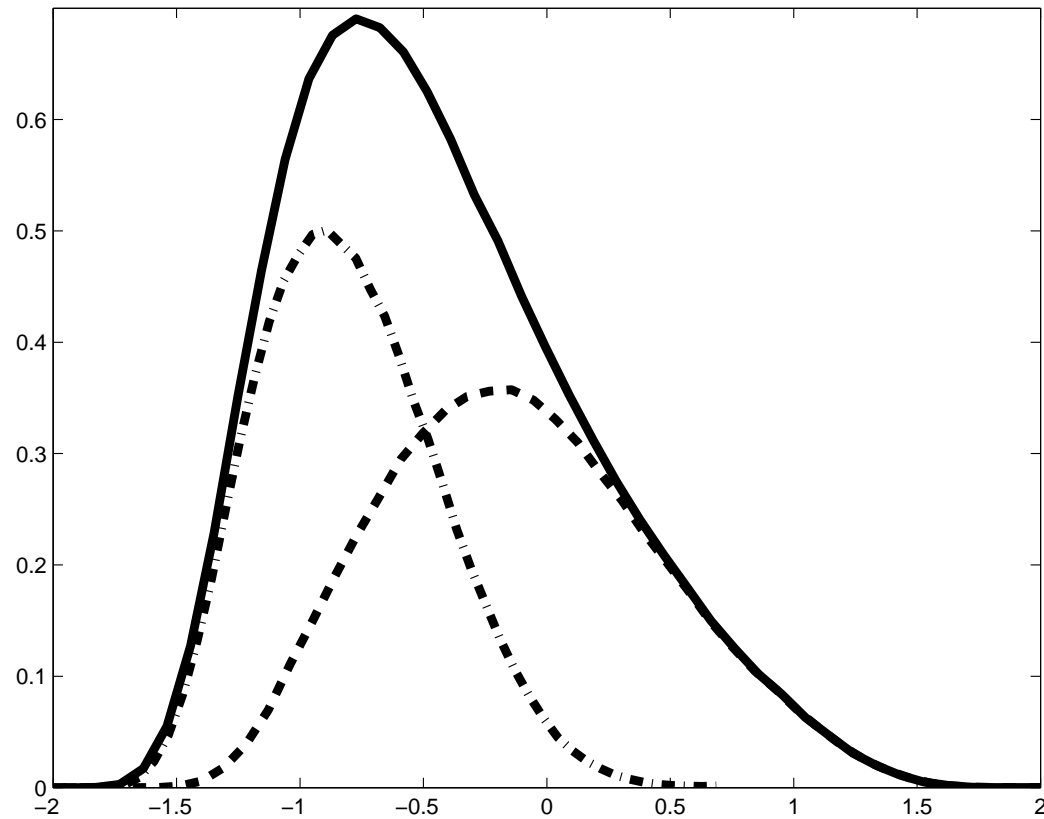
$$\text{Invariant distribution of } A^{0.2}: (0.529, 0.471) \quad (2)$$

Autocorrelation time scale of  $U$ : 5 time units

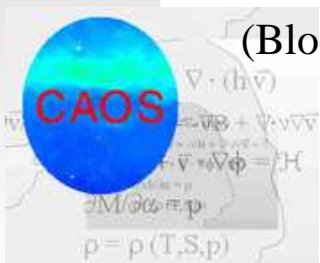
Residence time: H1 20 time units; H2 15 time units



# HMM Analysis for Metastable Regimes

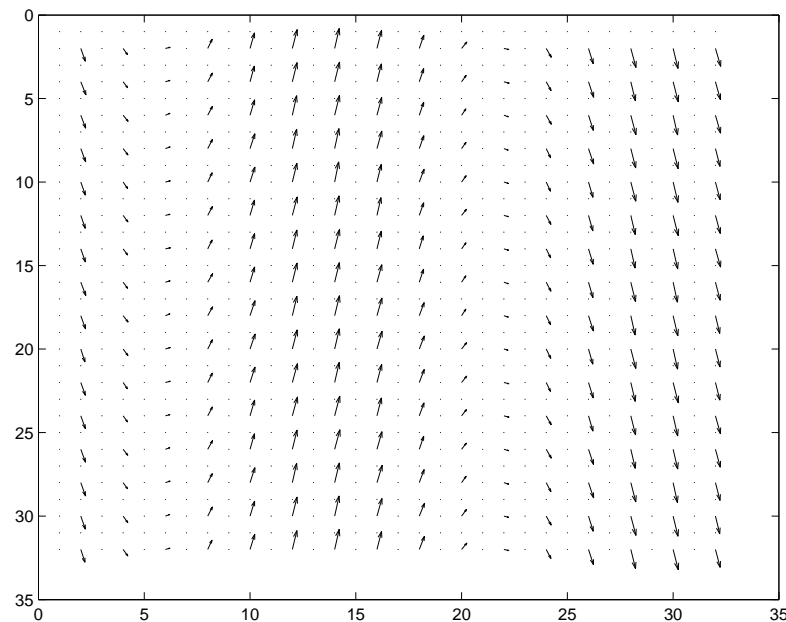


Climatological marginal PDF of  $U$  (solid line) and weighted conditional PDF's of hidden state 1 (Blocked flow, dashed line) and hidden state 2 (Zonal flow, dashed-dotted line).

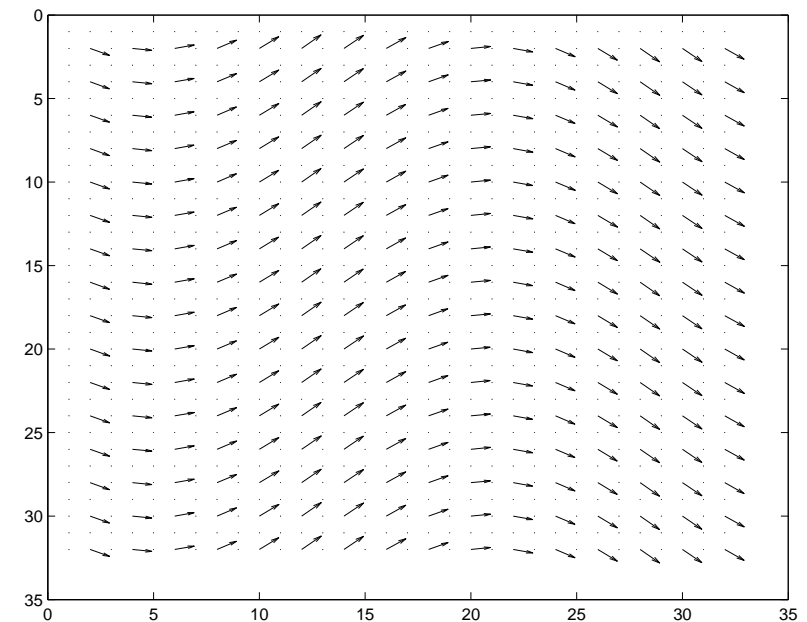


# HMM Analysis for Metastable Regimes

a) Hidden State 1: Blocked Flow



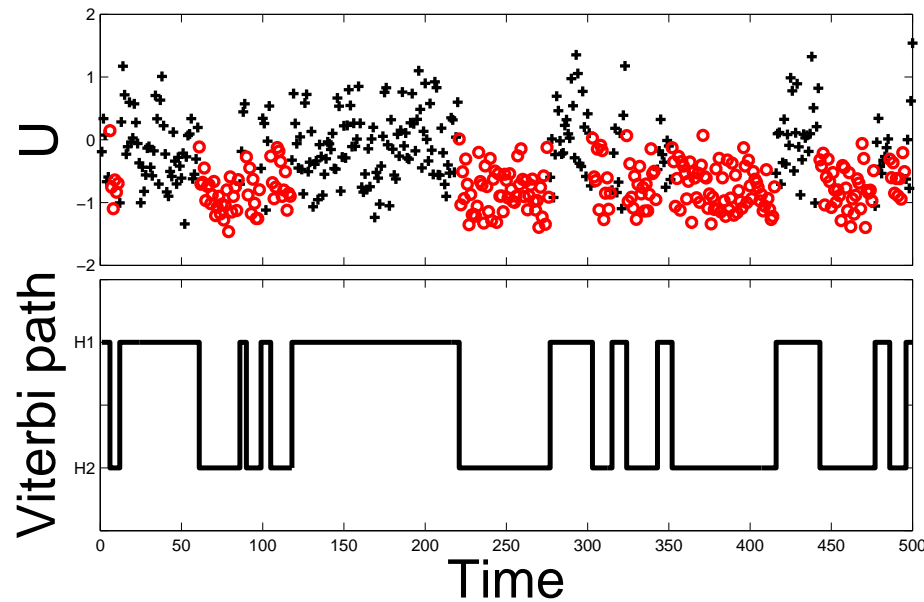
b) Hidden State 2: Zonal Flow



Velocity field conditioned on the Viterbi path of a HMM analysis in the subspace  $U$  for a) hidden state 1 (Blocked flow), and b) hidden state 2 (Zonal flow).



# HMM Analysis for Metastable Regimes



U (upper panel) and Viterbi path (lower panel). For  $U$  path black crosses and red circles denote states which correspond to hidden state 1 (Blocked flow) and 2 (Zonal flow), respectively.



## Why only 2 hidden states?

Analysis with 4 hidden states

eigenvalues of  $A^{0.2}$ :  $\lambda_1(A^{0.2}) = 1, \lambda_2(A^{0.2}) = 0.972,$

$$\lambda_3(A^{0.2}) = 0.930, \lambda_4(A^{0.2}) = 0.731$$

Invariant distribution of  $A^{0.2}$ : (0.137, 0.125, 0.345, 0.393)



## Empirical reduced equation for $U$

$$dU = B(U)dt + \sqrt{A(U)}dW$$

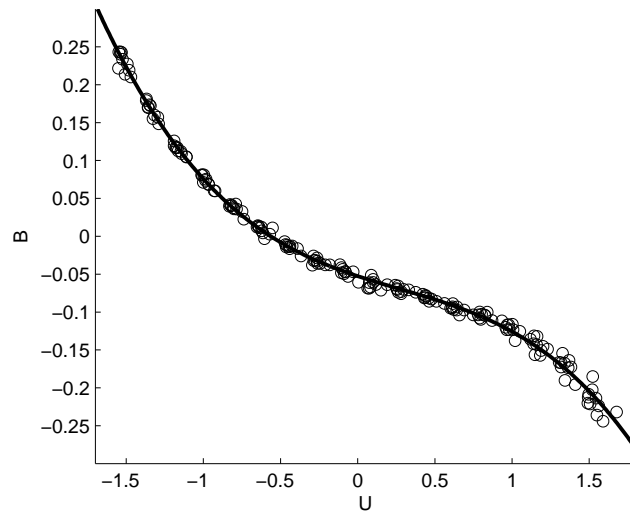
- $B(U)$  is the drift coefficient
- $\frac{A(U)}{2} > 0$  is the diffusion coefficient
- $A$  and  $B$  are estimated from observed  $U$
- $W$  is Brownian motion.

Crommelin, D. T., and E. Vanden-Eijnden (J. Comp. Phys.; Comm. Math. Sci. 2006)

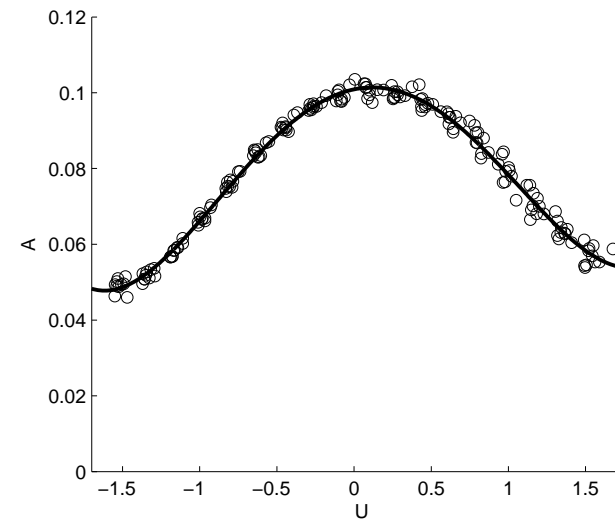


# Drift and Diffusion

a) Drift



b) Diffusion



Reconstructed a) Drift  $B$  and b) Diffusion  $A$  from time series variable  $U$ . The open circles are the result of the reconstruction, carried out 10 times on 10 different (non-overlapping) segments of the time series.





## Metastable Regimes

$$A^{0.2} = \begin{pmatrix} 0.990 & 0.010 \\ 0.016 & 0.984 \end{pmatrix},$$

$$B_1 = \mathcal{N}(-0.748, 0.086), B_2 = \mathcal{N}(0.209, 0.200)$$

eigenvalues of  $A^{0.2}$ :  $\lambda_2(A^{0.2}) = 0.9744$



## Summary and Conclusions

- HMM are utilized for objective atmospheric regime identification
- Two regimes are identified, which correspond to blocked and zonal flow
- Low-order stochastic models capture regime behavior
- This offers potential for using reduced stochastic models for long-range predictability



# Normal Forms for Reduced Stochastic Climate Models

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Majda, Franzke and Crommelin, 2009: Normal forms for reduced stochastic climate models. Proc. Natl. Acad. Sci. USA, 106, 3649-3653.



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# Normal Form for Reduced Stochastic Climate Models

- The systematic development of reduced low-dimensional stochastic climate models from observations or comprehensive high-dimensional climate models is an important topic for low-frequency variability, climate sensitivity, and improved extended range forecasting.
- The use of a few Empirical Orthogonal Functions (EOF) depending on observational data to span the low-frequency subspace requires the assessment of dyad interactions besides the more familiar triads in the interaction between the low- and high-frequency subspaces of the dynamics.
- For a single low-frequency variable the dyad interactions and climatological linear operator alone produce a normal form with Correlated Additive and Multiplicative (CAM) stochastic noise from advection of the large-scales by the small scales and simultaneously strong cubic damping. This normal form should prove useful for developing systematic regression fitting strategies for stochastic models of climate data.



## CAM Noise

The systematic stochastic mode reduction strategy (Majda et al. 1999, 2001, 2005, 2008) predicts the functional form of the deterministic and stochastic terms. In particular, it predicts two types of noises. One is a simple additive noise, which stems from the nonlinear driving of the resolved scales by the unresolved scales, and the other is CAM noise. CAM noise has the following structural form

$$\left( \vec{g} + \vec{f}(x) \right) d\vec{W} \quad (1)$$

where  $\vec{g}$  denotes a constant vector,  $\vec{f}(x)$  a function and  $\vec{W}$  a multivariate Wiener process. As (1) indicates, CAM noise acts both in a additive and multiplicative sense. The multiplicative noise stems from the nonlinear advection of the resolved scales by the unresolved scales while the additive noise part stems from the linear operator. This linear operator is derived by linearizing the equations of motion around the climatological basic state.



# Structural implications of energy conservation for the normal form

The dynamical core of comprehensive large-scale models for the climate has the form

$$\frac{du}{dt} = F + Lu + B(u, u), \quad u \cdot B(u, u) = 0 \quad (2)$$

where  $F$  is a constant forcing,  $L$  a linear and  $B$  a quadratically nonlinear operator and  $u \in \mathbf{R}^M$  denotes the state vector with  $M \gg 1$ .



# Structural implications of energy conservation for the normal form

The state vector  $u$  can be expanded into an orthonormal basis in the energy metric,  $u = \sum_{i=1}^M u_i(t) e_i$ . Then the conservation of energy constraint in (2) imposes the symmetries

$$e_i \cdot B(e_i, e_i) = 0 \quad (3a)$$

$$e_l \cdot B(e_j, e_k) + e_j \cdot B(e_k, e_l) + e_k \cdot B(e_l, e_j) = 0 \quad (3b)$$

for all indices  $l, j, k$  with  $1 \leq l, j, k \leq M$ . In a general basis, e.g. of EOF's arising from low-frequency data analysis,  $B(e_i, e_i) \neq 0$ , and there are nontrivial dyad interactions which satisfy (3) for  $k = l, j = i$

$$e_l \cdot (B(e_l, e_i) + B(e_i, e_l)) + e_i \cdot B(e_l, e_l) = 0 \quad (4)$$

for  $1 \leq l \leq M, i \neq l, 1 \leq i \leq M$ .



# Normal Form of 1D stochastic climate model

Repeating the explicit mode elimination procedure for  $\varepsilon \ll 1$  (Majda et al. 1999, 2001, 2005, 2008) and coarse graining time as  $t \rightarrow \frac{t}{\varepsilon}$ , which amounts to setting  $\varepsilon = 1$  yields the *normal form for scalar stochastic climate models* in Stratonovich form

$$\frac{dx}{dt} = \{L_{11}x + F_1\} \quad (6a)$$

$$+ \sum_p \left\{ -\frac{I_{1p}^{M^2}}{\gamma_p} x^3 + \frac{\sigma_p}{\gamma_p} (L_{1p} - I_{1p}^M x) \circ \dot{W}_p \right\} \quad (6b)$$

$$+ \sum_p \left\{ \frac{L_{1p}}{\gamma_p} F_p + \left( \frac{I_{1p}^M F_p}{\gamma_p} \right) x \right\} \quad (6c)$$

$$+ \sum_p \left\{ \left( \frac{L_{1p} L_{p1}}{\gamma_p} x \right) + \frac{I_{1p}^M L_{p1}}{\gamma_p} x^2 \right\} \quad (6d)$$

$$+ L_A x + \sigma_A \dot{W}_A \quad (6e)$$





# Normal Form of 1D stochastic climate model

For ease of notation we rewrite (6) in Ito form as

$$\frac{dx}{dt} = F + ax + bx^2 - cx^3 \quad (7a)$$

$$+ \sum_p \frac{\sigma_p}{\gamma_p} \left( L_{1p} - I_{1p}^M x \right) \dot{W}_p + \sigma_A \dot{W}_A \quad (7b)$$

where  $F = F_1 + \sum_p \left( \frac{L_{1p} F_p}{\gamma_p} - \frac{\sigma_p^2}{2\gamma_p^2} L_{1p} I_{1p}^M \right)$ ,

$a = L_{11} + \sum_p \left( \frac{I_{1p}^M F_p}{\gamma_p} - \frac{\sigma_p^2}{2\gamma_p^2} I_{1p}^{M2} \right) + L_A$ ,  $b = \sum_p \frac{I_{1p}^M L_{p1}}{\gamma_p}$ , and  $c = \sum_p \frac{I_{1p}^{M2}}{\gamma_p}$ . Here  $L_A$  is a linear damping coefficient and  $\sigma_A$  the variance of the white noise and both terms arise from additive dyad or triad interactions. The white noises  $W_A$  and  $W_p$  are mutually independent.



## Normal Form of 1D stochastic climate model

We call the terms

$$\sum_p \frac{\sigma_p}{\gamma_p} \left( L_{1p} - I_{1p}^M x \right) \dot{W}_p \quad (8)$$

Correlated Additive and Multiplicative (CAM) noise. The **key point** is that when these terms are non-zero they simultaneously produce a cubic damping term in (7):

$$c = \sum_p \frac{I_{1p}^{M^2}}{\gamma_p} > 0 \leftrightarrow \text{CAM} = \sum_p \frac{\sigma_p}{\gamma_p} \left( L_{1p} - I_{1p}^M x \right) \dot{W}_p \neq 0 \quad (9)$$

Thus, CAM noise requires the presence of the cubic term since both are associated with the same dyad interactions and thus arise from the same physical process. This property has important implications for the form of the PDF and especially for the decay of its tails.

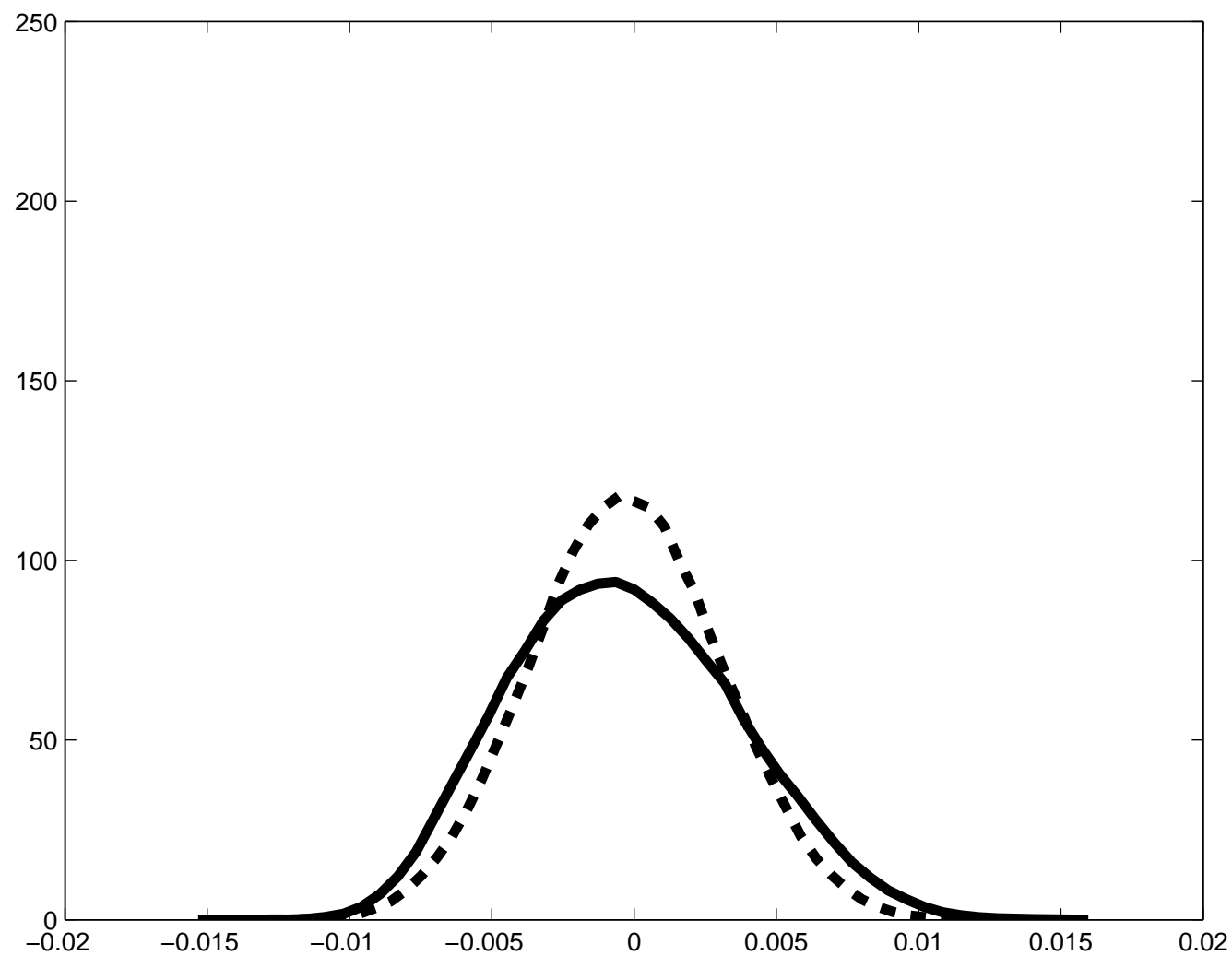


## Quasi-geostrophic model

- Global spectral model (Marshall and Molteni, JAS, 1993)
- T21 resolution ( $\sim 5.6^\circ \times 5.6^\circ$ )
- 3 Levels
- Forcing determined from ECMWF reanalysis data



# Probability Density Function



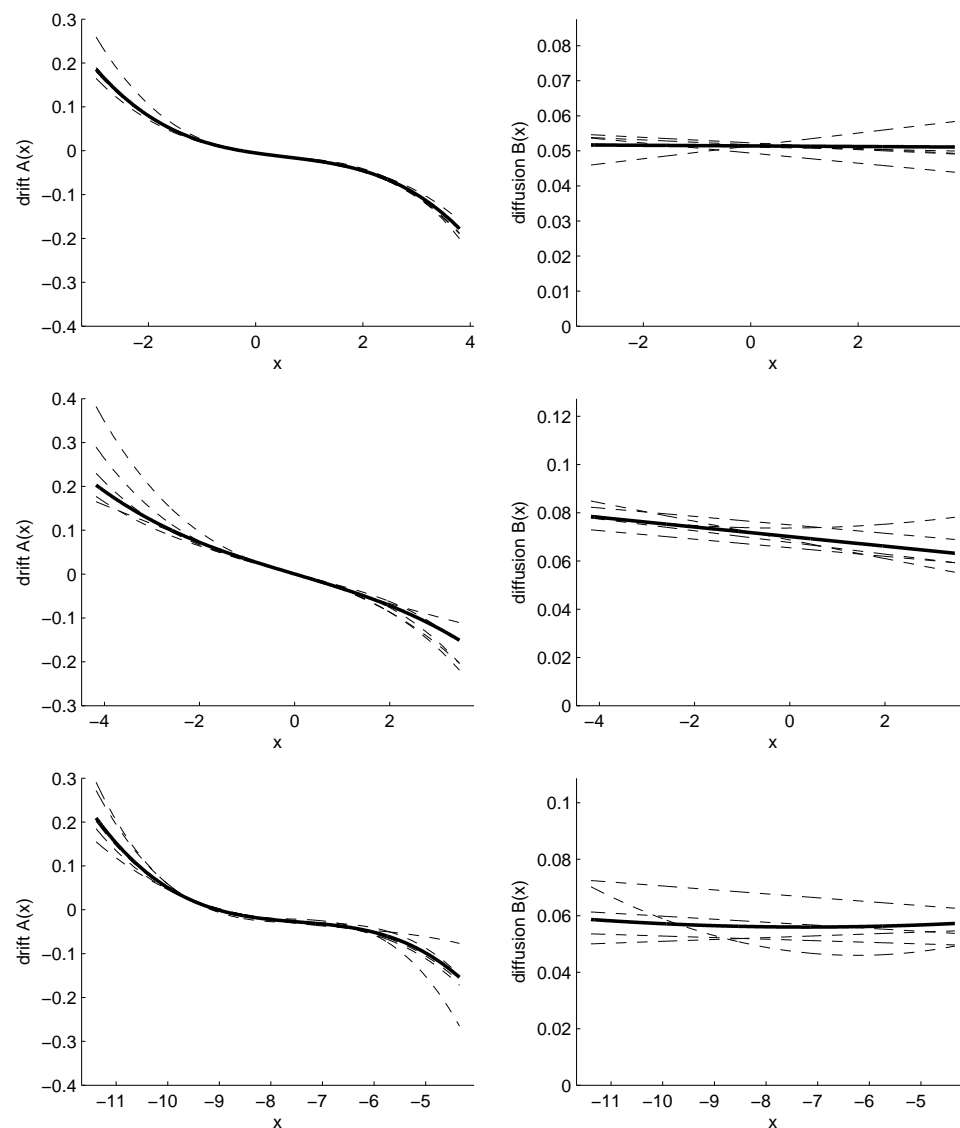
Solid line PC1 and dashed line PC2.



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# Drift and Diffusion



Drift and diffusion estimated from time series (Crommelin and Vanden-Eijnden 2006) for PC 1

# The normal form in general case of N-climate variables

It is straightforward to generalize the normal form to  $N$ -dimensions by using the above discussion together with that in including triad interactions. The result is given by

$$\frac{dx_i}{dt} = (\text{Bare truncation})_i + A_i \vec{x} \quad (10a)$$

$$+ F_i + \tilde{B}_i(\vec{x}, \vec{x}) + \sum_p \sigma_{ip}^A \dot{W}_{ip}^A \quad (10b)$$

$$- \sum_p \frac{I_{ip}^{M^2}}{\gamma_p} x_i^3 + \sum_{j \neq i, p} \frac{I_{ip}^M I_{pj}^M}{\gamma_p} x_i x_j^2 \quad (10c)$$

$$\sum_p \frac{\sigma_p}{\gamma_p} \left( L_{ip} - I_{ip}^M x_i \right) \circ \dot{W}_p \quad (10d)$$



# Constraints on the N-dimensional normal forms

$$\frac{1}{2} \frac{dE}{dt} = \sum_i \left( \sum_{j \neq i, p} \frac{I_{ip}^M I_{pj}^M}{\gamma_p} x_j^2 - \sum_p \frac{I_{ip}^{M^2}}{\gamma_p} x_i^2 \right) x_i^2 \quad (11a)$$

$$= \sum_i \left( \sum_{j \neq i} \tilde{A}_{ij} x_j^2 - \tilde{I}_i x_i^2 \right) x_i^2 \quad (11b)$$

with  $E = |\vec{x}|^2$ ,  $\tilde{A}_{ij} = \sum_p \frac{I_{ip}^M I_{pj}^M}{\gamma_p}$  and  $\tilde{I}_i = \sum_p \frac{I_{ip}^{M^2}}{\gamma_p}$ . Eq. (11) can be written in matrix notation as

$$\frac{1}{2} \frac{dE}{dt} = (\vec{x}^2)^T \left( \tilde{A} - \tilde{I} \mathbb{E} \right) \vec{x}^2 = (\vec{x}^2)^T Q \vec{x}^2 \quad (12)$$

where  $\mathbb{E}$  denotes the identity matrix. Eq. (12) is a quadratic form with  $\vec{x}^2 = (x_1^2, \dots, x_N^2)$ .

Stability of the original system now requires that  $\frac{1}{2} \frac{dE}{dt} < 0$ . This is fulfilled if the quadratic form  $Q$  is negative-definite on the positive cone in  $\mathbb{R}^N$ .



# Conclusions

- By using systematic principles new normal forms for reduced stochastic climate models for low-frequency teleconnection patterns have been developed here. Even for a scalar variable these normal forms predict a cubic nonlinear drift and a multiplicative correction to constant diffusion through CAM noise.
- The normal forms were applied in a parameter estimation strategy to fit the low-frequency patterns such as the NAO of a prototype climate model and the confirmation of the predicted nonlinear cubic drift was evident in these results.
- The normal forms also provide parameter constraints which will be useful for systematic parameter estimation procedures.

Majda, Franzke and Crommelin, 2009: Normal forms for reduced stochastic climate models.

Proc. Natl. Acad. Sci. USA, 106, 3649-3653.





# Systematic Identification of Metastable Atmospheric Regimes

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Franzke, Horenko, Majda and Klein, 2009: Systematic Metastable Atmospheric  
Regime Identification in an AGCM. J. Atmos. Sci., in press.



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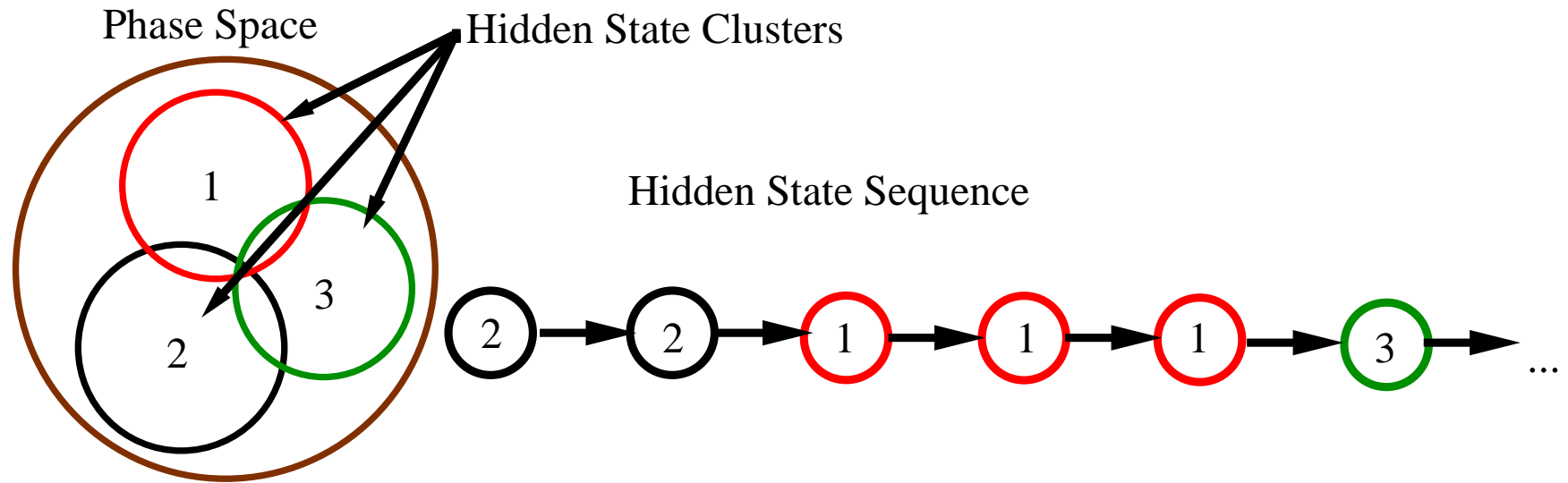
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## Why are Atmospheric Regimes important?

A pronounced characteristic of the atmospheric circulation is its irregularity with the daily change of the weather. Despite this chaotic behavior it is well known by synopticians that certain flow structures tend to occur over and over again. Synoptic meteorologists were the first to recognize the existence of persistent or recurrent weather patterns (Baur 1951), with blockings as one of the most pronounced examples (Rex 1950; Dole and Gordon 1983). An understanding of these weather patterns will be essential in successful extended range weather predictions.



# Hidden State Estimation



We **simultaneously** estimate the locations of the clusters and the most likely hidden state sequence. In previous studies *Hidden Markov Models* have been used (Majda et al. 2006 PNAS, Franzke et al. 2008 J. Climate). The new approach allows the relaxation of the Markov assumption, conditional independence relation, Gaussianity and low-dimensionality of the observed data.

## Multivariate Hidden State Estimation

- We are using a new approach to clustering of time series based on the minimization of the averaged clustering functional (Horenko 2009; SIAM J. Sci. Comp.)

$$\sum_{i=1}^K \gamma_i(t) g(x_t, \theta_i) \rightarrow \min_{\Gamma(t), \Theta} \quad (1)$$

subject to the constraints on  $\Gamma(t)$ :

$$\sum_{i=1}^K \gamma_i(t) = 1, \quad \forall t \in [0, T] \quad (2)$$

$$\gamma_i(t) \geq 0, \quad \forall t \in [0, T], i = 1, \dots, K. \quad (3)$$



## Multivariate Hidden State Estimation

We use a geometrical clustering approach (PCA algorithm) which is based on the iterative minimization of the distance from the data points to a set of  $K$  cluster centers which are recalculated in each iteration step:

$$g(x_t, \theta_i) = |x_t - \mathbf{T}_i^T \mathbf{T}_i x_t|^2 \quad (4)$$

where  $\mathbf{T}_i$  is an  $n \times m$  dimensional orthogonal projection matrix. The regularized minimization problem is solved by a **finite element** framework and the resulting hidden state sequence can be interpreted in Markovian sense (Horenko 2009; SIAM J. Sci. Comp.).



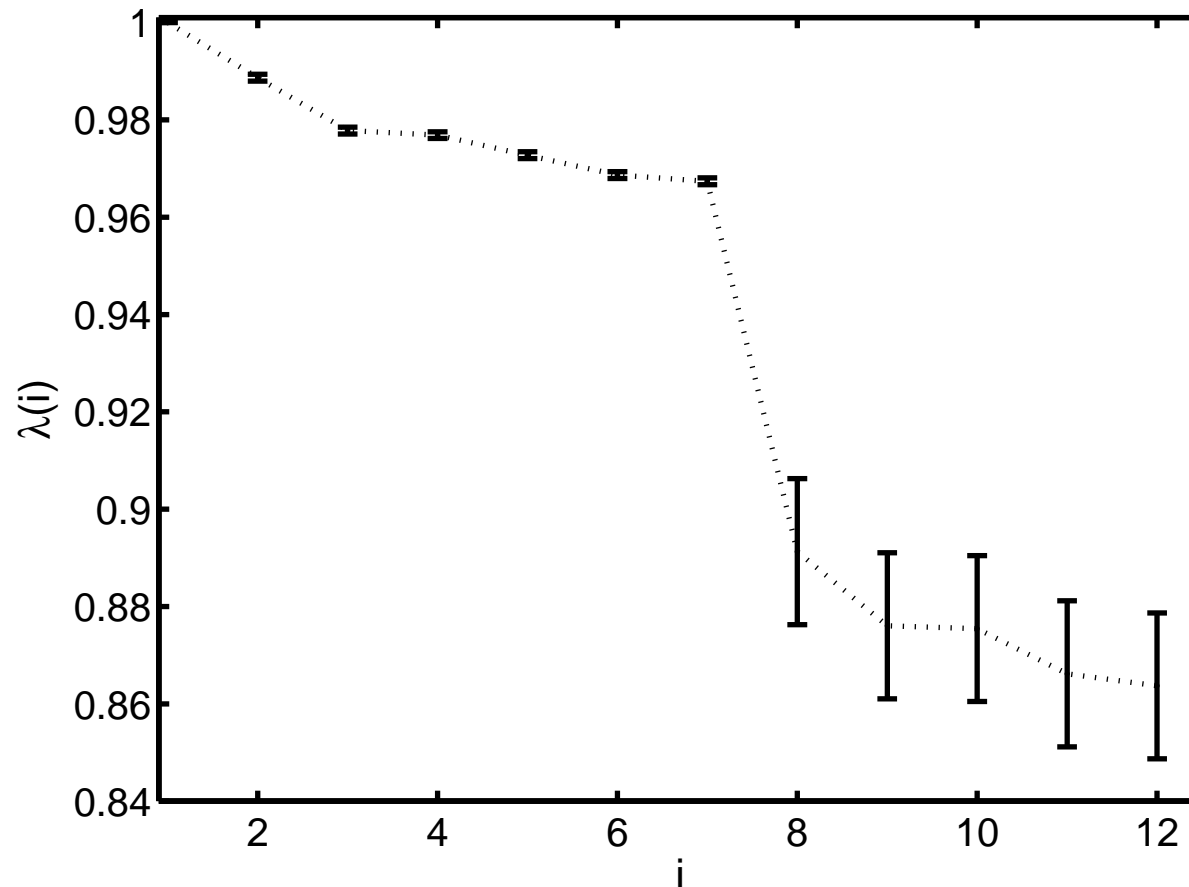
## Atmospheric GCM

- Global spectral model NCAR CCM0
- R15 resolution
- 9 Vertical Layers
- Perpetual January Boundary Conditions
- Even though the physical parameterizations of this model can no longer be regarded as state-of-the-art, its extra-tropical low-frequency variability is quite realistic.



# Eigenvalue Spectra

a)



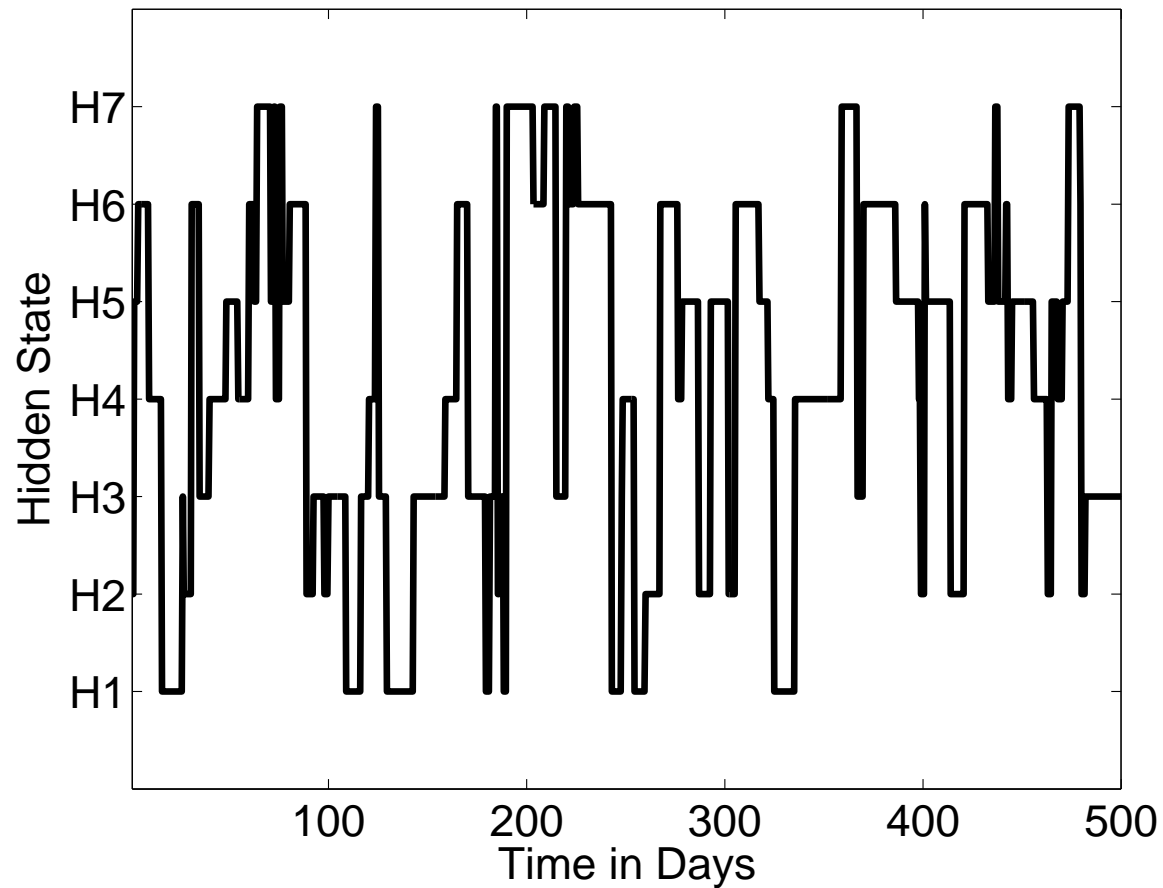
Eigenvalue spectrum and 95% significance intervals of Markov Transition matrix fitted to hidden state sequence.



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## Hidden State Sequence



Hidden state state sequence for representative 500 day sequence.





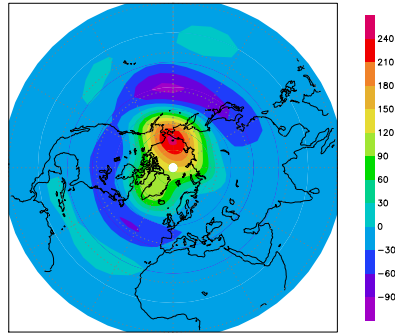
## Transition Matrix

$$A = \begin{pmatrix} 0 & 0.11 & \mathbf{0.52} & 0.13 & 0 & 0 & 0.23 \\ 0.04 & 0 & \mathbf{0.23} & 0.12 & 0.16 & \mathbf{0.33} & 0.12 \\ \mathbf{0.23} & 0.19 & 0 & \mathbf{0.22} & 0 & 0.17 & 0.18 \\ 0.13 & 0.19 & 0.09 & 0 & \mathbf{0.26} & 0.07 & \mathbf{0.24} \\ 0 & 0.20 & 0 & \mathbf{0.24} & 0 & \mathbf{0.33} & 0.21 \\ 0 & 0.18 & 0.18 & 0.13 & \mathbf{0.35} & 0 & 0.13 \\ 0.06 & 0.11 & \mathbf{0.24} & \mathbf{0.23} & 0.13 & 0.21 & 0 \end{pmatrix}$$

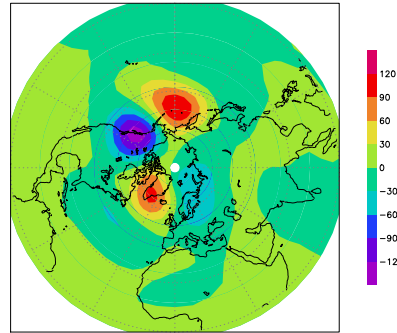
All matrix elements in bold are significant and describe preferred transitions.

# Conditional Mean States

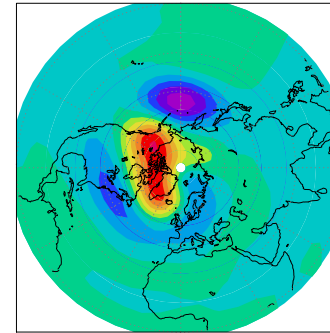
Hidden State 1



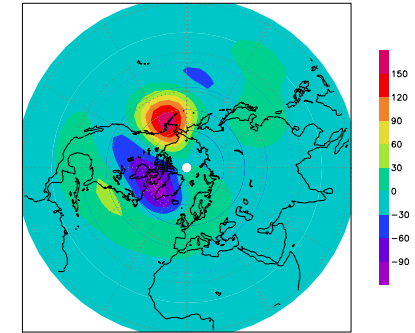
Hidden State 2



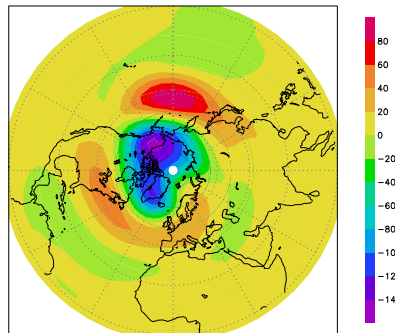
Hidden State 3



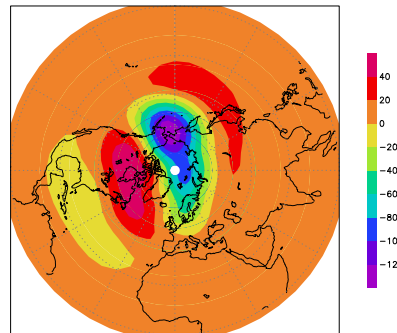
Hidden State 4



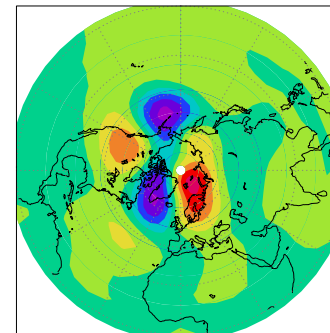
Hidden State 5



Hidden State 6



Hidden State 7



Conditional mean states of 500 hPa geopotential height composites.

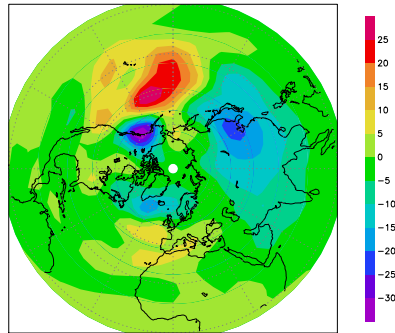


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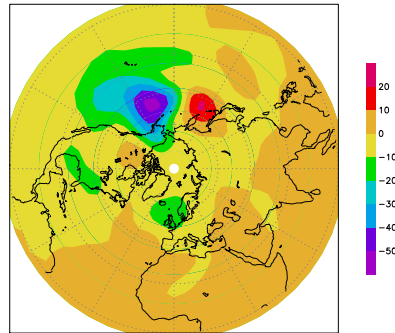
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# Conditional Transient Eddy Forcing

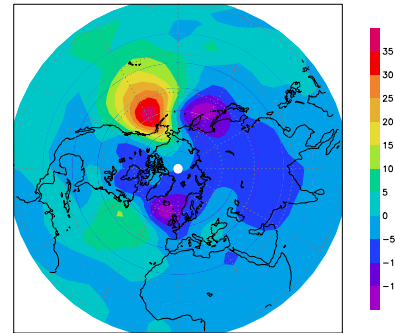
Hidden State 1



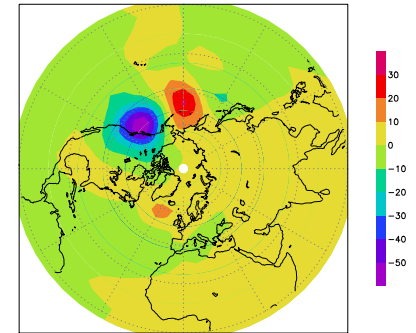
Hidden State 2



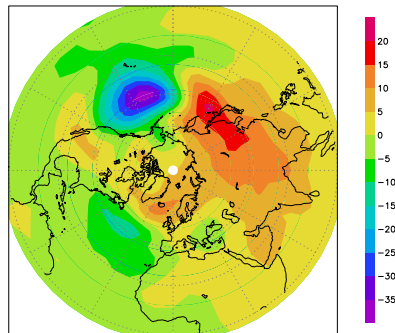
Hidden State 3



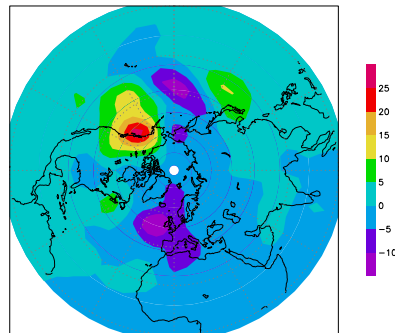
Hidden State 4



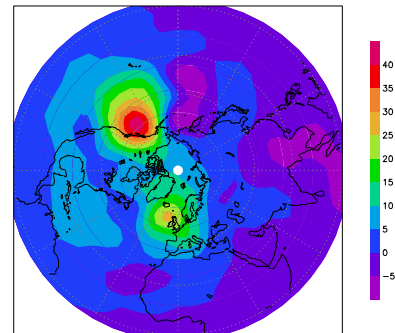
Hidden State 5



Hidden State 6



Hidden State 7



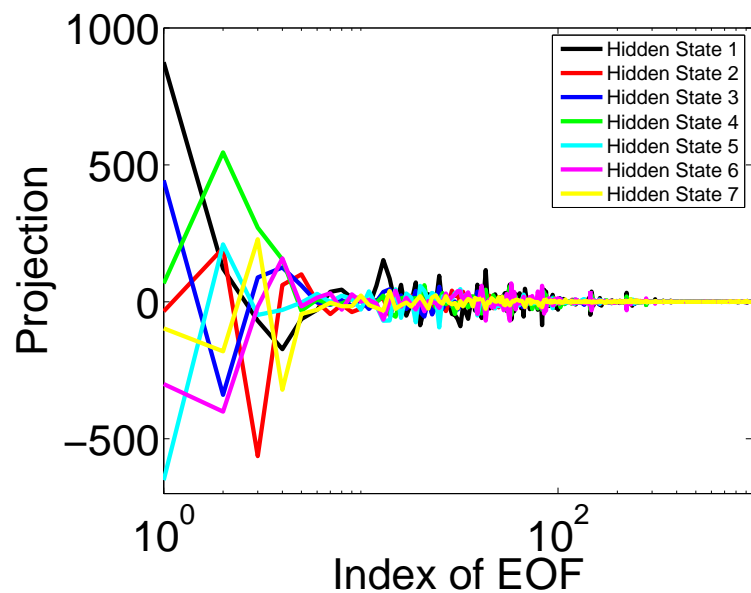
Conditional transient eddy forcing composites.



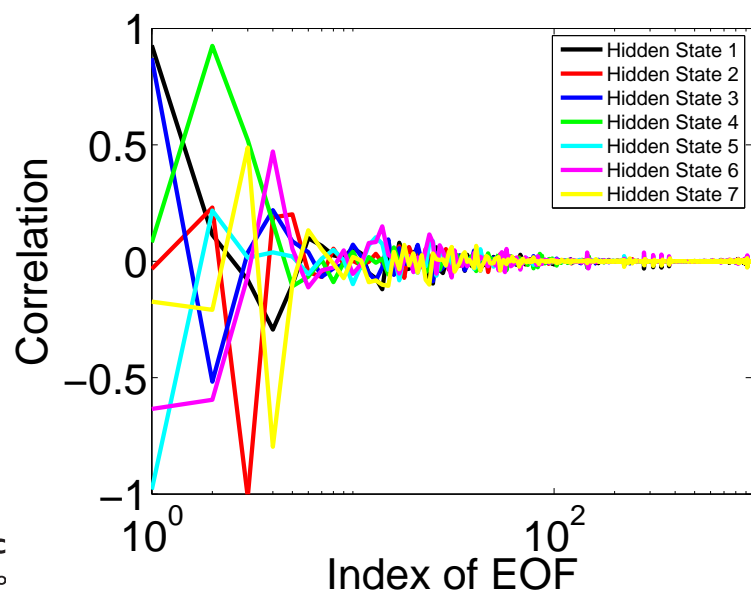
**British  
Antarctic Survey**

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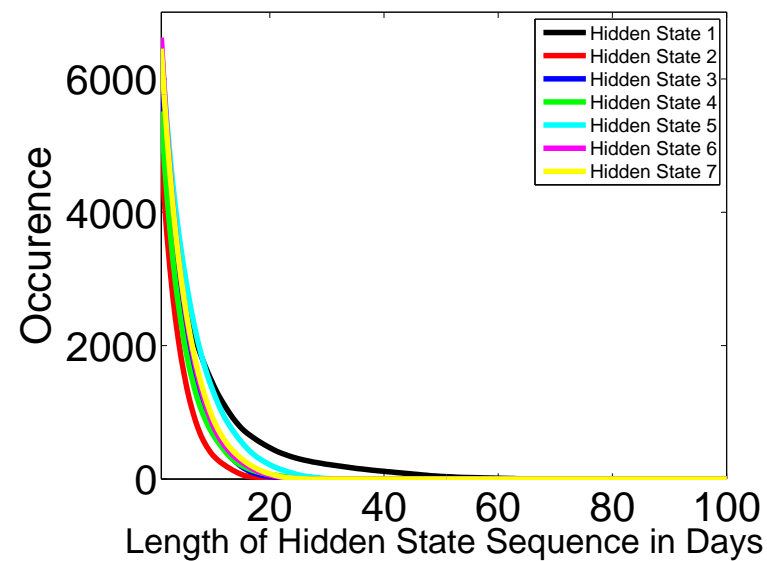
a) Projection



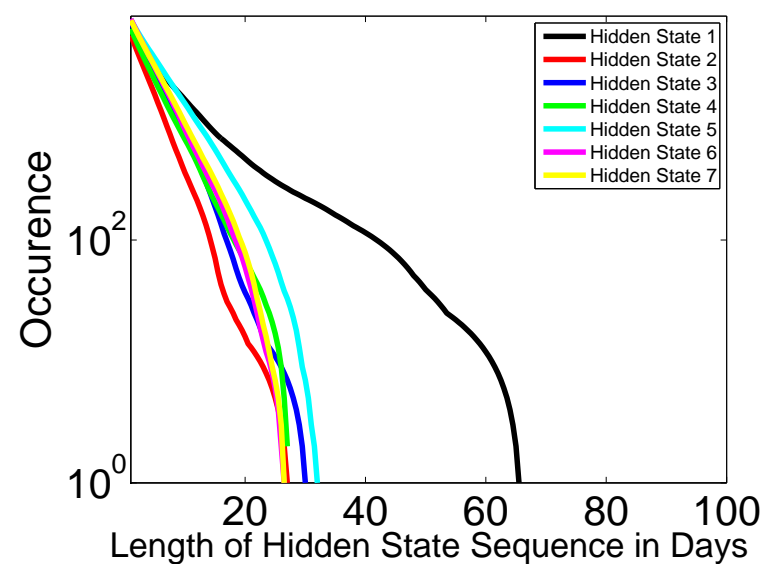
b) Correlation



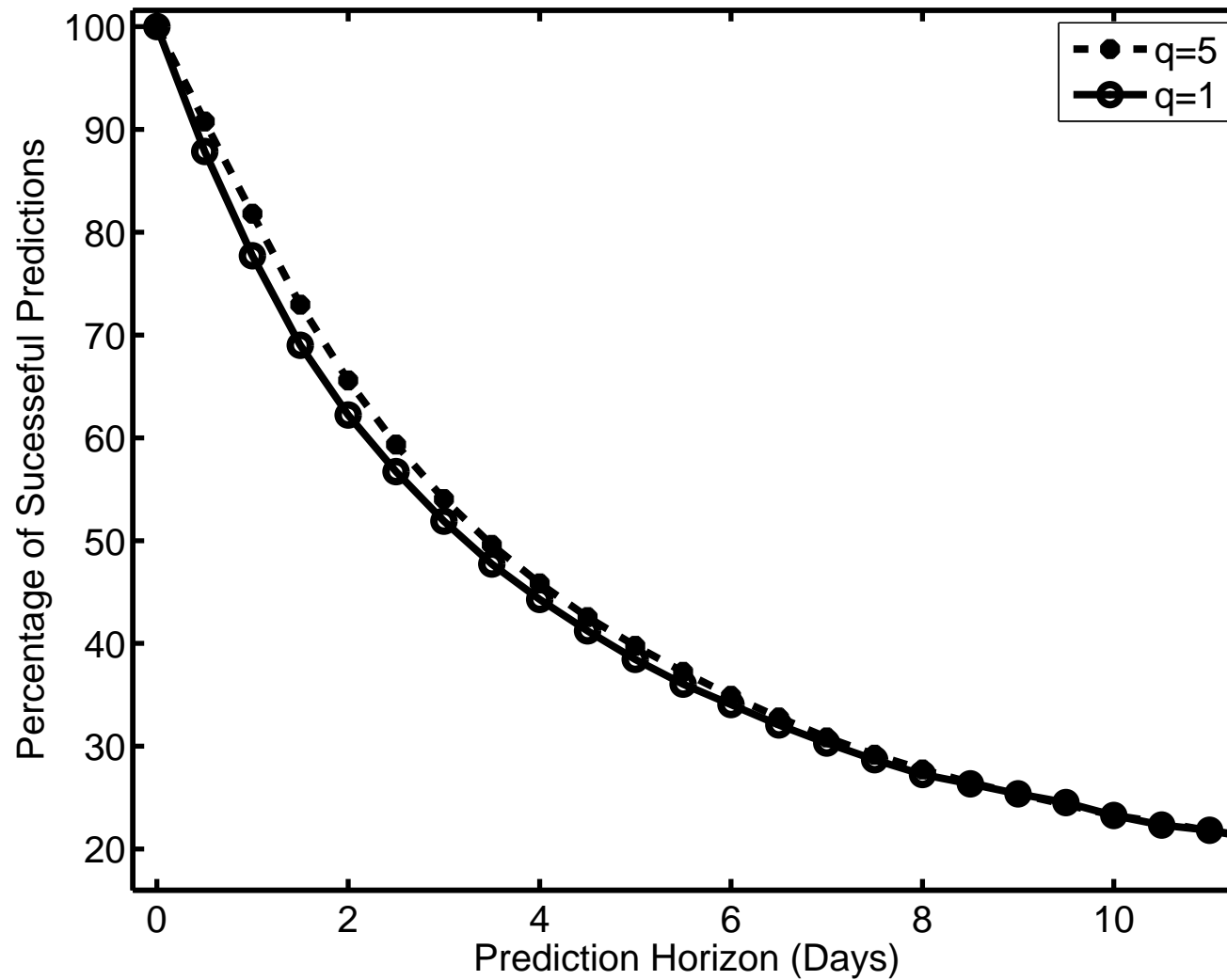
a)



b)



# Predictability of Regime States



# Conclusions

- In this study we applied the recently developed Finite Element Clustering method (Horenko 2009) in order to objectively identify metastable regimes in a high-dimensional data set produced by a comprehensive atmospheric GCM.
- The FEC method is able to identify in a 100 dimensional phase space of 500 hPa geopotential height seven dynamically significant metastable regimes. Some of the regimes correspond to the positive and negative phase of the Northern Annular Mode, respectively.
- Our predictability study shows that a simple Markov model for the evolution of the hidden states has predictive skill for about 6 days in successfully predicting the hidden state. This is about the same skill as the ECMWF Ensemble Prediction System in T255 resolution has in predicting the onset and decay of blocking situations (Pelly and Hoskins 2003) but with a much lower computational cost.

Franzke, Horenko, Majda and Klein, 2009: Systematic Metastable Atmospheric Regime Identification in an AGCM. J. Atmos. Sci., in press.

