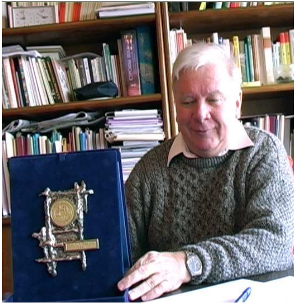
TBL-2009 Session in honor of
Vladimir E. Zakharov Birthday



ICTP@Trieste.Italy

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Turbulent Energy Spectra in Superfluids

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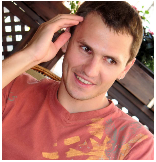

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- *Large scale, Eddy-dominated energy spectra and short scale energy spectra of Kelvin-waves are described uniformly, within differential approximation.*
- *Suggested natural physical hypotheses allow us to describe energy spectra at intermediate region of scales, and at finite temperatures without fitting parameters.*
- *Our model is in reasonable qualitative agreement with available experimental results.*

References: V.S.L'vov (VSL), S.V.Nazarenko (SVN), O.Rudenko (OR)

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- Eltsov, Golov, Graaf Hanninen, Krusius, VSL & Solntsev, *Phys. Rev. Letters*, **99**, 265301 (2007); ⇒ Helsinki ^{3}He turbulent front experiment
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- W.F. Vinen and R. J. Donnelly, *Physics today*, April (2007) ⇒ Overview on superfluid turbulence
- Walmsley, Golov, Hall, Levchenko & W. F. Vinen, *Phys. Rev. Letters*, **99**, 265302 (2007) ⇒ Manchester ^{4}He experiment.
- Kozik & Svistunov, *Phys. Rev. Lett.* **92**, 035301 (2004): ⇒ Kelwin wave spectrum

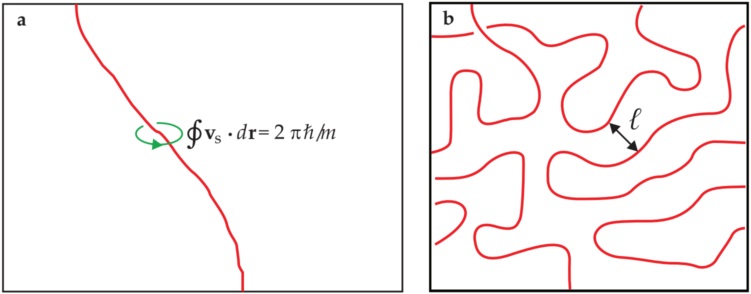
Outline

1. Story begins: Large– and small–scale turbulence in superfluids
   - Quasi-classical Two-fluid model & Mutual friction [7]
   - Kelvin waves of quantized vortex lines [9]
2. Helsinki ^{3}He-B experiment and bottleneck scenario
   - Turbulent front propagation and rate of energy dissipation [5]
   - Quasi-classical model of propagating turbulent front [6]
   - Eddy-wave Bottleneck scenario [3]
2. Manchester ^{4}He experiment and theory of bottleneck spectra
   - Manchester spin-down ^{4}He experiment [8]
   - Differential models for classical-quantum energy fluxes [4]
   - Bottleneck energy accumulation and effective viscosity [4]
   - Energy spectra at finite temperatures

Quantization of vortex lines, core radius a_0 and intervortex distance ℓ

$$\oint \boldsymbol{v}_s \cdot d\boldsymbol{r} = n\,\kappa, \quad \text{where } \kappa = \frac{2\pi\hbar}{M} \quad \text{is the circulation quant.}$$

$M = 4$ for ^{4}He and $M = 6$ for a pair of ^{3}He atoms.



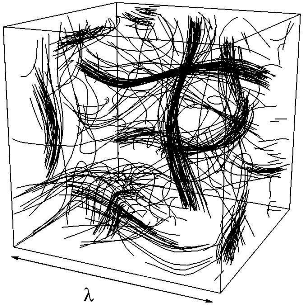
- ℓ is the mean intervortex distance,
- Vortex core radius $a_0 \simeq 1$ Å for ^{4}He & $a_0 \simeq 800$ Å at low p.

Two-fluid model (for scales $L \gg \ell$) & Mutual friction

"Coarse-grained" equation for the superfluid velocity $\boldsymbol{U}(\boldsymbol{r}, t)$ [1]:

$$\frac{\partial \boldsymbol{U}}{\partial \widetilde{t}} + (1-\alpha')(\boldsymbol{U}\cdot\nabla)\boldsymbol{U} + \nabla\mu = -\Gamma \boldsymbol{U}\,, \ \Gamma \equiv \alpha\,\omega_{\rm ef}\ .$$

- Chemical potential μ serves as the pressure,
- $\alpha'(T)\ \alpha(T)$ describe the mutual friction,
- Dissipative term Γ is taken in the simplified form, where $\omega_{\rm ef}$ is an effective vorticity.
- "Reynolds number" is $q^{-1} \equiv (1-\alpha')/\alpha$.



[1] E.B. Sonin, Rev. Mod. Phys. **59**, 87 (1987) and G. E. Volovik, JETP Lett. **78**, 533 (2002).

Kelvin waves of vortex line bending: $w(z) = x + i\,y$

- Energy (Hamiltonian) $\mathcal{H} = \dfrac{\kappa^2}{4\pi} \displaystyle\int \frac{\{1 + \mathrm{Re}\,[w'^*(z_1) w'(z_2)]\}\, dz_1 dz_2}{\sqrt{(z_1 - z_2)^2 + |w(z_1) - w(z_2)|^2}}$,

  - Hamiltonian form of Bio-Savart equations $i\,\kappa \dfrac{\partial w}{\partial t} = \dfrac{\delta \mathcal{H}\{w, w^*\}}{\delta w^*}$,
  - For small wave amplitudes, $w \ll \lambda$, $\quad \mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \mathcal{H}_6 + \dots$

- $\mathcal{H}_2 = \sum_k \omega_k |a_k|^2$, ($a_k = \sqrt{\kappa} w_k$) describes propagation of free KW

with the frequency $\omega_k = \dfrac{\kappa\Lambda}{4\pi} k^2$, $\Lambda = \ln\left(\dfrac{\ell}{a_0}\right) = \begin{cases} \simeq 15\,, & \text{for } ^4\text{He}\,, \\ \sim 10\,, & \text{for } ^3\text{He}\,. \end{cases}$

- $\mathcal{H}_4$ and $\mathcal{H}_6$ describe 4- and 6-wave interactions

$$\omega_1 + \omega_2 = \omega_3 + \omega_4\,, \quad \text{and} \quad \omega_1 + \omega_2 + \omega_3 = \omega_4 + \omega_5 + \omega_6$$

Effective 6-wave interaction coefficients $W_{\rm eff}$

$$\mathcal{H}_4 = \frac{1}{4}\sum_{1+2=3+4} T_{12,34} a_1^* a_2^* a_3 a_4\,, \quad \mathcal{H}_6 = \frac{1}{36}\sum_{1+2+3=4+5+6} W_{12,34} a_1^* a_2^* a_3^* a_4 a_5 a_6$$

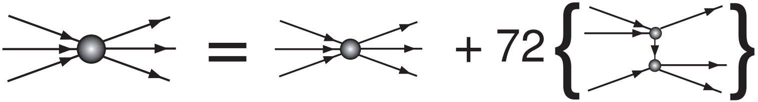
$$T_{12,34} = T^{\rm LIA}_{12,34} + \widetilde{T}_{12,34}\,, \qquad W_{123,456} = W^{\rm LIA}_{123,456} + \widetilde{W}_{123,456}\,.$$

LIA – Local Induction Approximation.

$$T^{\rm LIA}_{12,34} \simeq \Lambda\, k_1k_2k_3k_4 \sim \Lambda k^4\,, \qquad W^{\rm LIA}_{123,456} \simeq \Lambda\, k_1k_2k_3k_4k_5k_6 \sim \Lambda k^6\,,$$

$$\widetilde{T}_{12,34} \simeq k_1k_2k_3k_4 \sim k^4\,, \qquad \widetilde{W}_{123,456} \simeq k_1k_2k_3k_4k_5k_6 \sim k^6\,.$$

$2^{\rm nd}$ order perturbation theory:



$$W_{\rm eff} = W_{123,456} + 72\left\{T^2_{12,34}/\omega_k\right\}, \quad \omega_k \simeq \Lambda k^2\,.$$

Complete integrability $\Rightarrow$ ∞ # integrals of motion $\Rightarrow$

$$W^{\rm LIA}_{\rm eff} \equiv 0 \quad \Rightarrow \quad W_{\rm eff} \sim k^6 \ll \Lambda k^6\,.$$

Six-wave Kinetic Equation (Kozik-Svistunov [9])

$$\frac{\partial n_k}{\partial t} = \frac{\pi}{12}\int |W^{\text{eff}}_{k,1,2;3,4,5}|^2 \left[\mathcal{N}_{3,4,5;k,1,2} - \mathcal{N}_{k,1,2;3,4,5}\right]$$
$$\times\delta(\omega_k+\omega_1+\omega_2-\omega_3-\omega_4-\omega_5)\delta(k+k_1+k_2-k_3-k_4-k_5)\, dk_1 dk_2 \dots dk_5 \,,$$

$\mathcal{N}_{1,2,3;4,5,6} \equiv n_1 n_2 n_3 (n_4 n_5 + n_4 n_6 + n_5 n_6)$ has stationary solutions:

$n_k \propto \omega_k^{-1}$ $\Rightarrow$ energy-equipartition ;

$n_k^{\text{KS}} \simeq (\kappa^3\rho)^{-1/5}\,\epsilon^{1/5}\,|k|^{-17/5}$ $\Rightarrow$ constant-energy-flux ϵ along one vortex line.

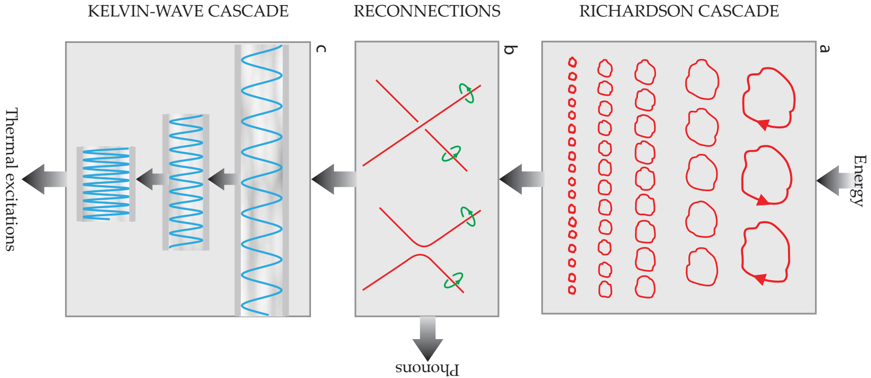
In 3D space with the vortex-line density ℓ^{-2} the KW energy density is

$$E^{\text{KW}} = \int \mathcal{E}^{\text{KW}}_k dk\,, \qquad \mathcal{E}^{\text{KW}}_k = \ell^{-2}\omega_k n_k \simeq \Lambda \ell^{-2}(\kappa^3\rho)^{-1/5}\,\epsilon^{1/5}\,|k|^{-7/5}$$

With the flux of energy density (per unite mass) $\varepsilon = \epsilon/\rho\ell^2$ we got [3]:

$$\mathcal{E}^{\text{KW}}_k \simeq \Lambda(\kappa^7\varepsilon/\ell^8)^{1/5}|k|^{-7/5}\,.$$

From Classical to Quantum energy cascade [7]



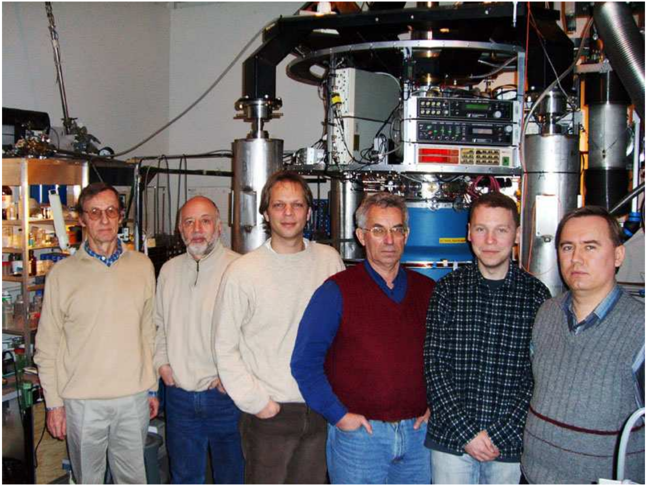
Very small scales:
$L \ll \ell$.
Vortex lines can be considered as independent

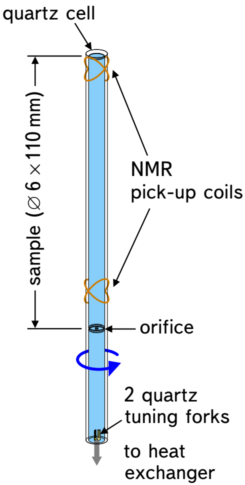
Intermediate, Intervortex scales:
$L \sim \ell$

Very large scales:
$L \gg \ell$

Helsinki rotating (below 10^{-3}K) cryostat [5]

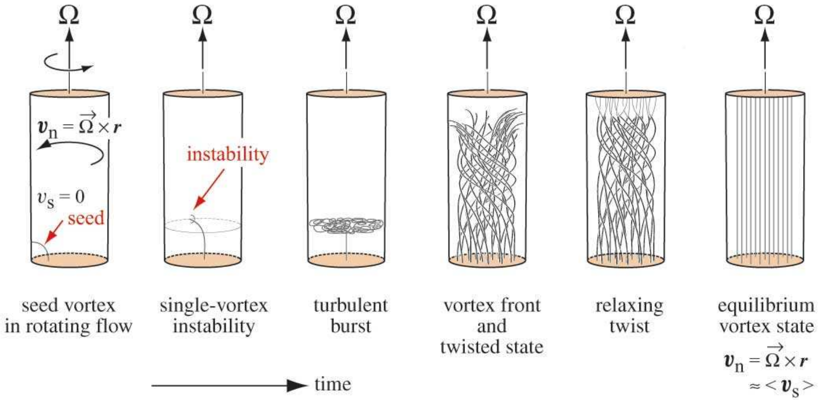
Low Temperature Laboratory, Helsinki University of Technology, Finland





M. Krusius, G. Volovik, R. de Graaf, VSL, R.E. Solntsev, V.B. Eltsov

Vortex instability and turbulence in a rotating ^{3}He-B [5]

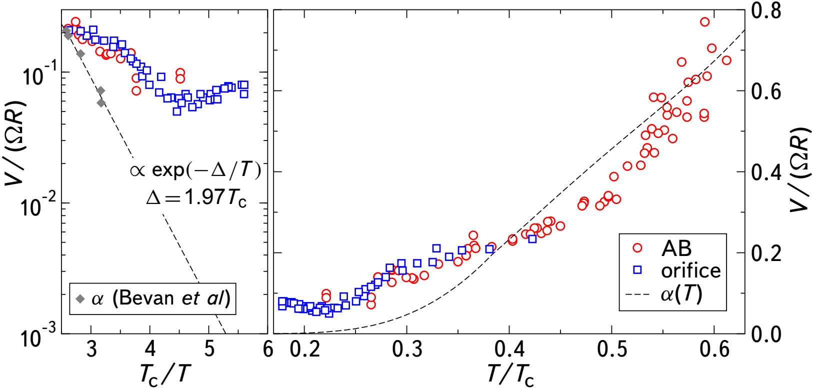


Resting (vortex-free) state of ^{3}He-B in rotating cell is meta-stable. A seed vortex loop is injected in the vortex-free flow and the subsequent evolution is depicted. Different transient states are traversed, until the stable rotating equilibrium vortex state is reached.

Velocity of the Front Propagation in rotating ^{3}He-B [5]

Front velocity V_f as a measure of the rate of energy dissipation $d\mathcal{E}/dt$:

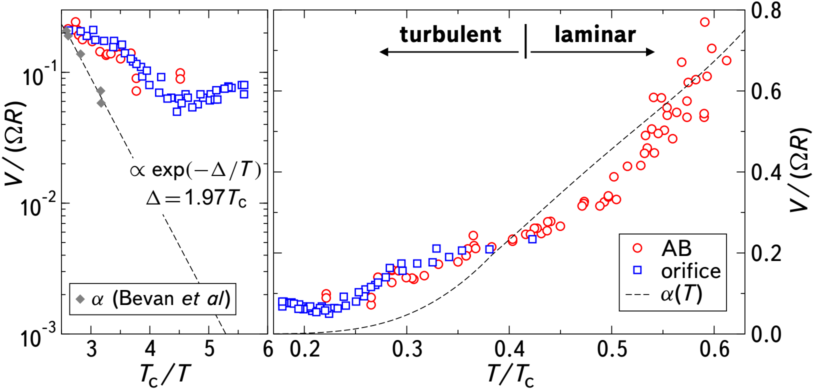
$$\frac{d\mathcal{E}}{dt} = -\frac{\pi}{4}\rho_\mathrm{s} V_\mathrm{f}\, \Omega^2 R^4\,, \quad \text{twist energy is neglected}$$



Dashed line: the mutual friction coefficient $\alpha(T)$.

Velocity of the Front Propagation in rotating ^{3}He-B [5]

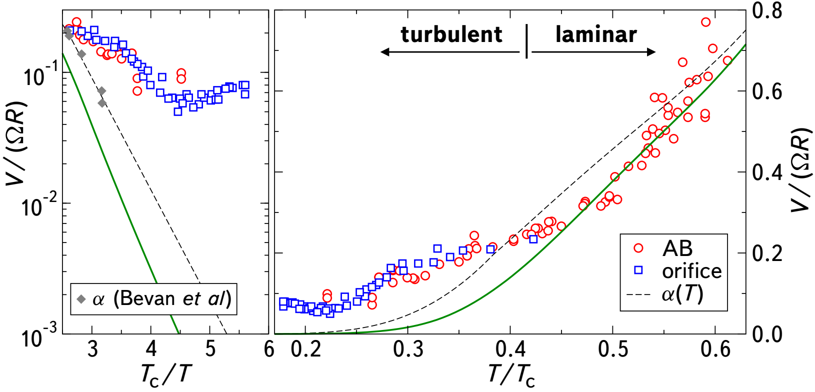
Two main regimes: high T: $V \simeq \alpha\,\Omega R$, low T: $V \to const \gg \alpha\,\Omega R$.



Dashed line: the mutual friction coefficient $\alpha(T)$.

Velocity of the Front Propagation in rotating ^{3}He-B [5]

High Temperature ($T > 0.4T_c$) **LAMINAR REGIME**



Vortex state behind the front is twisted: $\Rightarrow$ Free energy difference and front velocity are reduced by some factor: see green line.

Quasi-classical model of turbulent front propagation [6]

Global kinetic energy balance:

$$\frac{\pi}{4} V_{\rm f}\, \Omega^2 R^4 = 2\pi \int_0^R r\, dr \int_0^{\Delta(r)} dz \left[\frac{\widetilde{b}\, K^{3/2}(z,r)}{\Delta(r)} + \Gamma K(z,r) \right] .$$

$K \equiv \frac{1}{2}\langle |u|^2 \rangle$ turbulent kinetic energy density per unite mass, $\Delta(r)$ -effective front width (outer scale of turbulence) $\tilde{b} \equiv (1-\alpha')\, b_{\rm cl}$, in classical turbulent boundary layer $b_{\rm cl} \approx 0.27$,
$\Gamma = \alpha \omega_{\rm eff}$ – mutual friction damping.

Ignoring z-dependence of K and Γ one gets:

$$V_{\rm f}\, \Omega^2 R^4 = 8J\,, \quad J \equiv \int_0^R r\, dr\, [\widetilde{b}\, K^{3/2}(r) + \alpha\, \omega_{\rm eff}(r) \Delta(r) K(r)\, .$$

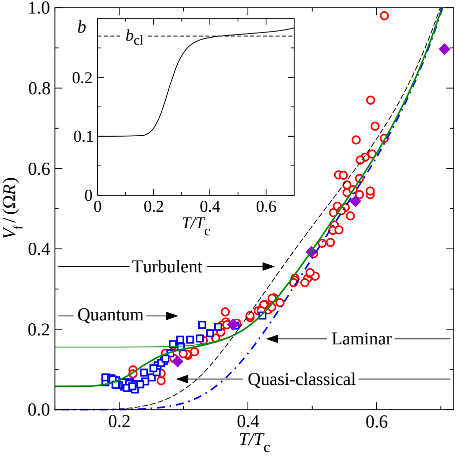
Quasi-classical model of turbulent front propagation [6]

Model comparison with Helsinki LTL experiment [5]

• Accounting for spatial turbulent diffusion of kinetic energy toward the centerline in the radial energy balance gives $\Delta(r) \propto r$, $K(r) \propto r^2$.
• Also one naturally can suggest $\omega_{\rm eff}(r)$ is r-independent.

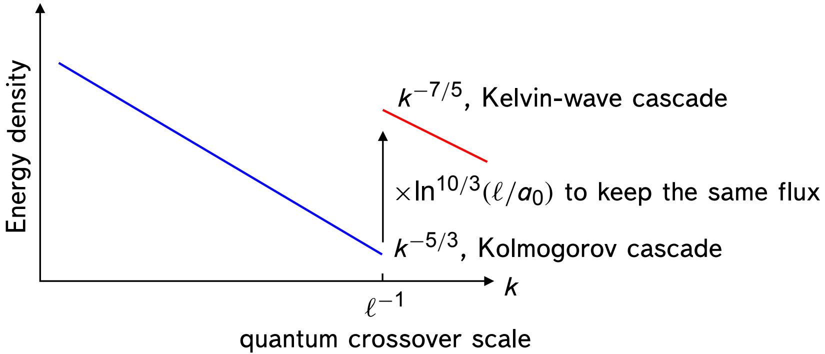
Taking $\Delta(r)\omega_{\rm eff}(r) = a\,\Omega r$, ($a \simeq 0.5$), $K(r) = c(\Omega r)^2/2$ ($c \simeq 1$) one gets Eqs. $\Downarrow$, show by —— $\Rightarrow$

$$\frac{V_{\rm f}}{\Omega R} \simeq \frac{4\,c}{5}\left[b(1-\alpha')\sqrt{\frac{c}{2}} + \alpha\, a\right]$$



Below 0.25 $T_{\rm c}$ data deviates down!

Eddy-wave Bottleneck scenario [2]



$$\mathcal{E}_k^{\text{K41}} \simeq \varepsilon^{2/3}|k|^{-5/3} \xrightarrow{\varepsilon = const} \mathcal{E}_k^{\text{KW}} \simeq \Lambda \left(\kappa^7 \varepsilon / \ell^8\right)^{1/5} |k|^{-7/5} .$$

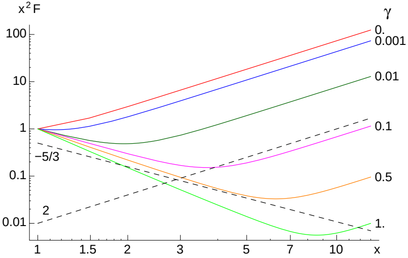
$$\Lambda \equiv \ln(\ell/a_0) \gg 1 .$$

Eddy-wave Bottleneck scenario [2]

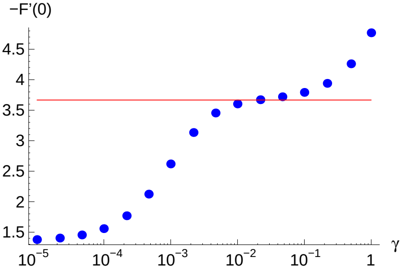
Balance of the energy density [1D energy spectrum $E = \int E_k dk$] in k-space:

$$\frac{d\varepsilon(k)}{dk} = -E_k\,, \qquad \varepsilon(k) \equiv (1-\alpha')\sqrt{k^{11}E_k}\,\frac{d(E_k/k^2)}{8\,dk}\,,$$

Here $\varepsilon(k)$ is taken in the Leiht-Nazarenko differential model



Resulting energy spectra E_k for different values of mutual friction parameter Γ



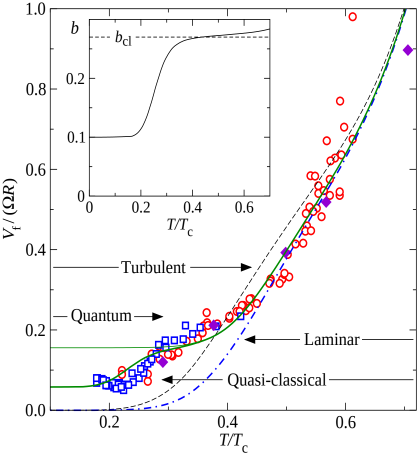
Resulting energy influx $\varepsilon(k_0)$ for fixed value E_{k_0} vs. Γ gives Γ-dependence of b in estimate $\varepsilon(k_0) = b(1-\alpha')K^{3/2}/\Delta$

Eddy-wave Bottleneck scenario vs. LTL experiment

- Using the same estimate

$$\frac{V_\mathrm{f}}{\Omega R} \simeq \frac{4\,c}{5}\left[b(T)(1-\alpha')\sqrt{\frac{c}{2}}+\alpha\,a\right],$$

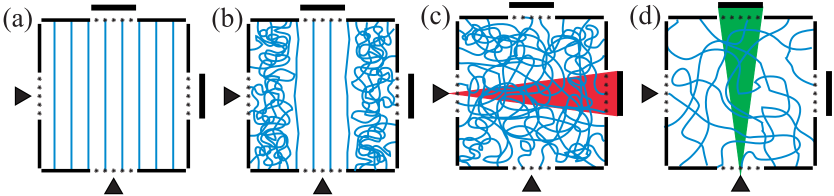
but with temperature dependent b(T), (see insert), which accounts for the bottleneck effect, one achieves a reasonable description of the temperature dependence of the front velocity in the quantum turbulent region, see green line ——⇒



Achieved agreement is an evidence that our model adequately reflects main physical features of the front propagation in ^{3}He-B in laminar, quasi-classical turbulent and quantum-turbulent regimes.

Manchester cube $(4.5\mathrm{cm})^3$ spin-down ^{4}He experiment

Measuring the time-decay of the vortex line density by negative-ion scattering



⇑ Cartoon of the vortex configurations ⇑.

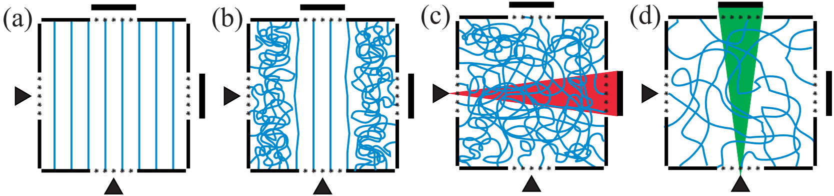
(a) Regular array of vortices at $\Omega =$const.;
(b) Immediately after stopping rotation;
(c) Homogeneous turb.: $\Omega t \sim 30 \div 300$;
(d) Almost decayed turbulence: $\Omega t > 10^3$.

Shaded areas indicate the paths of probe ions when sampling the vortex density.

Walmsley, Golov, Hall, Levchenko and Vinen,
PRL, **99**, 265302 (2007)

Manchester spin-down ^{4}He experiment [8]

Measuring the time-decay of the vortex line density by negative-ion scattering



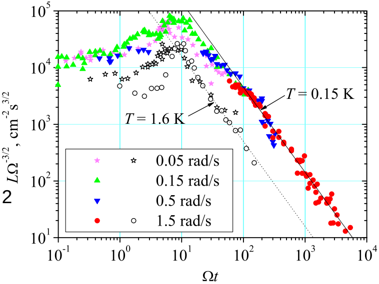
⇑ Cartoon of the vortex configurations ⇑

Vortex line density ($L\,\Omega^{-3/2}$) *vs.* ($\Omega\, t$) ⇒.

$$\frac{dE(t)}{dt} = \varepsilon(t) = \nu' \left\langle |\omega|^2 \right\rangle , \ \left\langle |\omega|^2 \right\rangle = (\kappa L)^2 ,$$

Turb. Energy $E \propto \varepsilon^{2/3} \Rightarrow E(t) \propto (t - t_*)^2$

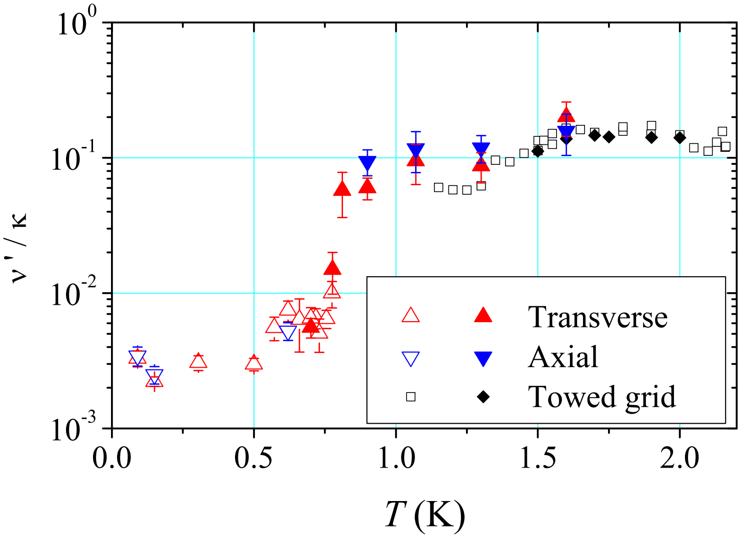
$$\Rightarrow L(t) \propto 1/[\kappa \sqrt{\nu'(t-t_*)^3}] \Rightarrow$$



Data on $L(t)$ allows to measure effective viscosity ν'

Temperature dependence of the effective viscosity ν'

Manchester spin-down experiment from $\Omega = 1.5$ rad/s in superfluid ^{4}He [8]



Differential model for classical⇒quantum energy flux

Leiht-Nazarenko differential model for Classical hydrodynamic (HD) energy flux [3]

$$\varepsilon_k = -\frac{1}{8}\sqrt{k^{13}F_k}\,\frac{dF_k}{dk}\,, \qquad F_k = \frac{\mathcal{E}_k^{\mathrm{HD}}}{k^2}\,,$$

F_k – 3-dimensional spectrum of turbulence.

- Generic spectrum with a constant energy flux is the solution to Eq. $\varepsilon_k = \varepsilon = \mathrm{const}$:

$$F_k^{\mathrm{HD}} = \left[\frac{24\,\varepsilon}{11\,k^{11/2}} + \left(\frac{T}{\pi\rho}\right)^{3/2}\right]^{2/3} \Rightarrow \begin{cases} (24/11)^{2/3}\ \varepsilon^{2/3}\,k^{-11/3}\,, \\ T/\pi\rho\,. \end{cases}$$

Low k region: K41 spectrum $\mathcal{E}^{\mathrm{HD}} \propto \varepsilon^{2/3}k^{-5/3}$,
Large k region: energy equipartition with an effective temperature T.

Differential model for classical⇒quantum energy flux

Leiht-Nazarenko differential model for Classical hydrodynamic (HD) energy flux [3]

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Large k region: energy equipartition with an effective temperature T.

Differential model for classical⇒quantum energy flux [4]

L'vov-Nazarenko-Rudenko [4] differential model for quantum Kelvin-wave energy flux

$$\varepsilon_{\rm KW}(k) = -\frac{5}{7}\frac{(k\ell)^8\mathcal{E}^4_{\rm KW}(k)}{\Lambda^5\kappa^7}\frac{d\mathcal{E}_{\rm KW}(k)}{dk}\,.$$

- Equation $\varepsilon_{\rm KW}(k) = \varepsilon = \text{const}$ has the solution

$$\mathcal{E}_{\rm KW}(k) = \Big[\frac{\Lambda^5\kappa^7}{\ell^8}\frac{\varepsilon}{k^7} + \Big(\frac{T}{\pi\rho}\Big)^5\Big]^{1/5} \Rightarrow \begin{cases} \Lambda\big(\kappa^7\varepsilon/\ell^8\big)^{1/5}k^{-7/5}, \\ T/\pi\rho\,. \end{cases}$$

Low k region: Kozik-Swistunov spectrum [9] of Kelvin waves (KW),
Large k region: equilibrium Rayleigh-Jeans spectrum.

Differential model for classical⇒quantum energy flux [4]

L'vov-Nazarenko-Rudenko [4] differential model for quantum Kelvin-wave energy flux

$$\varepsilon_{\rm KW}(k) = -\frac{5}{7}\,\frac{(k\ell)^8\mathcal{E}^4_{\rm KW}(k)}{\Lambda^5\kappa^7}\,\frac{d\mathcal{E}_{\rm KW}(k)}{dk}\,.$$

- Equation $\varepsilon_{_{\rm KW}}(k) = \varepsilon = \mathrm{const}$ has the solution

$$\mathcal{E}_{\rm KW}(k) = \Big[\frac{\Lambda^5\kappa^7}{\ell^8}\,\frac{\varepsilon}{k^7} + \Big(\frac{T}{\pi\rho}\Big)^5\Big]^{1/5} \Rightarrow \begin{cases} \Lambda\big(\kappa^7\varepsilon/\ell^8\big)^{1/5} k^{-7/5}\,, \\ T/\pi\rho\,. \end{cases}$$

Low k region: Kozik-Swistunov spectrum [9] of Kelvin waves (KW),
Large k region: equilibrium Rayleigh-Jeans spectrum.

Differential model for classical⇒quantum energy flux

Unified model for the eddy-wave total energy flux: Basic ideas

- Fluid motion is approximated as a mixture of “pure" HD and KW motions, with the portions of the energy density $g(k,\ell)$ & $1-g(k,\ell)$:

$$\mathcal{E}_k^{\rm HD}=\mathcal{E}_k\, g\,,\quad \mathcal{E}_k^{\rm KW}=\mathcal{E}_k\,(1-g)\,,\quad g(k,\ell)=\begin{cases}1\,, & k\ell \text{ small}\,,\\ 0\,, & k\ell \text{ large}\,.\end{cases}$$

- To find $g(dk,\ell)$ we define here HD and KW energies via velocities of k-bent, ℓ-separated parallel vortex lines $\boldsymbol{v}_{j,k}(\boldsymbol{r})$

$$\mathcal{E}_k=\mathcal{E}_k^{\rm HD}+\mathcal{E}_k^{\rm KW}=\frac{1}{2}\int d\boldsymbol{r}\Big|\sum_j v_{j,k}(\boldsymbol{r})\Big|^2,\quad \mathcal{E}_k^{\rm KW}=\frac{1}{2}\int d\boldsymbol{r}\sum_j\Big|v_{j,k}(\boldsymbol{r})\Big|^2.$$

After cumbersome calculations and controlled approximations this finally gives analytical formula for the blending function, that depends ONLY on $x=k\,\ell$

$$g(x)=g_0[0.32\,\ln(\Lambda+7.5)x]\,,\; g_0(x)=\left[1+\frac{x^2\exp(x)}{4\pi(1+x)}\right]^{-1}.$$

Differential model for classical⇒quantum energy flux

Unified model for the eddy-wave total energy flux: Basic ideas

• Fluid motion is approximated as a mixture of “pure" HD and KW motions, with the portions of the energy density $g(k,\ell)$ & $1-g(k,\ell)$:

$$\mathcal{E}_k^{\rm HD} = \mathcal{E}_k\, g\,, \quad \mathcal{E}_k^{\rm KW} = \mathcal{E}_k\,(1-g)\,, \quad g(k,\ell) = \begin{cases} 1\,, & k\ell \text{ small}\,, \\ 0\,, & k\ell \text{ large}\,. \end{cases}$$

• To find $g(dk,\ell)$ we define here HD and KW energies via velocities of k-bent, ℓ-separated parallel vortex lines $\boldsymbol{v}_{j,k}(\boldsymbol{r})$

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Unified model for the eddy-wave total energy flux : Basic ideas

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Additional contributions $\varepsilon^{\rm KW}_{\rm HD}(k)$ and $\varepsilon^{\rm HD}_{\rm KW}(k)$, originating from influence of KW on the HD-energy flux and vise versa, was found by some additional arguments, such as form of thermodynamical equilibrium.

All above reasoning finally give in the stationary case:

$$\varepsilon = \varepsilon(k) \quad = \quad -\Big\{\frac{1}{8}\sqrt{k^{11} g(k\ell)\mathcal{E}(k)} + \frac{5}{7}\frac{(k\ell)^8 k_*^2[1-g(k\ell)]^4\mathcal{E}(k)^4}{\Lambda^5\kappa^7}\Big\}$$
$$\times\frac{d}{dk}\Big\{\mathcal{E}(k)\Big[\frac{g(k\ell)}{k^2}+\frac{1-g(k\ell)}{k_*^2}\Big]\Big\}, \quad k_*\ell = \frac{6.64}{\ln(\Lambda+7.5)} . \qquad \text{(LNR)}$$

This is an ordinary differential equation, that allows to find energy spectrum $\mathcal{E}(k)$ (at given ε and ℓ) in the entire region of scales: classical HD, quantum KW and crossover scales $k\ell \sim 1$.

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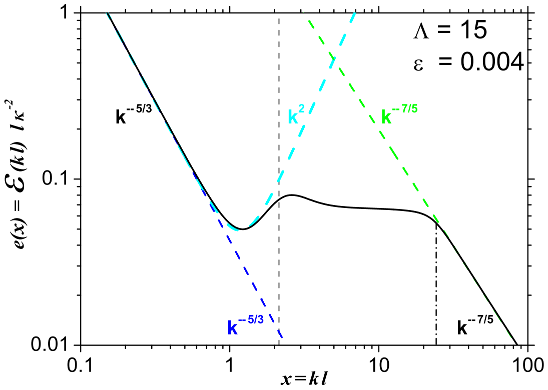
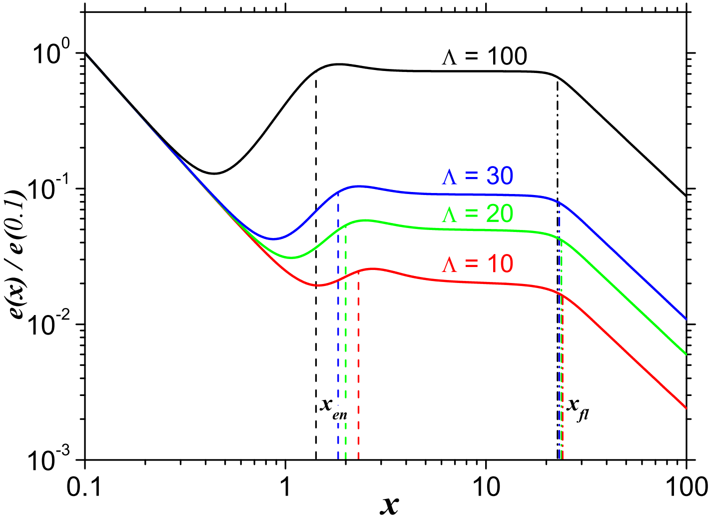


Black line – Spectrum $\mathcal{E}(k\ell)$ from solution of Eq. (LNR) $\varepsilon(k\ell) = \varepsilon$;
Dashed blue line – K41 HD energy spectrum $\mathcal{E}_{\mathrm{HD}}(k\ell) \propto k^{-5/3}$;
Dashed cyan line – general HD spectrum, including equilibrium $\propto k^2$;
Dashed green line – Energy spectrum of Kelvin waves $\mathcal{E}_{\mathrm{KW}} \propto k^{-7/5}$.
Vertical dashed lines – Left: energy crossover, $\mathcal{E}_{\mathrm{HD}}(k\ell) = \mathcal{E}_{\mathrm{KW}}(k\ell)$,
Right: flux crossover, $\varepsilon_{\mathrm{HD}}(k\ell) = \varepsilon_{\mathrm{KW}}(k\ell)$

Total energy spectra for different Λ

as the solutions of Eq. (LNR) $\varepsilon(k\ell) = \varepsilon$ for self-consistent values of ε: see next slide



Self-consistent estimate of ε and effective viscosity ν'

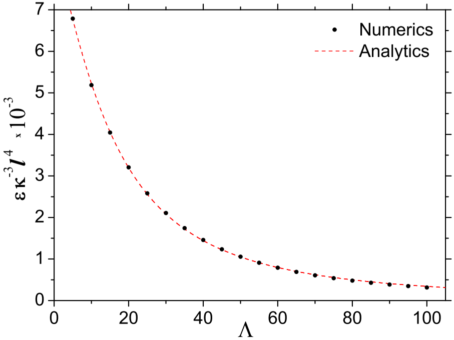
Eq. (LNR) gives $\mathcal{E}(k)$ at fixed ε and ℓ, related by: $(\kappa^2/\ell^4) = \langle|\omega|^2\rangle$

$$= 2\int_0^\infty k^2 g(k\ell)\mathcal{E}(k)\,dk \ .$$

This allows to numerically determine ε and ℓ as a function of Λ, see black dots on Fig. and the analytical fit:

$$\epsilon = \frac{\nu'}{\kappa} = \frac{8.65}{10^3 + 45.8\Lambda + 1.98\Lambda^2}\,,$$

shown as red dashed line.



For ^{4}He value of $\Lambda \simeq 15$ we got $\nu'_{\rm theor} \approx 0.004\,\kappa$, which is quite close to Manchester spin-down experimental value $\nu'_{exp} \approx 0.003\,\kappa$. Having in mind that our model does not contain fitting parameters, we consider this agreement as more than satisfactory.

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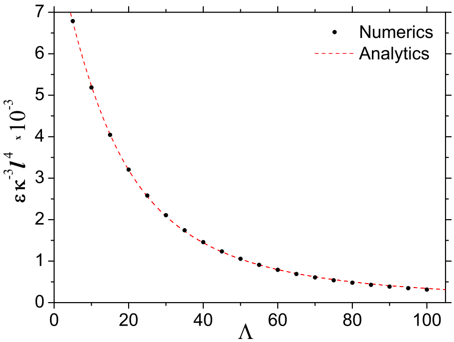

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Energy spectra at finite temperatures: UNDER CONSTRICTION

LNR model for the energy balance equation at finite temperature:

At finite temperatures the balance equation for superfluid energy $\mathcal{E}_\mathrm{s}$ includes:

• Large-scale (eddy-dominated) dissipation due to mutual friction, α_s , $\alpha_\mathrm{n} \neq 0$, that accounts for motion of normal component, $\mathcal{E}_\mathrm{n} \neq 0$,

• dissipation due to Kelvin waves

$$\frac{\partial \mathcal{E}_\mathrm{s}(k,t)}{\partial t} + \frac{\partial \varepsilon_\mathrm{s}(k)}{\partial k} = \alpha_\mathrm{s} \Big\{ \frac{\omega_{\mathrm{T}}[\alpha_\mathrm{n}\, g(k\ell)\mathcal{E}_\mathrm{s}(k) + \alpha_\mathrm{s}\, \mathcal{E}_\mathrm{n}(k)]}{\alpha_\mathrm{s} + \alpha_\mathrm{n} + [\nu\, k^2 + \gamma_\mathrm{n}(k) + \gamma_\mathrm{s}(k)]/\omega_{\mathrm{T}}}$$
$$-\Big(\omega_{\mathrm{T}} g(k\ell) + \kappa \Lambda k^2 [1 - g(k\ell)]/4\pi\Big)\, \mathcal{E}_\mathrm{s}(k,t)\Big\}\,,$$

Here LNR (ours) model for the energy flux is the same as for $T = 0$:

$$\varepsilon(k) \quad = \quad -\Big\{ \frac{1}{8}\sqrt{k^{11} g(k\ell)\mathcal{E}(k)} + \frac{5}{7}\frac{(k\ell)^8 k_*^2 [1 - g(k\ell)]^4 \mathcal{E}(k)^4}{\Lambda^5 \kappa^7} \Big\}$$
$$\times \frac{d}{dk}\Big\{ \mathcal{E}(k)\Big[\frac{g(k\ell)}{k^2} + \frac{1 - g(k\ell)}{k_*^2}\Big]\Big\}, \qquad k_*\ell = 6.64 \big/ \ln(\Lambda + 7.5).$$

Energy spectra at finite temperatures: UNDER CONSTRICTION

Superfluid and Normal Energy balance equations at finite temperature:

with $g(k\ell) = g_0\left[0.32\ \ln(+7.5)\ k\ell\right]$, $\quad g_0(k\ell) = \left[1 + \dfrac{(k\ell)^2 \exp(k\ell)}{4\pi(1+k\ell)}\right]^{-1}$.

At $T > 0$ the balance equation for normal energy $\mathcal{E}_\text{n}$ includes:

- LNR differential approximation for the energy flux,
- Energy source due to motion of superfluid component,
- Large-scale (eddy-dominated) dissipation due to mutual friction,
- viscous dissipation:

$$\frac{\partial \mathcal{E}_\text{n}(k,t)}{\partial t} - \frac{1}{8}\frac{\partial}{\partial k}\sqrt{k^{11}\mathcal{E}_\text{n}(k)}\,\frac{\partial}{\partial k}\frac{\mathcal{E}_\text{n}(k)}{k^2}$$

$$= \frac{\alpha_\text{n}\,\omega_{\text{T}}[\alpha_\text{n}\,g(k\ell)\mathcal{E}_\text{s}(k) + \alpha_\text{s}\,\mathcal{E}_\text{n}(k)]}{\alpha_\text{s} + \alpha_\text{n} + [\nu\,k^2 + \gamma_\text{n}(k) + \gamma_\text{s}(k)]/\omega_{\text{T}}} - \left(\alpha_\text{n}\omega_{\text{T}} + \nu k^2\right)\mathcal{E}_\text{n}(k,t)\,,$$

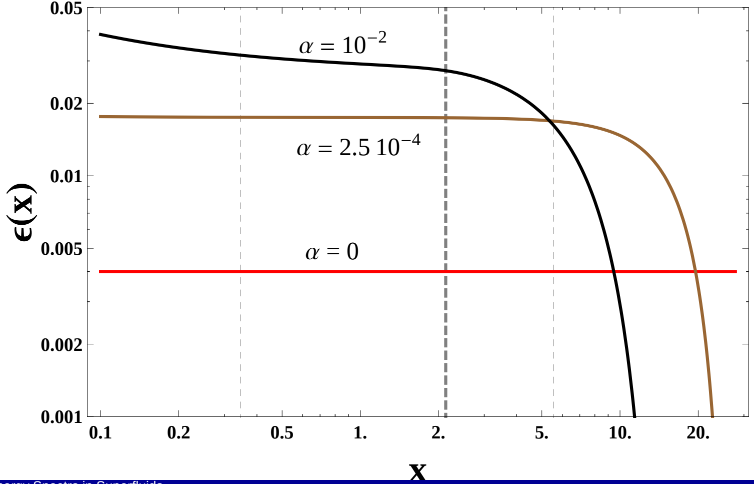
where

$$,\quad \gamma_\text{n}(k) = k^{3/2}\sqrt{\mathcal{E}_\text{n}(k)}\,,\quad \gamma_\text{s}(k) = k^{3/2}\sqrt{g(k\ell)\,\mathcal{E}_\text{s}(k)}\,.$$

Energy spectra at finite temperatures: UNDER CONSTRICTION

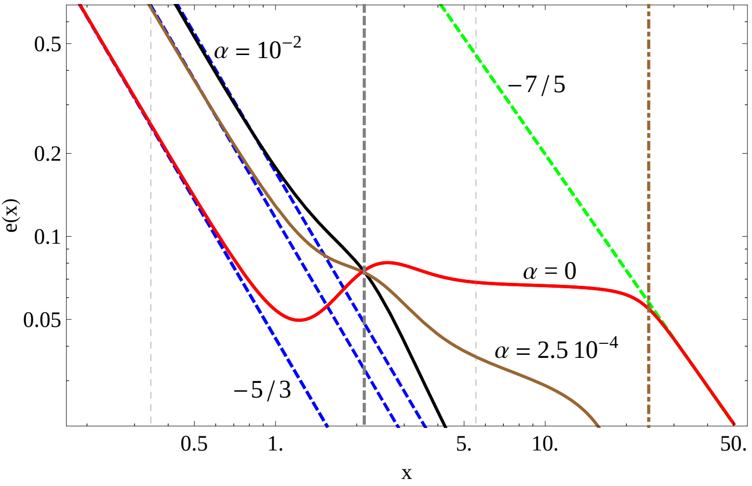
Stationary (numerical) solutions: the energy flux at finite temperature:

Stationary (numerical) solutions of this equation (at $\Lambda = 15$) and different α are as follows:



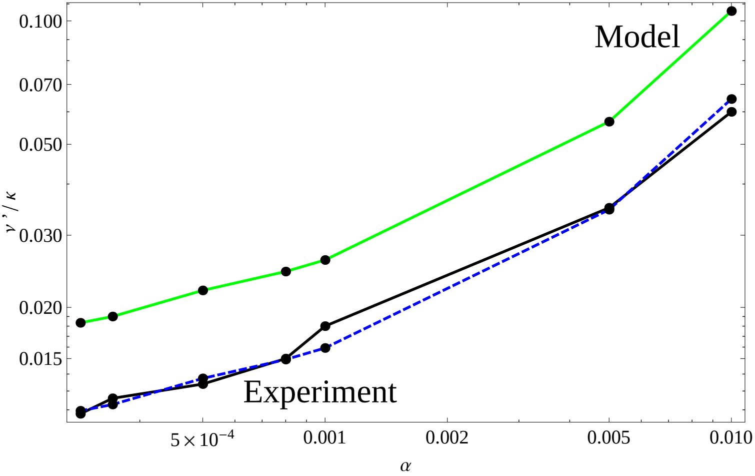
Energy spectra at finite temperatures: UNDER CONSTRICTION

Stationary (numerical) solutions the energy balance equation at finite temperature:



Energy spectra at finite temperatures: UNDER CONSTRICTION

Temperature dependence of the effective viscosity:



Summary and road ahead

• Achieved agreement with the Helsinki ^{3}He front-propagation and Manchester ^{4}He spin-down experiments is an evidence that
Our theory adequately reflects main physical features of superfluid ^{3}He and ^{4}He turbulence in laminar, quasi-classical turbulent, quantum-turbulent and crossover regimes.

• Our feeling is that
The model assumptions and approximations we made do not essentially affect the resulting physical picture of superfluid turbulence.

• The theory predicts not only the temperature dependence of ν' but the Entire energy spectrum at zero and finite temperatures, consisting (at $T \to 0$) of: K41 HD energy spectrum with constant energy flux $\propto k^{-5/3}$, HD equilibrium $\propto k^2$, a KW equilibrium $\simeq$ const. and a KW-spectrum with constant energy flux, $\propto k^{-7/5}$.

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