Long time behavior of Stochastic Flows

Joint work with D. Dolgopyat and V. Kaloshin

Consider motion in a random flow in $\mathbb{R}^n$. Study long time behavior.

$$dx_t = v_\omega(x_t, t)dt \quad \text{or}$$

$$dx_t = v_\omega(x_t, t)dt + \sigma(x_t)dW_t.$$

For now, not specific about the right hand side.

1. One point motion. Typical result:

$$\frac{x_t}{\sqrt{t}} \to N(0, D) \quad \text{as } t \to \infty.$$

Indeed,

$$x_{NT} = (x_T - x_0) + (x_{2T} - x_T) + (...) + ...$$

(almost independent terms).

Examples: $dx_t = v(x_t)dt + dW_t$, v-periodic (Kozlov).

$$dx_t = v_\omega(x_t)dt + dW_t,$$

v_ω - stationary, ergodic, incompressible. (Kozlov, Papanicolaou-Varadhan, Zhikov,...)

result: diffusive behavior for almost all ω. (Often formulated in terms of PDE's with rapidly occilating coefficients.)

$$dx_t = v_\omega(t, x_t)dt,$$

v_ω - incompressible, Markovian in time, Gaussian. (Koralov, Fannjiang-Komorowski).

2. Follow the evolution of sets (measures) carried by the flow.

Example: Linear growth of the diameter of a connected (non-trivial) set.
(Lisei, Scheutzow, Cranston, Steinsaltz)

Results: For various flows, upper and lower bounds on the linear growth of the diameter.

Our results (Dolgopyat, Kaloshin, Koralov):

Motion:

$$dx_t = \sum_{k=1}^{d} X_k(x_t) \circ dW_k(t) + X_0(x_t)\, dt,$$

where, $x_t \in \mathbb{T}^n$, can be viewed as motion on $\mathbb{R}^n$,
X_k- measure preserving (for simplicity) vector fields;
non-degeneracy assumptions.

1. CLT for multi-point motion:

$$\frac{(x_t^1, \cdots, x_t^m)}{\sqrt{t}} \longrightarrow \text{Gaussian}$$

2. CLT for measures (for almost all realizations of the randomness). .

μ is a measure on Ω
(e.g. Legesgue meassure)

$$\int \frac{\mu(x)\mu(y)}{|x-y|^p} < \infty, \quad p > 0$$

(i.e. Hausdorff dimension $(\Omega) > 0$).
μ_t is a scaled image of μ.
$\mu_t(A) = \mu(x_0 : x_t \in \sqrt{t}A)$.
Theorem. $\mu_t \to N(0, D)$ *for almost every realization of randomness.*

3. Limit shape theorem (in $\mathbb{R}^2$)
$x \in \mathcal{W}_t$ if $x \in \Omega_s$ for some $s \leq t$. .

Theorem. $\exists\ B-$ *convex non-random set, such that*

$$tB(1-\epsilon) \subseteq \mathcal{W}_t \subseteq tB(1+\epsilon)$$

almost surely for sufficiently large t. .

Non-degeneracy assumptions:
(a) Strong Hormander condition (hypoellipticity)

$$Lie(X_1, \cdots, X_d)(x) = T_x \mathbb{T}^n$$

(b) The same for 2-point motion on $\mathbb{T}^n \times \mathbb{T}^n \backslash \Delta$.
(c) The same for the induced flow on the unit tangent bundle.
Notice: (a) $\Rightarrow$ Lyapunov exponents exist, are non-random

$$\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \log ||Dx_t(x)||$$

(d) positive Lyapunov exponent $\lambda_1 > 0$.
(d) follows from (a)-(c) for measure preserving flows (Baxendale)

Markovian flows:

$$\dot{x}_t = v(x_t, t),$$

$\mathrm{div} v = 0$; $v-$stationary, Markovian in time, Gaussian.

Example:

$$v(x,t) = \sum_{i=1}^{k} y^i(t) v_i(x), \quad v_i - \text{periodic}; \;\; y^i - \text{OU processes},$$

thus

$$dx_t = \sum_i v_i(x_t) y_t^i \; dt$$

$$dy_t^i = \alpha^i \, dW_t^i - \beta^i y_t^i \; dt$$

on $\mathbb{T}^2 \times \mathbb{R}^n$.

CLT $+$ transition from finite to infinite number of modes (Koralov)

$\lambda_1 > 0$ (Carmona, Xu, Molchanov)