Nonlinear Gyrokinetics:

A powerful tool for the description of microturbulence in magnetized plasmas [a]

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Presented at

"Turbulent Mixing and Beyond"

ICTP, Trieste, Italy

July 29, 2009

[a]Work supported by U.S. Dept. of Energy Contract No. DE–AC02–09CH11466.

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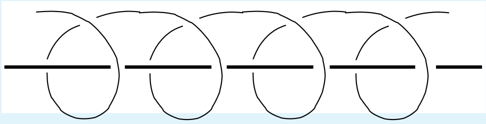

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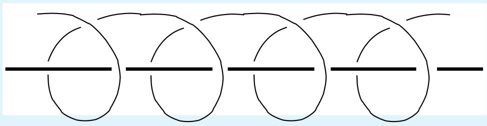

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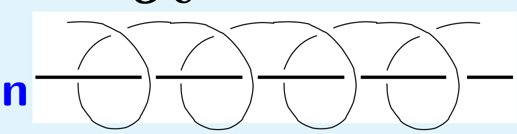

  [6 dimensions (!!!) + time]. A generalization of **Boltzmann’s eq’n**.

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Plasmas motions span
greatly disparate length and time scales.

Consider the ITER device...

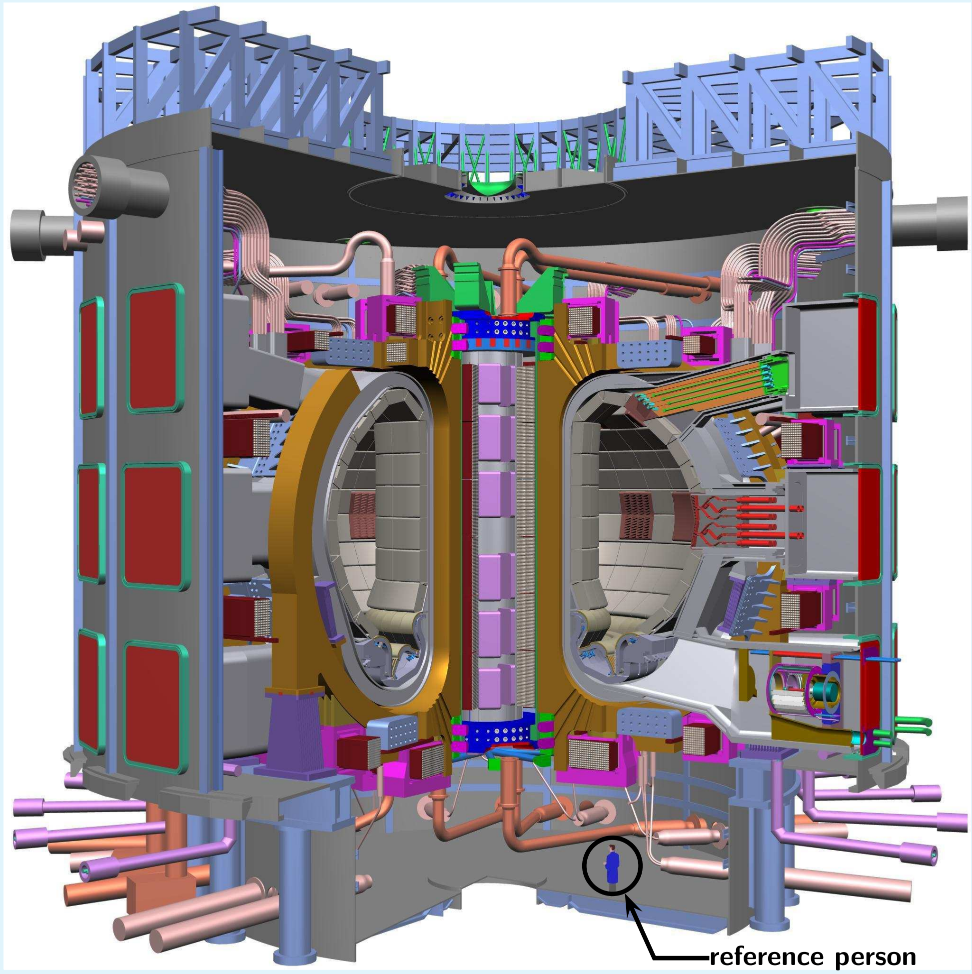

Fig. 5. The ITER device being constructed in Cadarache, France.

- Cost $\sim \mathrm{E}\,10^{10}$.
- $B_T = 5$ Tesla $= 5 \times 10^4$ Gauss
- a (minor radius) > 2 meters $\ggg$ gyroradius ρ_i
- Pulse length $>$ 1000 sec $\ggg$ turbulence autocorrelation time.

The disparate time and space scales in this device present a very difficult challenge for numerical simulation.

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In a strong magnetic field, the large gyrofrequency complicates direct numerical solution of the plasma kinetic equation.

The gyrofrequency for species s ($s = e$ or i) is

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Typical gyroperiods:

$$2\pi/\omega_{ce} < 10^{-10}\,\text{sec}; \quad 2\pi/\omega_{ci} < 10^{-7}\,\text{sec} = 0.1\,\mu\text{sec}. \tag{4}$$

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- Fortunately, for microturbulence in magnetically confined fusion devices, the collective modes of interest (e.g., *drift waves*) are low frequency:

$$\frac{\omega}{\omega_{ci}} \lesssim \frac{\rho_i}{L_n} \sim \frac{0.2\,\text{cm}}{2 \times 10^2\,\text{cm}} \approx 10^{-3} \ll 1. \tag{5}$$

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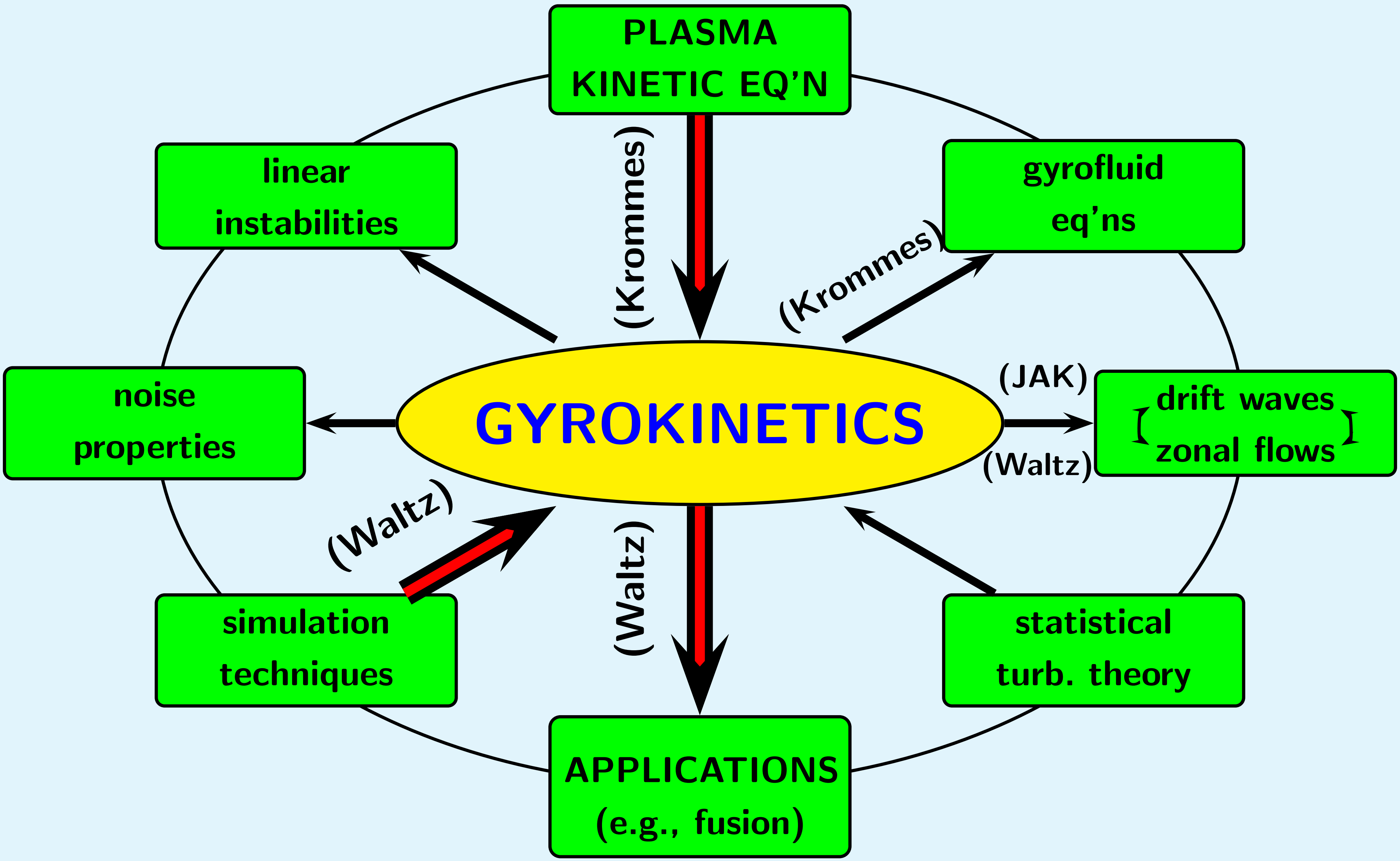
- Therefore, one can make some analytical reductions by averaging the motion over the rapid gyration (gyrophase ζ).

Gyrokinetics is the central formalism that describes the low-frequency physics of magnetized plasma.



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For $\omega/\omega_{ci} \ll 1$ and $k_\perp \rho_i \ll 1$, particles move on average via the drift equations.

$$\frac{d\vec{X}}{dt} = v_\parallel \hat{\mathrm{b}} + \vec{V}_E + \vec{V}_d, \tag{6a}$$

$$\frac{dv_\parallel}{dt} = \frac{q}{m} E_\parallel + \cdots, \tag{6b}$$

$$\frac{d\mu}{dt} = 0 \quad \text{(magnetic moment is conserved!)}. \tag{6c}$$

Here the **guiding-center drifts** are

$$\vec{V}_E \doteq c\vec{E} \times \hat{\mathrm{b}}/B \quad (\vec{E} \times \vec{B} \text{ drift}), \tag{7a}$$

$$\vec{V}_d \doteq \frac{v_\perp^2/2}{\omega_c} \hat{\mathrm{b}} \times \vec{\nabla} \ln B + \frac{v_\parallel^2}{\omega_c} \hat{\mathrm{b}} \times (\hat{\mathrm{b}} \cdot \vec{\nabla} \hat{\mathrm{b}}) \quad \text{(magnetic drifts)}, \tag{7b}$$

and the **magnetic moment**[a] is $\mu \approx \frac{1}{2} m v_\perp^2 / \omega_c(\vec{x})$. Note

- $\mu \propto \oint \vec{p} \cdot d\vec{q}$

  (special case of general theory of "adiabatic invariants").

[a] This formula for μ is *the lowest-order approximation* to the true magnetic moment $\overline{\mu}$. See later discussion.

For a distribution of particles and $k_\perp \rho_i \ll 1$, we may write the drift kinetic equation:

$$\frac{\partial F(\vec{X}, v_\parallel, \mu, t)}{\partial t} + \underbrace{v_\parallel \nabla_\parallel F}_{\text{parallel streaming}} + \underbrace{(\vec{V}_E + \vec{V}_d) \cdot \vec{\nabla} F}_{\text{guiding-center drifts}} + \underbrace{\frac{q}{m}(E_\parallel + \cdots)\frac{\partial F}{\partial v_\parallel}}_{\text{parallel acceleration}} = 0. \quad (8)$$

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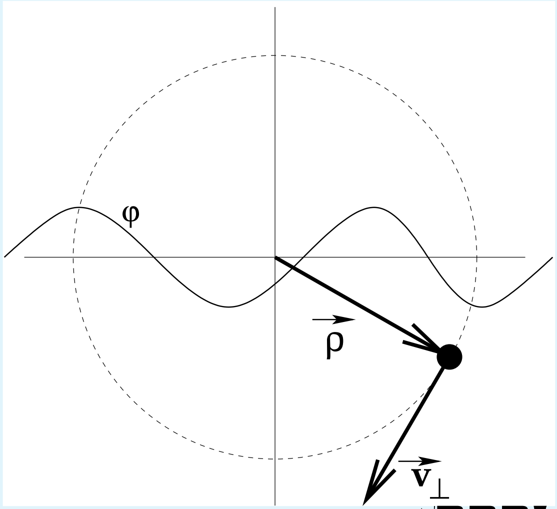
- For $k_\perp \rho \ll 1$, the particle position $\vec{x}$ and the guiding-center position $\vec{X}$ are essentially coincident.
- But for $k_\perp \rho = O(1)$, one must introduce the notion of the gyrocenter, the *average* position of the particle. Gyrocenters feel *effective* potentials.



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Gyrocenters feel effective electromagnetic fields. They move according to the gyrokinetic equation.

Let $\vec{E} = -\vec{\nabla}\varphi$ (electrostatics). Since $\varphi(\vec{x}) = (2\pi)^{-3}\int d\vec{k}\,\varphi_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}$,

$$\langle\varphi(\vec{x})\rangle_\zeta \equiv \langle\varphi\rangle(\vec{X}) = \frac{1}{2\pi}\int_0^{2\pi} d\zeta \int \frac{d\vec{k}}{(2\pi)^3}\,\varphi_{\vec{k}} e^{i\vec{k}\cdot[\vec{X}+\vec{\rho}(\zeta)]} \tag{9a}$$

$$= \int \frac{d\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{X}} \underbrace{J_0(k_\perp\rho)\varphi_{\vec{k}}}_{\langle\varphi\rangle_{\vec{k}}}. \tag{9b}$$

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The gyrokinetic equation is the workhorse of research on modern fusion microturbulence.

In the GK formalism, the magnetic moment μ is assumed to be conserved. It is just a parameter in the GKE.

The structure of the GKE is

$$\frac{\partial F(\vec{X}, v_\parallel, \mu, t)}{\partial t} + \dot{\vec{X}} \cdot \vec{\nabla} F + \dot{v}_\parallel \frac{\partial F}{\partial v_\parallel} + \underbrace{\dot{\mu}}_{0!} \frac{\partial F}{\partial \mu} + \dot{\zeta} \underbrace{\frac{\partial F}{\partial \zeta}}_{0!} = 0. \quad (11)$$

The coefficients of the gradients are the characteristic equations of motion:

- $\widehat{\mathrm{b}} \cdot \dot{\vec{X}}$ — parallel streaming
- $\dot{\vec{X}}_\perp$ — perpendicular drifts
- $\dot{v}_\parallel$ — parallel acceleration
- $\dot{\mu} = 0$ — μ is conserved.

To close the gyrokinetic equation, one needs the fields. Thus we must consider the GK Maxwell equations.

For definiteness, consider electrostatics. Then one has $\vec{E} = -\vec{\nabla}\varphi$ and one must solve Poisson's equation

$$-\nabla^2\varphi(\vec{x},t) = 4\pi \underbrace{\rho(\vec{x},t)}_{\text{charge density}} . \tag{12}$$

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- $F(\vec{Z}) \stackrel{T}{\Rightarrow} \rho(\vec{x}) \stackrel{\text{Poisson}}{\Rightarrow} \varphi(\vec{x}) \stackrel{\vec{\nabla}}{\Rightarrow} \vec{E}(\vec{x}) \stackrel{T^{-1}}{\Rightarrow} \vec{E}(\vec{X}) \stackrel{\text{GKE}}{\Rightarrow} F(\vec{Z})$.

The most important physical distinction between particles and gyrocenters lies in the polarization drift (of the ions).

Iterative solution of $m\dfrac{d\vec{u}}{dt} = q(\vec{E} + c^{-1}\vec{u}\times\vec{B}) + \cdots$ for $B = \text{const}$ leads to $\vec{u} = \vec{u}_{\vec{E}} + \vec{u}^{\text{pol}} + \cdots$, where the **polarization drift velocity** is

$$\vec{u}^{\text{pol}} \doteq \frac{1}{\omega_c}\frac{\partial}{\partial t}\left(\frac{c\vec{E}_\perp}{B}\right). \qquad (13)$$

The polarization drift leads to a **polarization charge density** ρ^{pol}:

$$\partial_t \rho^{\text{pol}} = -\vec{\nabla}\cdot(nq\vec{u}^{\text{pol}}). \qquad (14)$$

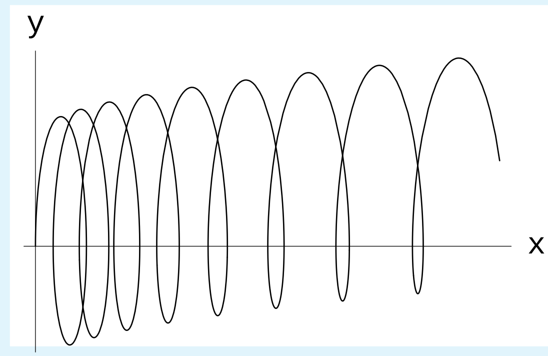

Fig. 7. $\vec{E} = E(t)\widehat{\mathrm{y}}$, $\vec{B} = B_0\widehat{\mathrm{z}}$. $\vec{V}_E \propto \widehat{\mathrm{x}}$, $\vec{V}^{\text{pol}} \propto \widehat{\mathrm{y}}$.

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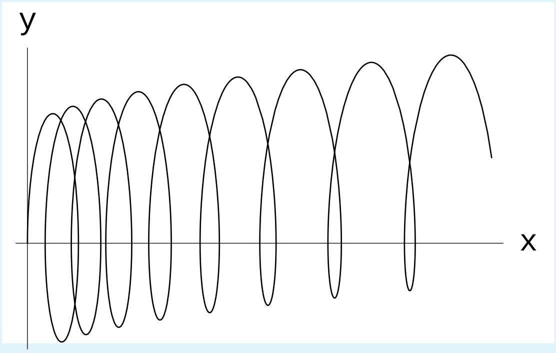

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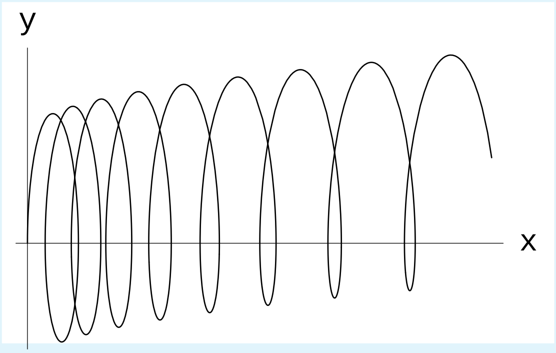

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Ion polarization leads to an interesting form of the GK Poisson equation.

One finds (for $T_i = 0$) $\rho_i^{\mathrm{pol}} = (nq)_i \rho_{\mathrm{s}}^2 \nabla_\perp^2 \Phi$, where $\Phi \doteq e\varphi/T_e$ and

$$\rho_{\mathrm{s}} \doteq c_{\mathrm{s}}/\omega_{ci} \quad (\text{“sound radius”};\ \rho_{\mathrm{s}} = \rho_i \text{ for } T_i = T_e), \tag{15a}$$

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or

$$-\nabla^2\Phi = k_{De}^2\left(\frac{n_i^{\text{G}}}{\overline{n}_i} + \rho_s^2\nabla_\perp^2\Phi - \frac{n_e^{\text{G}}}{\overline{n}_e}\right) \tag{17}$$

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Then Poisson's equation becomes

$$-\nabla^2 \varphi = 4\pi\rho = 4\pi[(\rho_i^{\mathrm{G}} + \rho_i^{\mathrm{pol}}) + (\rho_e^{\mathrm{G}} + \underbrace{\rho_e^{\mathrm{pol}}}_{\text{negligible}})] \tag{16}$$

or

$$-\nabla^2 \Phi = k_{De}^2 \left(\frac{n_i^{\mathrm{G}}}{\overline{n}_i} + \rho_{\mathrm{s}}^2 \nabla_\perp^2 \Phi - \frac{n_e^{\mathrm{G}}}{\overline{n}_e} \right) \tag{17}$$

or

$$-\Bigg[\underbrace{\nabla^2}_{\substack{\text{original}\\ \text{Poisson}}} + \underbrace{\left(\frac{\rho_{\mathrm{s}}^2}{\lambda_{De}^2} \right) \nabla_\perp^2}_{\substack{\text{ion}\\ \text{polarization}}} \Bigg] \Phi = k_{De}^2 \Bigg(\underbrace{\frac{n_i^{\mathrm{G}}}{\overline{n}_i} - \frac{n_e^{\mathrm{G}}}{\overline{n}_e}}_{\substack{\text{Note: GKE contains}\\ \text{no polarization drift}}} \Bigg). \tag{18}$$

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Ion polarization is intimately related to the vorticity of the plasma flow.

Across $\vec{B}$, the plasma moves predominantly with the $\vec{E} \times \vec{B}$ flow $\vec{u}_{\vec{E}} \propto \widehat{\mathrm{b}} \times \vec{\nabla}\varphi$. The vorticity associated with that flow is

$$\vec{\varpi} = \vec{\nabla} \times \vec{u}_{\vec{E}} \propto \nabla_{\perp}^{2}\varphi\,\widehat{\mathrm{b}} \quad \text{(for constant } B\text{)}. \tag{19}$$

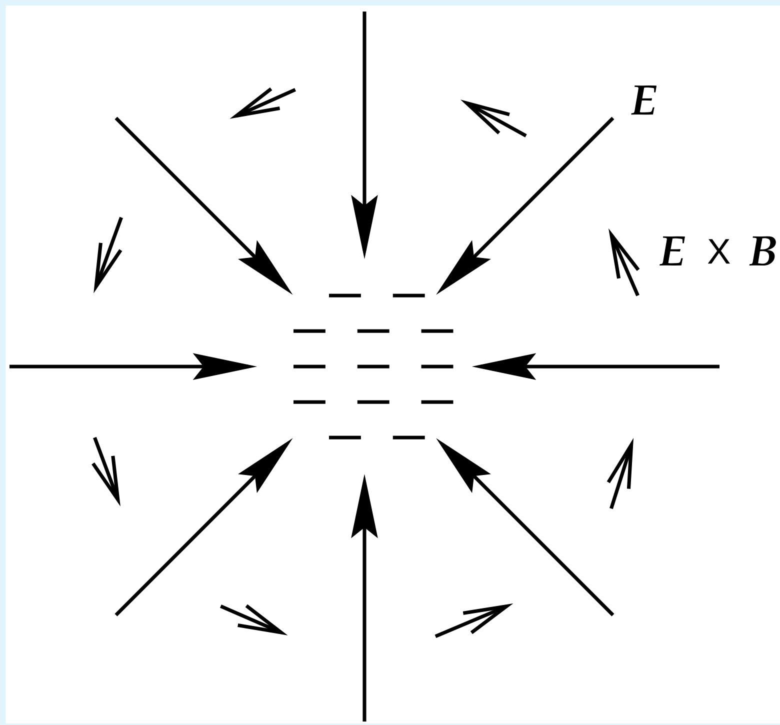

Fig. 8. Illustration of the intimate relationship between vorticity and polarization charge for a strongly magnetized plasma. A concentration of charge leads to an electric field; the plasma then rotates due to $\vec{E} \times \vec{B}$.

Ion polarization is responsible for the "dielectric constant of the gyrokinetic vacuum."

We had

$$-\Bigg[\underbrace{\nabla^2}_{\substack{\text{original}\\ \text{Poisson}}} + \underbrace{\left(\frac{\rho_s^2}{\lambda_{De}^2}\right)\nabla_\perp^2}_{\substack{\text{ion}\\ \text{polarization}}}\Bigg]\Phi = k_{De}^2\left(\underbrace{\frac{n_i^{\mathrm{G}}}{\overline{n}_i} - \frac{n_e^{\mathrm{G}}}{\overline{n}_e}}_{\substack{\text{Note: GKE contains}\\ \text{no polarization drift}}}\right). \quad (20)$$

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Compare with $\varepsilon\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ for a dielectric medium. Define

$$\varepsilon^G \doteq \frac{\rho_s^2}{\lambda_{De}^2} = \frac{\omega_{pi}^2}{\omega_{ci}^2} \qquad \left(\begin{array}{l}\text{dielectric constant of the "GK vacuum",}\\ \text{analogous to permittivity } \varepsilon_0 \text{ of free space}\end{array}\right). \quad (21)$$

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Then

$$-\lambda_{De}^2\Big(\overbrace{\underbrace{\nabla^2}_{\text{original Poisson}}}^{\text{small}} + \overbrace{\underbrace{\varepsilon^G\nabla_\perp^2}_{\text{ion polarization}}\Big)\Phi = \underbrace{\frac{n_i^G}{\overline{n}_i} - \frac{n_e^G}{\overline{n}_e}}_{\text{gyrocenter (charge) density}}}^{\text{quasineutrality}}. \qquad (22)$$

For fusion applications, $\varepsilon^G \gg 1$, so we can neglect the original ∇^2 and deal with the quasineutrality condition:

The GK Poisson equation is a statement of quasineutrality.

The quasineutrality condition $n_i \approx n_e$ can be expressed as

$$-\underbrace{\rho_s^2 \nabla_\perp^2 \Phi}_{\text{ion polarization}} = \underbrace{\frac{n_i^{\mathrm{G}}}{\overline{n}_i} - \frac{n_e^{\mathrm{G}}}{\overline{n}_e}}_{\text{net gyrocenter density}} . \tag{23}$$

(This form is correct only for $T_i = 0$, but it can be generalized.)

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We have now achieved closure: the gyrokinetic–Poisson system:

GKE:

$$\partial_t F = \underbrace{\cdots + \langle \vec{V}_{\vec{E}} \rangle [\Phi] \cdot \vec{\nabla} F + \cdots}_{\text{right-hand side depends on potential(s)}}; \tag{24}$$

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Knowledge of the gyrokinetic distribution function enables one to calculate turbulent transport.

Example — Particle transport:

$$\frac{\partial \overline{n}}{\partial t} + \frac{\partial \overline{\Gamma}}{\partial x} = 0, \tag{26}$$

where the turbulent particle flux is $\overline{\Gamma} = \overline{\delta V_{E,x}\, \delta n}$. Also, heat, momentum, ...

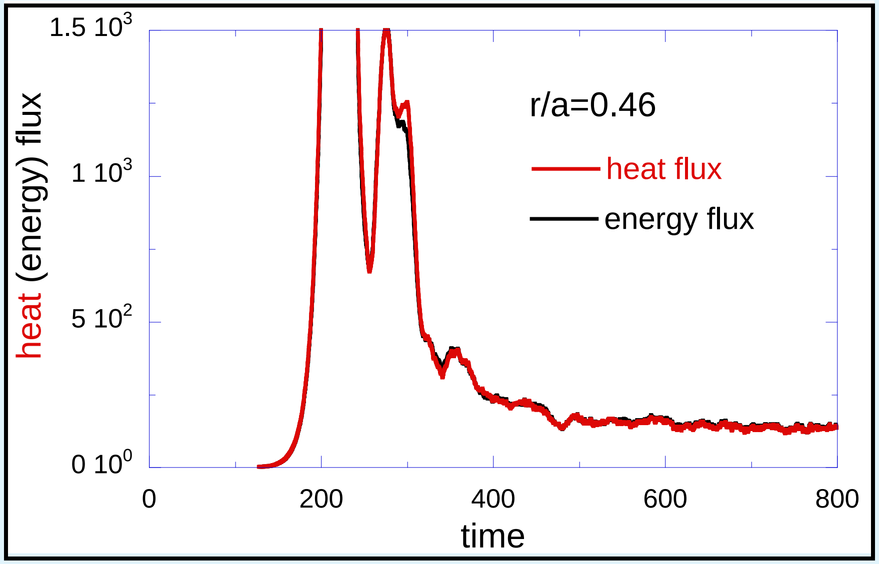

Fig. 9. A randomly selected simulation result showing the approach to saturated heat flux from arbitrary initial conditions. From Wang et al., Phys. Plasmas 13, 092505 (2006).

The GK formalism leads readily to the drift wave.

- Continuity equation for ion gyrocenters ($T_i = 0$):

$$\partial_t n_i^{\mathrm{G}} + \vec{\nabla}_\perp \cdot (\vec{u}_{\vec{E}} n_i^{\mathrm{G}}) + \underbrace{\nabla_\parallel (u_{\parallel i} n_i^{\mathrm{G}})}_{\text{neglect}} = 0. \tag{27}$$

Linearize assuming $L_n^{-1} \doteq -\partial_x \ln \overline{n}(x)$:

$$\partial_t (\delta n_i^{\mathrm{G}} / \overline{n}) + \underbrace{\delta \vec{u}_{\vec{E}} \cdot \vec{\nabla} \ln \overline{n}}_{V_* \partial_y \delta\Phi} = 0, \text{ where } \underbrace{V_* \doteq \rho_{\mathrm{s}} c_{\mathrm{s}} / L_n}_{\text{“diamagnetic velocity”}}. \tag{28}$$

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Thus $\partial_t [\overbrace{(1 - \rho_{\mathrm{s}}^2 \nabla_\perp^2) \delta\Phi}] + V_* \partial_y \delta\Phi = 0$, or, with $\omega_* \doteq k_y V_*$,

$$\boxed{\Omega_{\vec{k}} = \frac{\omega_*(\vec{k})}{1 + k_\perp^2 \rho_{\mathrm{s}}^2} \quad \text{(drift wave).}} \tag{29}$$

The drift wave propagates in the poloidal direction.

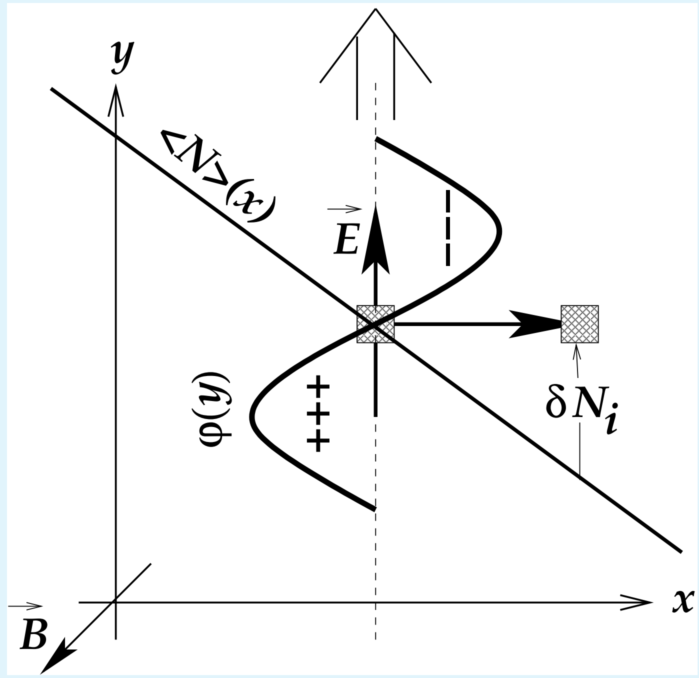

Fig. 10. Essential physics of the drift wave (polarization ignored).

- Assume $\varphi \sim \cos(k_y y - \Omega t)$.
- $\vec{E} \times \vec{B} \propto \widehat{\mathrm{x}}$ creates ion density fluctuations by advecting the background density profile.
- Those must be neutralized by electron flow along the field lines.
- For consistency, the wave must propagate upwards.

Nonlinearly, the Hasegawa–Mima equation emerges.

- Continuity equation for ion gyrocenters ($T_i = 0$):

$$\partial_t n_i^{\mathrm{G}} + \vec{\nabla}_\perp \cdot (\vec{u}_{\vec{E}} n_i^{\mathrm{G}}) + \underbrace{\nabla_\parallel (u_{\parallel i} n_i^{\mathrm{G}})}_{\text{neglect}} = 0. \tag{30}$$

$$\frac{\partial(\delta n_i^{\mathrm{G}}/\overline{n})}{\partial t} + \underbrace{\delta\vec{u}_{\vec{E}} \cdot \vec{\nabla} \ln \overline{n}}_{V_* \partial_y \delta\Phi} + \delta\vec{u}_{\vec{E}} \cdot \vec{\nabla}(\delta n_i^{\mathrm{G}}/\overline{n}) - \langle \dots \rangle = 0. \tag{31}$$

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Thus we obtain the Hasegawa–Mima equation:

$$\boxed{(\underbrace{1}_{\substack{\text{adiab.}\\ \text{elect.}}} - \underbrace{\rho_{\mathrm{s}}^2 \nabla_\perp^2}_{\substack{\text{ion}\\ \text{polarization}}})\frac{\partial \delta\Phi}{\partial t} + \underbrace{V_* \frac{\partial \delta\Phi}{\partial y}}_{\substack{\text{linear}\\ \text{drift wave}}} + \underbrace{\vec{u}_{\vec{E}} \cdot \vec{\nabla}(-\rho_{\mathrm{s}}^2 \nabla_\perp^2 \delta\Phi)}_{\substack{\text{nonlinear } \vec{E}\times\vec{B}\\ \text{advection of vorticity}}} = 0.}$$

(3

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The Hasegawa–Mima equation is not suitable for the treatment of zonal flows!

Zonal flow: $\varphi(r,\theta,\phi) \to \varphi(r,\cancel{\theta},\cancel{\phi}) \Rightarrow \vec{V}_E = V_{\vec{E},\mathrm{pol}}(r)\widehat{\theta}$.

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- The Hasegawa–Mima equation must be *modified* in order to describe zonal flows (which do not permit adiabatic electron response because $k_\parallel = 0 \Rightarrow \omega/k_\parallel v_{te} \gg 1$, i.e., *fluid* regime). Let

$$\widehat{\alpha} \doteq \begin{cases} 1 & (k_\parallel \neq 0) \\ 0 & (k_\parallel = 0). \end{cases} \tag{33}$$

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$$\widehat{\alpha} \doteq \begin{cases} 1 & (k_\parallel \neq 0) \\ 0 & (k_\parallel = 0). \end{cases} \tag{33}$$

Then we obtain the modified Hasegawa–Mima equation:

$$\boxed{\Big(\underbrace{\widehat{\alpha}}_{\text{elect. response}} - \underbrace{\rho_s^2 \nabla_\perp^2}_{\text{ion polarization}}\Big)\frac{\partial \delta\Phi}{\partial t} + \underbrace{V_* \frac{\partial \delta\Phi}{\partial y}}_{\text{linear drift wave}} + \underbrace{\vec{u}_{\vec{E}} \cdot \vec{\nabla}[(\widehat{\alpha} - \rho_s^2 \nabla_\perp^2)\delta\Phi]}_{\text{nonlinear } \vec{E}\times\vec{B} \text{ advection of ion density}} = 0.} \tag{34}$$

Zonal flows are driven by drift-wave Reynolds stresses.

For zonal flows, the dominant terms in the modified Hasegawa–Mima equation are

$$\frac{\partial(\nabla_\perp^2 \varphi^Z)}{\partial t} + \vec{\nabla} \cdot (\vec{V}_E^D\, \nabla_\perp^2 \varphi^D) = 0. \tag{35}$$

Average over (periodic) poloidal direction y:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \overline{\varphi^Z}}{\partial x^2}\right) + \frac{\partial}{\partial x}\left[\overline{-\frac{\partial \varphi^D}{\partial y}\left(\frac{\partial^2 \varphi^D}{\partial x^2} + \frac{\partial^2 \varphi^D}{\partial y^2}\right)}\right] = 0. \tag{36}$$

Note that $\partial\varphi/\partial x = u_y$. Integrate in x. Manipulate to find

$$\underbrace{\frac{\partial \overline{u}_y^Z(x)}{\partial t}}_{\text{sheared zonal flows}} + \frac{\partial}{\partial x}\Big(\underbrace{\overline{\delta u_x^D \delta u_y^D}}_{\text{Reynolds stresses}}\Big) = 0. \tag{37}$$

Self-consistent saturated turbulent states involve coupled drift waves and zonal flows.

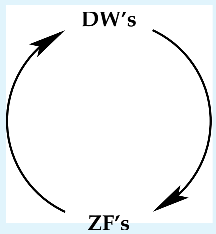

Fig. 11. Drift waves and zonal flows form a self-regulating coupled system.

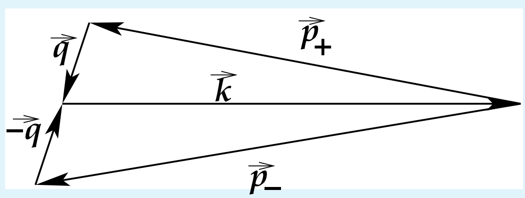

Fig. 12. Drift-wave ($\vec{k}$) sidebands ($\vec{p}_+$ and $\vec{p}_-$) couple to drive zonal flows ($\vec{q}$).

Fig. 13. Zonal-flow shearing in New York City. Zonal flows destroy drift-wave eddies $\Rightarrow$ smaller turbulent transport.

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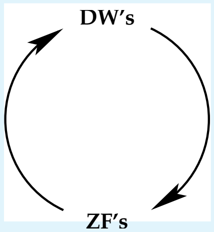

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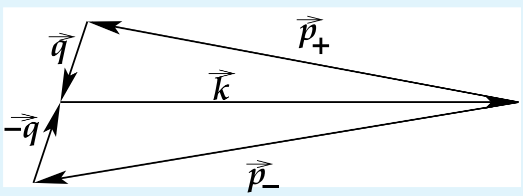

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- **Statistical closure theory can be used to calculate the zonal flow generation rate.**

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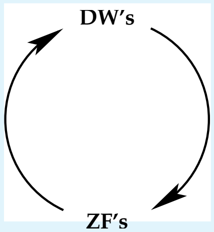

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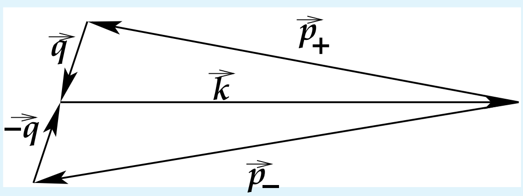

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- Statistical closure theory can be used to calculate the zonal flow generation rate.
- Kraichnan's theory of 2D Navier–Stokes eddy viscosity can be recovered as a special limit.

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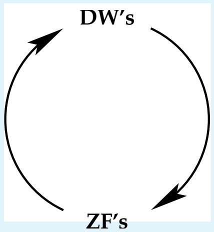

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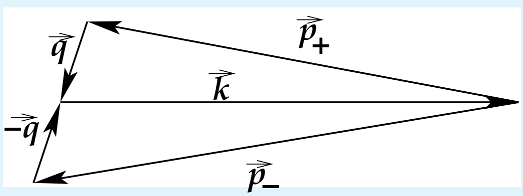

Fig. 12. Drift-wave ($\vec{k}$) sidebands ($\vec{p}_+$ and $\vec{p}_-$) couple to drive zonal flows ($\vec{q}$).

Fig. 13. Zonal-flow shearing in New York City. Zonal flows destroy drift-wave eddies $\Rightarrow$ smaller turbulent transport.

- **Statistical closure theory can be used to calculate the zonal flow generation rate.**
- **Kraichnan's theory of 2D Navier–Stokes eddy viscosity can be recovered as a special limit.**
- **Connections to field-theoretic Hamiltonian theory and Casimir invariants can be demonstrated. . . . (But let's proceed with more discussion of GKs. . .)**

Zonal flows can eat drift waves.

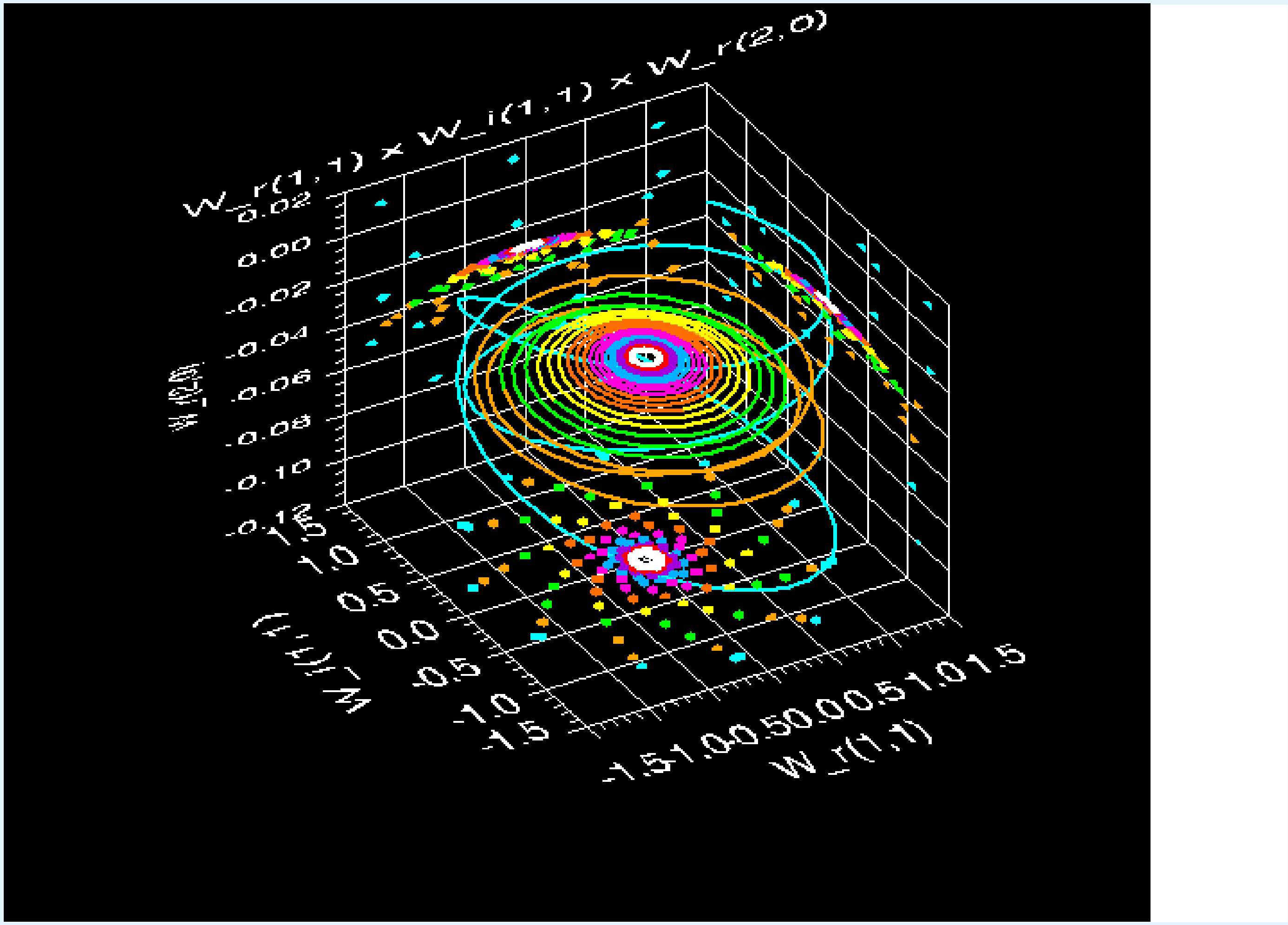

Fig. 14. Projection of a 10D center-manifold reduction of PDEs for ion-temperature-gradient-driven (ITG) turbulence onto two ITG amplitudes (x and y axes) and a zonal amplitude (z axis). Only the zonal mode survives for these parameters.

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Zonal-flow suppression of drift waves can delay the onset of turbulence.

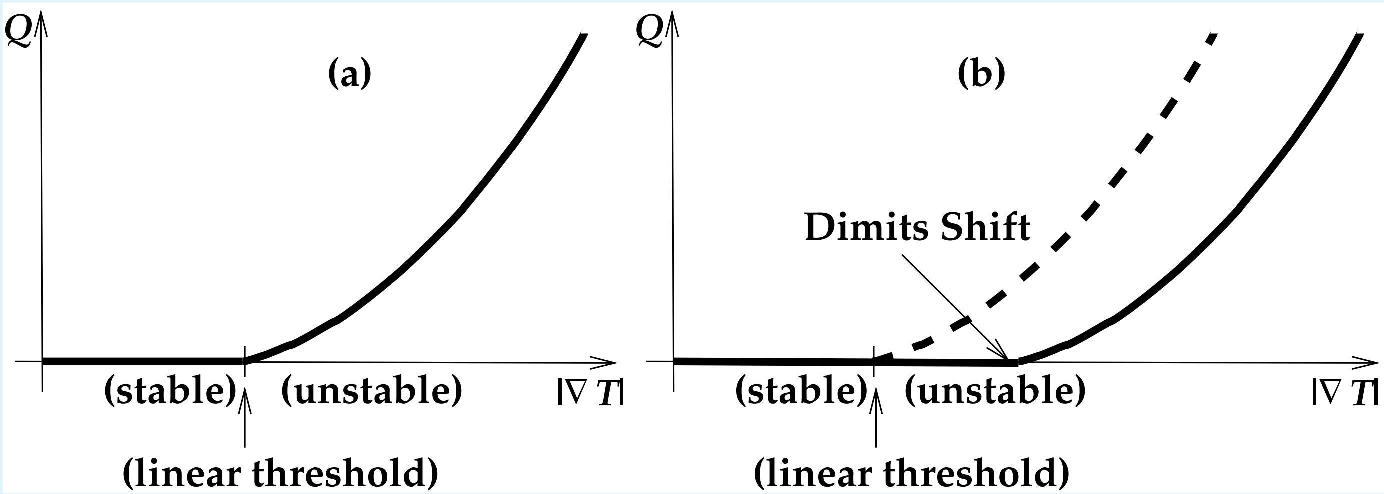

Fig. 15. (a) One might expect that linear instability would drive ITG turbulence and heat flux. (b) In actuality, zonal modes suppress turbulence onset for some region (the *Dimits shift*) above linear threshold.

Summary of Heuristic Gyrokinetics

- Gyrocenters move (only) with effective $\vec{E} \times \vec{B}$ drifts and magnetic drifts.

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So far we have been doing handwaving. BUT: Various things have been swept under the rug!

- Most fundamentally, the formalism assumes that μ is conserved. But in reality, what most people call μ, namely $\mu \doteq \frac{1}{2}mv_\perp^2/\omega_c(\vec{x})$, **is not exactly conserved**. (It is not even Galilean invariant!) *What quantity $\overline{\mu}$ is really conserved?*

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- When appropriately formulated, gyrokinetics is a **Hamiltonian system** (very useful to know both analytically and numerically). However, that is not apparent from the "derivation" given so far.
- Finally, are we sure that the gyrokinetic system conserves the appropriate things (e.g., energy or momentum)?

Gyrokinetics has an interesting intellectual history.

Catto (1978) — Linearized gyrokinetics

Frieman & Chen (1982) — First **nonlinear gyrokinetic equation (GKE)**

Lee (1983) — Reformulated GKE for **particle simulation**

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Goals:

- Systematic derivation of gyrokinetics from first principles (Newton's laws and Maxwell's equations).
- Asymptotic construction of the "true" adiabatic invariant $\overline{\mu}$ (in complicated geometry with $\vec{\nabla} B \neq \vec{0}$ and also $\partial_t \vec{E} \neq \vec{0}$).

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Remark: Much of analytical physics consists of **finding the "best" variables**. For example, contrast

- $\vec{x}$ — particle position
- $\vec{X}$ — *lowest-order* gyrocenter position (for circular motion)
- $\overline{\vec{X}}$ — "true" gyrocenter position

We'll transform $\{\vec{x}, \vec{v}\} \Rightarrow \{\vec{X}, U, \mu, \zeta\} \Rightarrow \{\overline{\vec{X}}, \overline{U}, \overline{\mu}, \overline{\zeta}\}$.

The dynamical equations of particle motion follow from Lagrange's variational principle.

$$0 = \delta \int_{t_0}^{t_1} L \, dt \quad (L \doteq \vec{p} \cdot \dot{\vec{q}} - H) \tag{39}$$

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- The variational principle does not care about the particular variables used to express γ.
- Therefore, we are free to use "better" variables (chosen to satisfy some criterion); then the variational principle gives us the equations of motion for those variables.

One can cast γ into a representation-free form.

Define

$$z^\nu \doteq \{t, \vec{x}, \vec{p}\}, \tag{42a}$$

$$\gamma_\nu \doteq \{-H, \vec{p}, \vec{0}\}. \tag{42b}$$

Then the one-form can be written covariantly as

$$\gamma = \gamma_\nu dz^\nu \quad \text{(summation convention)}. \tag{43}$$

Obviously this can be transformed to any set of variables one pleases:

$$\gamma(z) = \overline{\gamma}(\overline{z}) = \overline{\gamma}_\nu d\overline{z}^\nu. \tag{44}$$

For any z^ν's, the Euler–Lagrange equations of motion are

$$\omega_{\sigma\nu}\frac{dz^\nu}{dt} = 0, \tag{45}$$

where the symplectic two-form is

$$\omega_{\sigma\nu} \doteq \frac{\partial\gamma_\nu}{\partial z^\sigma} - \frac{\partial\gamma_\sigma}{\partial z^\nu}. \tag{46}$$

Symmetries of the Lagrangian are related to conservation laws. In particular, Noether's Theorem is fundamental.

Noether's Theorem:

If all of the γ_ν's are independent of a particular coordinate z^α, then γ_α is conserved.

I.e.,

$$\gamma = \gamma_1(z^1, \dots, \cancel{z^\alpha}, \dots, z^n)dz^1 + \cdots + \underbrace{\boxed{\gamma_\alpha(z^1, \dots, \cancel{z^\alpha}, \dots, z^n)}}_{\text{conserved!}} dz^\alpha + \cdots + \gamma_n(z^1, \dots, \cancel{z^\alpha}, \dots, z^n)dz^n. \tag{47}$$

Proof: *A simple exercise left to the reader.*

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A special application of Noether's Theorem leads to the conserved magnetic moment $\overline{\mu}$.

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Gyrokinetics is derived by removing gyrophase dependence from $\overline{\gamma}$. As a byproduct, one obtains formulas for the conserved $\overline{\mu}$ and the other gyrocenter variables.

How does gyrophase dependence enter the one-form?

In canonical variables $(\vec{x}, \vec{p})$, one has

$$\gamma = \vec{p} \cdot d\vec{x} - H\,dt, \tag{48}$$

with $\vec{p} \doteq m\vec{v} + (q/c)\vec{A}(\vec{x})$ and $H \doteq p^2/2m + q\varphi(\vec{x}, t)$. With this form, it is not obvious that the particle gyrates around a magnetic field line; the gyrophase ζ is not apparent. *It's better to use noncanonical variables...*

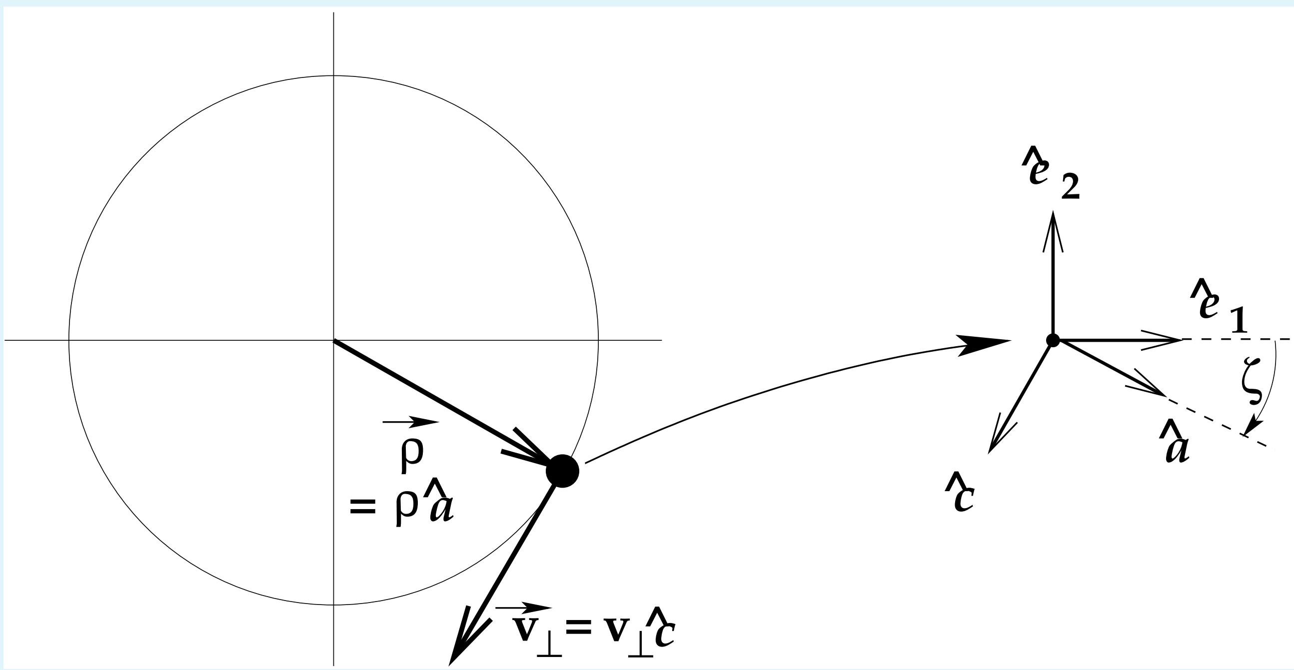

Fig. 16. Illustration of the lowest-order gyrokinetic variables. $\widehat{\mathrm{a}}$ and $\widehat{\mathrm{c}}$ rotate with the particle. Vectors can be decomposed with respect to either of the orthonormal sets $\{\widehat{\mathrm{a}}, \widehat{\mathrm{b}}, \widehat{\mathrm{c}}\}$ or $\{\widehat{\mathrm{e}}_1, \widehat{\mathrm{e}}_2, \widehat{\mathrm{b}}\}$.

Gyration is more apparent by using the lowest-order gyrocenter variables.

Instead of working with $\vec{z} \doteq \{\vec{x}, \vec{v}\}$, let's use the *lowest-order gyrocenter variables* $\vec{Z} \doteq \{\vec{x}, U, \mu, \zeta\}$. Then

gyrophase dependence

$$\gamma = \underbrace{\frac{q}{c}\vec{A}\cdot d\vec{x}}_{O(\epsilon^{-1})} + \underbrace{m[U\widehat{\mathrm{b}}(\vec{x}) + v_\perp \widehat{\mathrm{c}}(\zeta, \vec{x})]\cdot d\vec{x}}_{O(1)}$$

$$- \underbrace{\left(\frac{1}{2}mU^2 + \mu\omega_c(\vec{x})\right)dt}_{O(1)} + \underbrace{q\varphi(\vec{x}, t)dt}_{O(\epsilon)}. \qquad (49)$$

We used

$$\frac{mv_\perp}{qA/c} = \frac{v_\perp}{L_B[q(A/L_B)/mc]} = \frac{v_\perp/\omega_c}{L_B} = \frac{\rho}{L_B} = \epsilon_B. \qquad (50)$$

- The fundamental expansion parameter ϵ is
  - the size of the magnetic inhomogeneity, $\epsilon = \epsilon_B \doteq \rho/L_B$;
  - the size of the fluctuating fields, e.g., $\epsilon = \epsilon_\varphi \doteq e\delta\varphi/T_e$.

We are now set to remove ζ dependence (perturbatively) order by order.

We have

$$\gamma = \gamma^{(-1)} + \underbrace{\gamma^{(0)}}_{\zeta\text{-dependent}} + \gamma^{(1)}. \tag{51}$$

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- Then $\overline{\gamma}_{\overline{\zeta}} \equiv \overline{\mu}$ is conserved (by Noether's theorem).

How does the one-form γ transform?

The value of γ is independent of coordinate system: $\overline{\gamma}(\overline{z}) = \gamma(z)$. Recall that $\overline{z} = Tz$. Then

$$\underbrace{T}_{\text{“pull-back”}}\overline{\gamma}(\vec{z}) = \gamma(\vec{z}), \tag{53a}$$

$$\overline{\gamma}(z) = \underbrace{T^{-1}}_{\text{“push-forward”}}\gamma(z). \tag{53b}$$

- Also, note that the differential of a scalar function S does not contribute to the equations of motion: $\delta \int_{t_0}^{t_1} dS = 0$.
- The “gauge scalar” S gives us extra freedom that helps us achieve the goal of removing gyrophase dependence.
- Thus the basic formula is

$$\boxed{\overline{\gamma}(\cancel{\zeta}) = \underbrace{T^{-1}(\zeta)}_{\text{push-forward transformation}}\ \gamma(\zeta) + \underbrace{dS(\zeta)}_{d\text{(gauge scalar)}}\ .} \tag{54}$$

We will use a Lie transformation $T(\epsilon)$.

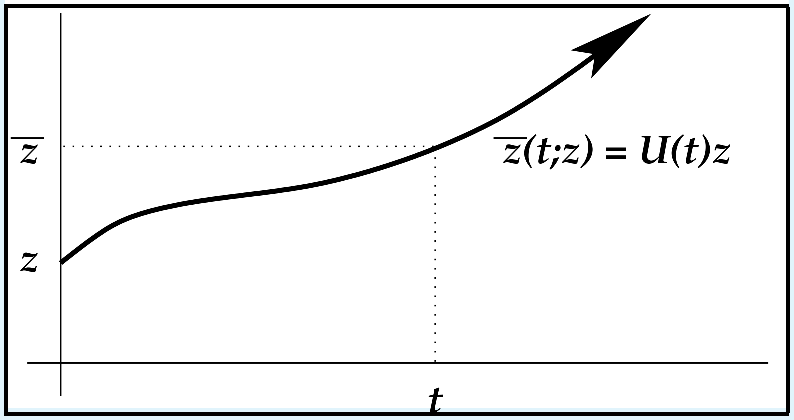

Fig. 17. Evolution in time t: The solution of $\partial_t \overline{z} = V(\overline{z})$ [$\overline{z}(0) = z$] can be thought of as the Lie transformation $\overline{z}(t; z) = U(t)z$.

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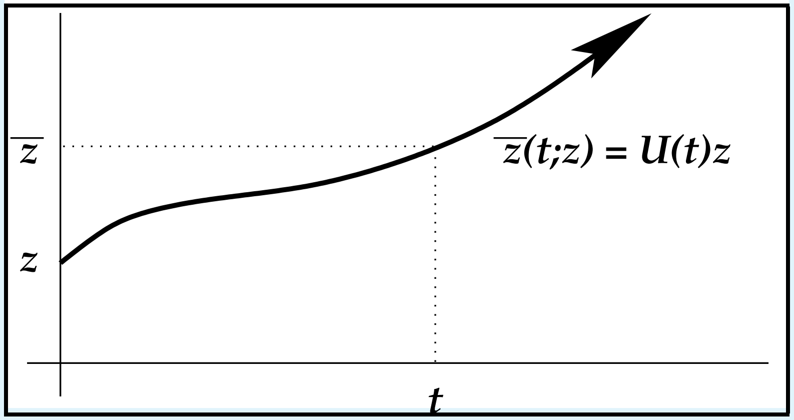

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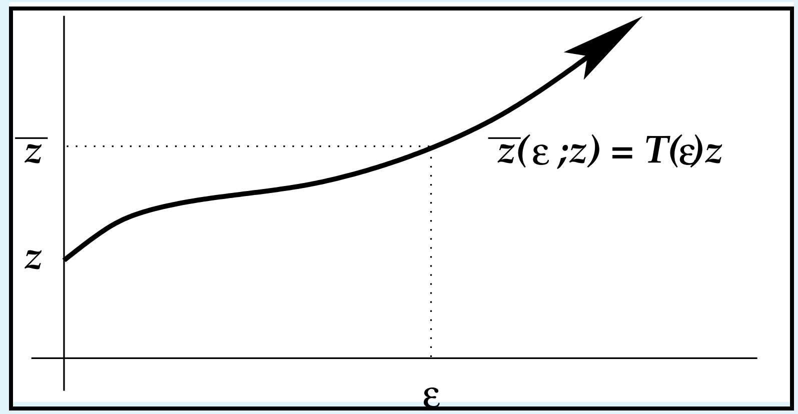

Fig. 18. "Evolution" in perturbation strength ϵ: The solution of $\partial_\epsilon \overline{z} = g(\overline{z})$ [$\overline{z}(0) = z$] can be thought of as the Lie transformation $\overline{z}(\epsilon; z) = T(\epsilon)z$.

By analogy with time-dependent evolution, we use a Lie transformation in the perturbation parameter ϵ: $\overline{z} = T(\epsilon)z$, where

$$T(\epsilon) = e^{\epsilon L_g} \approx 1 + \epsilon L_g + \cdots . \tag{55}$$

Here

$$L_g \doteq \underbrace{g(z)}_{\text{generating function}} \frac{\partial}{\partial z}. \tag{56}$$

For use in perturbative calculations, Lie transforms can be compounded.

Consider the set of generating functions $\{g_n \mid n = 1, 2, \ldots\}$ $[g_n = O(\epsilon^n)]$, and define $L_n \equiv L_{g_n}$. Then

$$T = e^{L_1} e^{L_2} e^{L_3} e^{L_4} \ldots, \tag{57a}$$

$$T^{-1} = \ldots e^{-L_4} e^{-L_3} e^{-L_2} e^{-L_1} \tag{57b}$$

$$= 1 - L_1 + \left(-L_2 + \frac{1}{2} L_1^2\right) + \left(-L_3 + L_2 L_1 - \frac{1}{6} L_1^3\right) + \left(-L_4 + L_3 L_1 + \frac{1}{2} L_2^2 - \frac{1}{2} L_2 L_1^2 + \frac{1}{24} L_1^4\right) + \cdots . \tag{57c}$$

We can use these results and the formula[a]

$$L_g \gamma = g^\sigma \omega_{\sigma\nu} dz^\nu \tag{58}$$

to expand the basic law $\overline{\gamma} = T^{-1}\gamma + dS$ order by order.

[a] The differential geometers will recognize that there is a fundamental swindle here; please see me after class!

From $\overline{\gamma} = T^{-1}\gamma + dS$, we are thus led to the equations of one-form perturbation theory:

$$\overline{\gamma}^{(-1)} = \gamma^{(-1)} + dS^{(-1)}, \tag{59a}$$

$$\overline{\gamma}^{(0)} = \gamma^{(0)} + dS^{(0)} - L_1\gamma^{(-1)}, \tag{59b}$$

$$\overline{\gamma}^{(1)} = \gamma^{(1)} + dS^{(1)} - L_1\gamma^{(0)} + (\tfrac{1}{2}L_1^2 - L_2)\gamma^{(-1)}, \tag{59c}$$

$$\overline{\gamma}^{(2)} = \gamma^{(2)} + dS^{(2)} - L_1\gamma^{(1)} + (\tfrac{1}{2}L_1^2 - L_2)\gamma^{(0)} + (-L_3 + L_2L_1 - \tfrac{1}{6}L_1^3)\gamma^{(-1)}, \tag{59d}$$

$$\overline{\gamma}^{(3)} = \gamma^{(3)} + dS^{(3)} - L_1\gamma^{(2)} + (\tfrac{1}{2}L_1^2 - L_2)\gamma^{(1)} + (-L_3 + L_2L_1 - \tfrac{1}{6}L_1^3)\gamma^{(0)} + (-L_4 + L_3L_1 + \tfrac{1}{2}L_2^2 - \tfrac{1}{2}L_2L_1^2 + \tfrac{1}{24}L_1^4)\gamma^{(-1)}. \tag{59e}$$

$$\overline{\gamma}^{(4)} = \cdots\,. \tag{59f}$$

The algebra is now straightforward, if tedious.

- At nth order, one needs to determine g_n^σ and $S^{(n)}$.
- The goal is to remove ζ dependence from $\overline{\gamma}_\sigma^{(n)}$.
- There is more than enough freedom to do this.

An example taken from the midst of the algebra:

$$\overline{\gamma}^{(2)}(\vec{Z}) = (\dots)\cdot d\vec{x} + (\dots)dt + (\dots)dU + (\dots)d\mu + \left(f_\zeta - g_1^\mu + \frac{\partial S^{(2)}}{\partial \zeta}\right) d\zeta. \tag{60}$$

Write $S = \langle S\rangle + \delta S$, such that $\langle \delta S\rangle = 0$. ($S$ must be periodic in ζ in order to avoid secularities.) δg_1^μ was determined at $O(\epsilon)$. From

$$\langle \overline{\gamma}_\zeta^{(2)}\rangle = \langle f_\zeta\rangle - \langle g_1^\mu\rangle, \tag{61}$$

choose $\langle g_1^\mu\rangle$ to eliminate $\langle \overline{\gamma}_\zeta^{(2)}\rangle$. Now the condition $\delta\overline{\gamma}_\zeta^{(2)} = 0$ determines $\delta S^{(2)} = \int d\zeta\,(-\delta f_\zeta + \delta g_1^\mu)$.

Everything is now determined (through some chosen order).

- The procedure gives us the new ($\overline{\zeta}$-independent) one-form $\overline{\gamma}$. Through second (relative) order,

$$\overline{\gamma} \approx [(q/c)\vec{A} + m\overline{U}\widehat{\mathrm{b}} - \overline{\mu}\vec{K}^*] \cdot d\overline{\vec{x}} + \overline{\mu}\, d\overline{\zeta} - \underbrace{(\tfrac{1}{2}m\overline{U}^2 + \overline{\mu}\omega_c + q\langle\varphi\rangle)}_{\text{gyroaveraged Hamiltonian}} dt, \qquad (62)$$

where

$$\vec{K}^* \doteq \vec{K} + \frac{1}{2}(\underbrace{\widehat{\mathrm{b}} \cdot \vec{\nabla} \times \widehat{\mathrm{b}}}_{\text{torsion}})\widehat{\mathrm{b}}, \quad \underbrace{\vec{K} \doteq (\vec{\nabla}\widehat{e}_1) \cdot \widehat{e}_2}_{\text{gyrogauge vector}}. \qquad (63)$$

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- From $\overline{\gamma}$, we can calculate the symplectic two-form $\overline{\varpi}$; thus we obtain the equations of motion for the gyrocenter:

$$\overline{\varpi}_{\sigma\nu}\frac{d\overline{z}^\nu}{dt} = 0. \tag{64}$$

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- The procedure gives us the new ($\bar{\zeta}$-independent) one-form $\bar{\gamma}$. Through second (relative) order,

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- And the g's give us $T(\epsilon)$, i.e., the asymptotic expansion of the proper ($\overline{\mu}$-conserving) gyrocenter variables: $\overline{\vec{Z}} = T(\epsilon)\vec{Z}$.

From the equations of motion, we obtain the (collisionless) gyrokinetic equation.

- The Vlasov equation has the covariant form

$$\frac{\partial f(\vec{z},t)}{\partial t} + \dot{z}^\nu \frac{\partial f}{\partial z^\nu} = 0. \tag{65}$$

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- Thus it retains its form in any system of coordinates:

$$\frac{\partial\ \widetilde{\overline{F}}(\vec{\overline{Z}},t)}{\partial t} + \dot{\overline{Z}}^{\nu}\frac{\partial \widetilde{\overline{F}}}{\partial \overline{Z}^{\nu}} = 0, \tag{66}$$

or

$$\frac{\partial \widetilde{\overline{F}}(\vec{\overline{X}},\overline{U},\overline{\mu},\overline{\zeta},t)}{\partial t} + \dot{\vec{\overline{X}}}\cdot\vec{\overline{\nabla}}\widetilde{\overline{F}} + \dot{\overline{U}}\frac{\partial \widetilde{\overline{F}}}{\partial \overline{U}} + \underbrace{\dot{\overline{\mu}}}_{0}\frac{\partial \widetilde{\overline{F}}}{\partial \overline{\mu}} + \dot{\overline{\zeta}}\frac{\partial \widetilde{\overline{F}}}{\partial \overline{\zeta}} = 0. \tag{67}$$

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- We can trivially average this over $\overline{\zeta}$, defining $\overline{F} \doteq \langle \widetilde{\overline{F}} \rangle_{\overline{\zeta}}$:

$$\boxed{\frac{\partial \overline{F}}{\partial t} + \dot{\vec{\overline{X}}}\cdot\vec{\overline{\nabla}}\overline{F} + \dot{\overline{U}}\frac{\partial \overline{F}}{\partial \overline{U}} = 0.} \tag{68}$$

This is what is usually called the (collisionless) gyrokinetic equation.

Summary of Systematology, Part I:
The gyrokinetic equations of motion

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- By Noether's theorem, $\overline{\mu}$ is conserved.
- Use the characteristic gyrocenter equations of motion to write the gyrokinetic equation for the gyrocenter distribution $\overline{F}$.

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Systematic Gyrokinetics, Part II: The Gyrokinetic Poisson Equation

To obtain the gyrokinetic Poisson equation, we must express the charge density of the particles in terms of the gyrocenter distribution.

Following Dubin, Krommes, Oberman, and Lee (1983), we must consider various distribution functions:

$$f(\vec{z}, t) \text{ — particle PDF} \tag{69a}$$

$$\widetilde{F}(\vec{Z}, t) \text{ — particle PDF in lowest-order gyrocenter coordinates} \tag{69b}$$

$$\widetilde{\overline{F}}(\overline{\vec{Z}}, t) \text{ — particle PDF in “true” gyrocenter coordinates} \tag{69c}$$

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Poisson's equation is naturally written in the particle (laboratory) coordinate system:

$$-\nabla^2 \varphi(\vec{x}, t) = 4\pi \sum_s (\overline{n} q)_s \int d\vec{v}' \underbrace{f_s(\vec{x}, \vec{v}', t)}_{\substack{\text{must obtain} \\ \text{from } \widetilde{\overline{F}}}}. \tag{70}$$

Now

$$I \doteq \int d\vec{v}'\, f(\vec{x},\vec{v}',t) = \underbrace{\int d\vec{z}'}_{\text{integral over complete phase space}} \quad \delta(\vec{x}-\vec{x}')f(\vec{z}',t) \tag{71a}$$

$$= \int J(\vec{Z})d\vec{Z}\,\delta(\vec{x}-\vec{x}'(\vec{Z}))\widetilde{F}(\vec{Z},t), \tag{71b}$$

where the Jacobian is

$$J \doteq \frac{\partial(\vec{z}')}{\partial(\vec{Z})}. \tag{72}$$

Now recall the pull-back transformation

$$\widetilde{F} = T\widetilde{\overline{F}}. \tag{73}$$

Therefore

$$I = \int \overline{J}\,d\overline{\vec{Z}}\,\delta(\vec{x}-\vec{x}'(\overline{\vec{Z}}))T \quad \underbrace{\widetilde{\overline{F}}(\overline{\vec{Z}},t)}_{\text{still depends on } \zeta} \quad . \tag{74}$$

[*This is still formally exact (assuming $\overline{\mu}$ is conserved).***]**

Use the pull-back transformation $T(\epsilon)$ to close Poisson's equation in terms of the gyrocenter distribution $\widetilde{\overline{F}}$.

$$\frac{\partial \widetilde{\overline{F}}}{\partial t} + \dot{\overline{\vec{X}}} \cdot \overline{\vec{\nabla}} \widetilde{\overline{F}} + \dot{\overline{v}}_{\parallel} \frac{\partial \widetilde{\overline{F}}}{\partial \overline{v}_{\parallel}} + \underbrace{\dot{\overline{\mu}}}_{0} \frac{\partial \widetilde{\overline{F}}}{\partial \overline{\mu}} + \dot{\overline{\zeta}} \frac{\partial \widetilde{\overline{F}}}{\partial \overline{\zeta}} = -C[\widetilde{\overline{F}}], \qquad (75)$$

$$-\nabla^2 \varphi = 4\pi \sum_s (\overline{n} q)_s \int \overline{J}\, d\overline{\vec{Z}}\, \delta(\vec{x} - \vec{x}'(\overline{\vec{Z}})) \underbrace{T}_{\text{pull-back}} \widetilde{\overline{F}}(\overline{\vec{Z}}, t). \qquad (76)$$

Important insight (Dubin, Krommes, Oberman, and Lee, 1983):

- By construction, $\dot{\overline{\vec{X}}}$, $\dot{\overline{v}}_{\parallel}$, and $\dot{\overline{\zeta}}$ are all independent of $\overline{\zeta}$.
- Therefore, $\overline{\zeta}$ dependence enters only from
  - initial conditions,
  - collisional effects.
- For *collisionless* theory, the kinetic equation does not couple the evolution of $\overline{F} \doteq \langle \widetilde{\overline{F}} \rangle_{\overline{\zeta}}$ and $\delta \overline{F} \doteq \widetilde{\overline{F}} - \langle \overline{F} \rangle$.

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For collisionless theory, we make the GK closure:

$$\widetilde{\overline{F}}(\overline{\vec{z}}, t) \approx \underbrace{\overline{F}(\overline{\vec{Z}}, t)}_{\text{obeys GKE}} . \tag{77}$$

$\partial_t \overline{F} + \cdots = 0.$

If we write $T = 1 + \delta T$, then

$$I \approx \int \overline{J}\, d\overline{\vec{Z}}\, \delta(\vec{x} - \vec{x}'(\overline{\vec{Z}}))(\underbrace{1}_{\substack{\text{gyrocenter}\\ \text{density}}} + \underbrace{\delta T}_{\substack{\text{polarization}\\ \text{density}}})\overline{F}(\overline{\vec{Z}}, t). \tag{78}$$

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- δT is complicated; it must be truncated at some order in ϵ!

The coupled gyrokinetic-Poisson equations comprise **a new nonlinear dynamical system** with unique properties.

The gyrokinetic nonlinear dynamical system describes an ensemble of gyrocenters placed into a $gyrokinetic\ vacuum$ possessing large dielectric permittivity ε^G. They move with the $\vec{E} \times \vec{B}$ and magnetic drifts.

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- is gyrogauge-invariant (physical quantities like $\overline{\mu}$ are independent of the choice of perpendicular unit vectors $\widehat{\mathrm{e}}_1$ and $\widehat{\mathrm{e}}_2$).

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A form of Noether's theorem guarantees that *any* approximation to S leads to an energetically consistent theory!

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- Alternatively, derive both the GKE and the GK Poisson equation from a single variational principle employing an approximate Lagrangian (thereby *guaranteeing* energetic consistency).

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The Current Status of Gyrokinetics

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- Analytical theory of drift waves and related modes:
  - linear theory in general magnetic geometry is well developed;
  - wave (gyro)kinetic equations for weak turbulence are also studied.

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- Simulation techniques:
  - particle-in-cell (PIC) / Monte Carlo sampling
  - "continuum" or "Vlasov" approach (direct solution of the 5D GKE; see, for example, the GYRO code of Candy & Waltz)

There remain some serious concerns.

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- Is gyrokinetics ill-posed for general 3D magnetic fields (torsional or stochastic), as has recently been asserted? NO (but subtle: anholonomic frame fields, global coordinate systems, etc.).
- Is the conventional gyrokinetic closure adequate for the study of long-wavelength phenomena such as
  - macroscopic electric fields and plasma rotation,
  - zonal flows?

Perhaps not (Parra & Catto, 2008, 2009)...

Physics on the transport time scale: GK momentum conservation, rotation, etc.

- Gyrokinetics is good for studies of low-frequency microturbulence on the nonlinear saturation time scale ($\ll$ plasma confinement time).
- What about transport-time-scale physics?
  - profile relaxation
  - radial diffusion of momentum
- There might be interchanges of limits ($\epsilon \to 0$; $t \to \infty$), inconsistent truncations [GKE vs GK Poisson; $k_\perp \rho_i \to 0$ although originally derived for $k_\perp \rho_i = O(1)$], etc.

Following F. Parra (2009, unpublished), consider a slab limit with constant $\vec{B}$ (as did Dubin et al., 1983):

x — "radial" direction (of profile gradients)

y — "poloidal" direction

z — direction of $\vec{B}$.

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Standard gyrokinetic truncation may introduce a spurious momentum source.

Let $\overline{Q}(x) \doteq (L_y L_z)^{-1} \int_0^{L_y} dy \int_0^{L_z} dz\, Q(x,y,z)$ denote the "flux-surface average." Then an *exact* moment of the Vlasov equation leads to

$$\frac{\partial(n_i m_i \overline{u}_{i,y})}{\partial t} = -\frac{\partial}{\partial x}\Bigg(\underbrace{\overline{\pi}_{xy}}_{\substack{\text{off-diagonal}\\ \text{component of}\\ \text{stress tensor}}} \Bigg). \tag{79}$$

Here

$$\overline{\pi}_{xy} \sim n_i m_i \overline{V_{E,x} V_{E,y}} \quad \text{(Reynolds stresses).} \tag{80}$$

Parra's result is that when slab gyrokinetics is truncated to $O(\epsilon^2)$,

$$\frac{\partial(n_i m_i \overline{u}_{i,y})}{\partial t} = -\frac{\partial \overline{\pi}_{xy}}{\partial x} + \underbrace{(\text{spurious momentum source})}_{O(\epsilon^3),\ \text{but nonconservative!}}. \tag{81}$$

(There are also further problems with collisions and recovery of neoclassical theory.) Final resolutions still pending...

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Summary

- Gyrokinetics is the key formalism for the study of low-frequency microturbulence in magnetized plasmas.
- It has an elegant and systematic development in terms of Hamiltonian / Lagrangian formalism and Lie perturbation theory.
- It has enjoyed many practical successes (see next-week's talk by Ron Waltz).
- There are some interesting outstanding problems.
  - The magnetic moment $\overline{\mu}$ is not always conserved.
  - Currently, there is some confusion about the role and proper expression of momentum conservation in gyrokinetics.

Gyrokinetics has had an interesting intellectual development; we should pay tribute to the pioneers.

Some key players on gyrokinetic *theory*:

- Catto
- Frieman & Chen
- Lee
- Dubin et al.
- Hahm
- Brizard
- Qin

More recently,

- Sugama
- Scott

. . . and especially Robert Littlejohn.

Companion talk on the application of nonlinear gyrokinetics to magnetic fusion:

"Gyrokinetic Simulation of Turbulent Transport in Fusion Plasmas"

R. E. Waltz

Tuesday morning, Aug. 4