Evidence of turbulence power laws from image data

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  - direct physical-based observation model, *prior* regularity model on motion
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▶ Power law priors in motion estimation

Motion estimation with power law priors

Motion $\mathbf{v}$ estimation by minimization of a global energy on the image :

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but depends on weight α and disconnected from physics !

Probability Distribution Function (PDF) of velocity increments :

- velocity increments :

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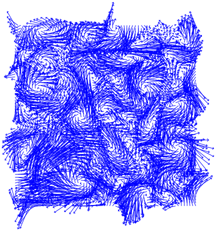
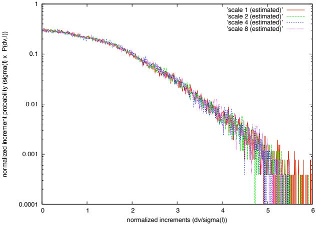
- self-similarity of the PDF (homogeneous and isotropic) at scale ℓ : $p_\ell(\delta v_{||}(\ell))$





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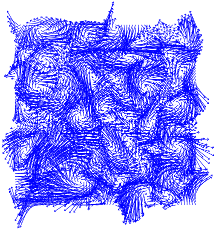

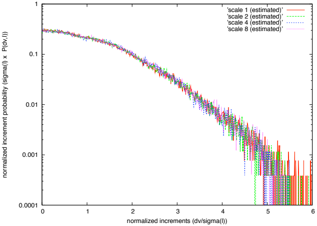

- Third order structure function (3-rd order moment) follows a power law :

$$\begin{aligned} \mathbb{E}[\delta v_{||}(\ell)^3] &= \int_{\mathbb{R}} \delta v_{||}(\ell)^3 p_\ell(\delta v_{||}(\ell)) d\delta v_{||}(\ell) \\ &\propto \beta_3 \ell^{\zeta_3} \end{aligned}$$

where β_3 et ζ_3 are the scaling law parameters

In particular,

- K41law of *Kolmogorov41* for 3D Navier-Stokes :

$$\mathbb{E}[\delta v_{||}(\ell)^3] = -\frac{4}{5}\epsilon\ell, \text{ \textit{in the inertial zone}}$$

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- Model of *Lindborg01* for atmospheric flows :

$$\mathbb{E}[\delta v_{||}(\ell)^3] = -\epsilon\ell + \frac{1}{8}\epsilon_\omega \ell^3,$$

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Strict self-similarity :

$$\beta\ell^\zeta \sim \mathrm{E}[\delta v_{||}(\ell)^2] = \mathrm{E}[\delta v_{||}(\ell)^3]^{\frac{2}{3}} \sim \beta\ell^{\frac{2\zeta_3}{3}}$$

- but intermittency $\Rightarrow$ non-strict self-similarity

Self-similar constraint at scale ℓ :

- 2-nd order moment : mean over the image support Ω and over directions $\mathbf{n}$ (horizontal, vertical and diagonal) :

$$\mathrm{E}[\delta v_{||}(\ell)^2] \simeq \frac{1}{|\mathbf{n}||\Omega|} \int_{\Omega} \int_{\mathbf{n}} \left(\delta v_{||}(\ell, \mathbf{s}, \mathbf{n}) \right)^2 ds d\mathbf{n}$$

- The turbulent velocity field $\mathbf{v}$ must respect the contraint :

$$g_\ell(\mathbf{v}, \beta, \zeta) = \frac{1}{2} (\mathrm{E}[\delta v_{||}(\ell)^2] - \beta \ell^{\zeta}) = 0$$

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Problem (P) : minimization of image observation model subject to contraints $\{g_\ell(\mathbf{v},\beta,\zeta)\}$:

$$(P)\begin{cases} \min_{\mathbf{v}}\ f_d(I,\mathbf{v}) \\ s.t.\ : \\ g_\ell(\mathbf{v},\beta,\zeta)=0, \quad \forall \ell \in \mathbf{I} \\ \mathbf{v}\in\mathbb{R}^n \end{cases}.$$

where $\mathbf{I}$ is the power law scale range.

- Lagrangian associated to problem (P) :

$$L(\mathbf{v}, \boldsymbol{\lambda}, \beta, \zeta) = f_d(I, \mathbf{v}) + \sum_{\ell \in \mathbf{I}} \lambda_\ell g_\ell(\mathbf{v}, \beta, \zeta), \quad \boldsymbol{\lambda} = \{\lambda_\ell\}.$$

Dual problem (D) : find the "saddle point" $(\mathbf{v}^*, \boldsymbol{\lambda}^*)$ of the lagrangian

$$L(\mathbf{v}^*, \boldsymbol{\lambda}^*, \beta, \zeta) = \max_{\boldsymbol{\lambda}}\{\min_{\mathbf{v}} L(\mathbf{v}, \boldsymbol{\lambda}, \beta, \zeta)\}$$

- Solved with Uzawa algorithm details

Objective : remove the prior dependance by

- selecting the most likely prior power law (β, ζ) directly from the image !
- Thus, characterize turbulence
  - flow regularity
  - flux across scales

Evidence of power law priors

3-level bayesian hierarchical model :

$$\underbrace{I}_{\textit{image}} \longrightarrow \underbrace{\mathbf{v}}_{\textit{motion}} \longrightarrow \underbrace{\beta,\zeta}_{\textit{multi-scale model}}$$

- **1-st inference level :** *a posteriori* estimation of motion $\mathbf{v}$ knowing (β,ζ)

$$\begin{aligned} p(\mathbf{v}|I,\zeta,\beta) &= \frac{p(I|\mathbf{v},\zeta,\beta)p(\mathbf{v}|\zeta,\beta)}{p(I|\zeta,\beta)} \\ \text{a posteriori} &= \frac{\text{likelihood} \times \text{prior}}{\text{evidence}} \propto \text{likelihood} \times \text{prior} \end{aligned}$$

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- **2-nd inference level :** most likely model (β, ζ) selection given I
  - Evidence by marginalization of motion $\mathbf{v}$: huge dimension !

$$p(I|\zeta, \beta) = \int_{\mathbb{R}^n} p(I|\mathbf{v}, \zeta, \beta)p(\mathbf{v}|\zeta, \beta)\mathrm{d}\mathbf{v}$$

  - But evidence = normalization constant of the first inference level

$$p(I|\zeta, \beta) = \frac{p(I|\mathbf{v}, \zeta, \beta)p(\mathbf{v}|\zeta, \beta)}{p(\mathbf{v}|I, \zeta, \beta)}$$

- Solving (P) equivalent to minimizing over $\mathbf{v}$ the Gibbs posterior energy :

$$\underbrace{-\log p(\mathbf{v}|I,\zeta,\beta)}_{\textit{posterior energy}=L(\mathbf{v},\boldsymbol{\lambda}^*,\beta,\zeta)} \propto -\log p(I|\mathbf{v},\zeta,\beta) - \log p(\mathbf{v}|\zeta,\beta)$$

$$\propto \underbrace{f_d(I,\mathbf{v})}_{\textit{likelihood energy (observation model)}} + \underbrace{\sum_\ell \boldsymbol{\lambda}_\ell^*(\beta,\zeta) g_\ell(\mathbf{v},\beta,\zeta)}_{\textit{self similar prior energy}},$$

where the likelihood and the prior are quadratic Gibbs Random Fields :

$$f_d(I,\mathbf{v}) = \underbrace{\frac{1}{2}\mathbf{v}^T A_0 \mathbf{v} - \mathbf{b}_0^T \mathbf{v} + c_0.}_{\textit{gaussian energy}},$$

$$\sum_\ell \lambda_\ell g_\ell(\mathbf{v},\beta,\zeta) = \sum_\ell \lambda_\ell^*(\beta,\zeta)(\underbrace{\frac{1}{2}\mathbf{v}^T A_\ell \mathbf{v} - \mathbf{b}_\ell^T \mathbf{v} + c_\ell(\beta,\zeta)}_{\textit{gaussian energy}}),$$

back

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  $\Rightarrow$ Bayesian evidence = ratio of partition functions

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- We show that a Gaussian approximation of the log evidence reads

$$-\log p(I|\zeta,\beta) \propto \underbrace{f_d(\mathbf{v}^*, I)}_{\textit{likelihood energy at the MAP}} \underbrace{+\frac{1}{2}\log\frac{\det(A_0+\sum_\ell \lambda_\ell^* A_\ell)}{\det(\sum_\ell \lambda_\ell^* A_\ell)}}_{\textit{log Occam factor}}$$

NB : the Occam factor is a uncertainty ratio (variance ratio in 1D) between the self-similar *prior* and the *posterior*

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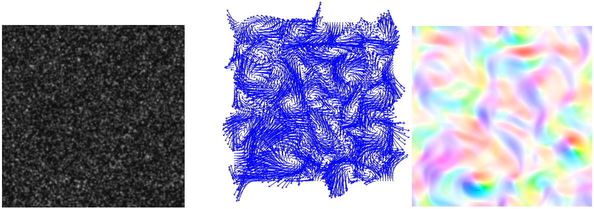
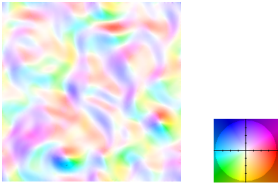
- Sampling of (β,ζ) to maximize the evidence $p(I|\zeta,\beta)$

Experimental evaluation

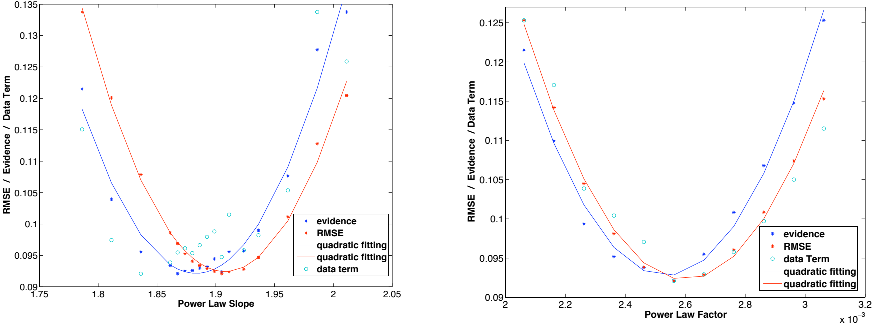
- Forced 2D DNS ($Re = 3000$), dissipative or enstrophy cascade at small scales : $\zeta \sim 2$
- Synthesis of a particule image sequence *[Carlier05]*
- Power law priors in the scale range of [1,10] pixels





Left : *particle image obtained by DNS of 2D Navier-Stokes equations & true velocity field.*

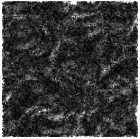
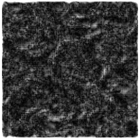
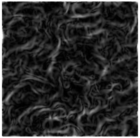
Right : *estimated velocity field.*



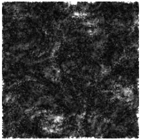
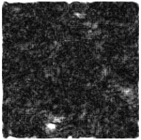
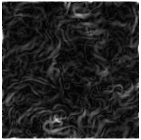
Power law model evidence. *Behavior of data term and minus the logarithm of the evidence w.r.t slope* ζ *(***left***) and factor* β *(***right***) in comparison to the RMSE.*

Experimental evaluation - 2D turbulence

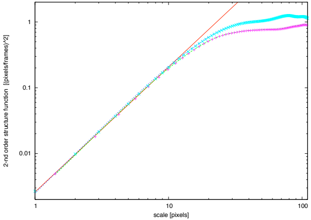
Barron angular error : 4.2656° 4.3581° 3.0485° 2.8836°

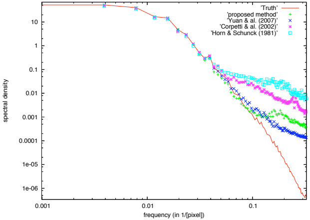

RMSE : 0.138501 0,13402 0.09602 0.09141


Horn & Schunck (1981) (gradient penalization) | Corpetti & al. (2002) (div-curl reg.) | Yuan & al. (2007) (zero div & curl reg.) | proposed method (self-similar reg.)



2-nd order structure function



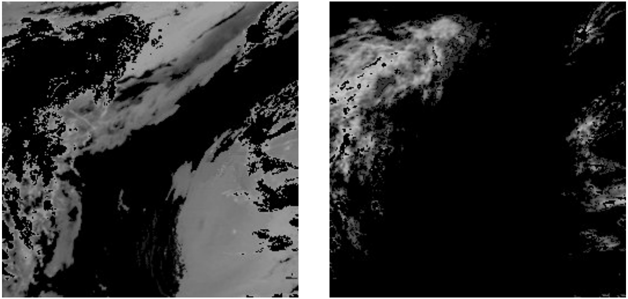
Energy spectrum

- MSG images and direct physical observation model *[Heas&al. 07]*
- Atmospheric turbulence : energy cascades at small scales $[1, 10]$ km *[Lindborg01]* :

$$\mathrm{E}[\delta v(\ell)^2] \sim \beta \ell^{\frac{2}{3}}, \ \textit{avec } \beta = C\epsilon^{\frac{2}{3}} \ \textit{et } \epsilon = \textit{energy flux across scales (or dissipation rate)}$$

- Self-similarity constraints in the scale range $[1, 4]$ pixels

Lower layer	Intermediate layer



Pressure difference images

Selection by evidence maximization of the flux ϵ in the energy cascade :

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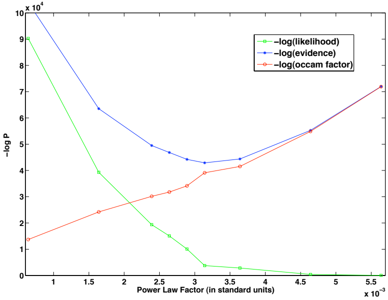
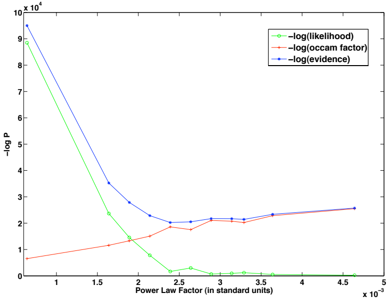
	

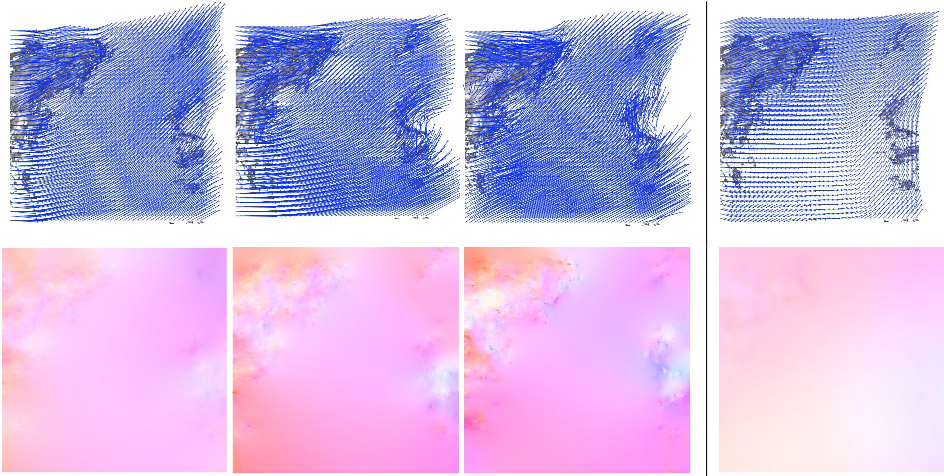
Scaling law model evidence. *Behavior of data term, minus the log evidence and of minus the log of occam factor w.r.t factor* β

- Minimum of evidence $\Rightarrow \hat{\beta}$, yields energy flux across scales estimates :

$$\begin{cases} \hat{\epsilon}^{mid} \simeq 0.79 \times 10^{-5} m^2 s^{-3} \\ \hat{\epsilon}^{low} \simeq 1.20 \times 10^{-5} m^2 s^{-3}. \end{cases}$$

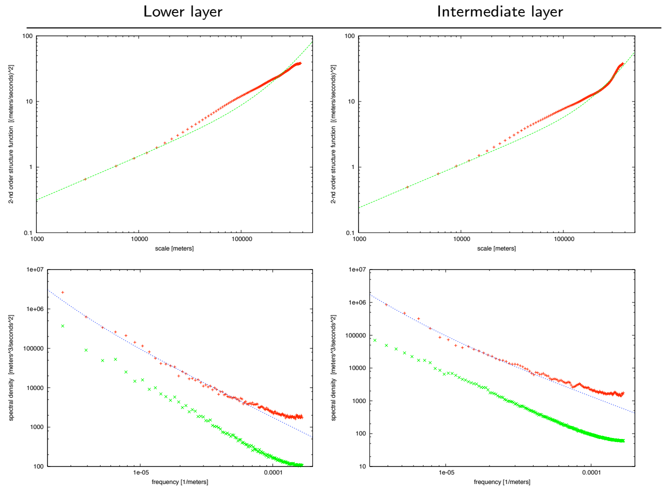
- Same order of magnitude as observed *in situ* by *[Dewan97, Lindborg01]*

Experimental evaluation - Atmospheric turbulence



$\epsilon = 0.11e-5$ $\hat{\epsilon} = 0.79e-5$ $\epsilon = 2.15e-5$ *[Heas&al. 07]*

Intermediate wind fields for increasing energy flux (left) in comparison to [Heas&al. 07] (right)



Second order structure functions & spectra

Summary :

- Efficient physical-based inverse method for motion estimation
- Power law model selection by bayesian evidence
- Tool for characterization of physics of turbulence (regularity, flux exchanges, dissipation) from images

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Perspectives :

- Bayesian evidence : what happens when observation model is wrong ?
- Non stationary fields, multifractal models : localized statistics

Lagrangian minimisation w.r.t. $\mathbf{v}$:

- Cancelling the functional (quadratic form) gradient

$$\nabla_{\mathbf{v}} L(\mathbf{v}, \boldsymbol{\lambda}) = \nabla_{\mathbf{v}} f_d(I, \mathbf{v}) + \sum_{\ell} \lambda_\ell \nabla_{\mathbf{v}} g_\ell(\mathbf{v}) = 0.$$

or resolution of a large linear system

$$(A_0 + \sum_{\ell} \lambda_\ell A_\ell)\mathbf{v} = \mathbf{b}_0 + \sum_{\ell} \lambda_\ell \mathbf{b}_\ell,$$

▶ solution $\mathbf{v}^*$ obtained by the Conjugate Gradient Squared (CGS) algorithm.

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$$w(\lambda) = L(\mathbf{v}^*, \lambda)$$

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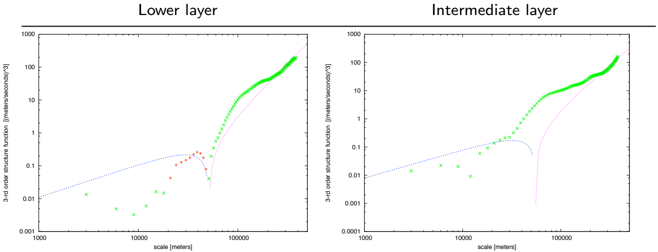
$\Rightarrow$ **velocity $\mathbf{v}^*$ and optimal weights λ^* of multi-scale regularization.** back



Third order structure functions

Least square estimationof the flux ϵ_ω in the enstrophy cascade :

$$\mathrm{E}[\delta v(\ell)^3] \quad = \quad -\epsilon\ell + \frac{1}{8}\epsilon_\omega \ell^3$$

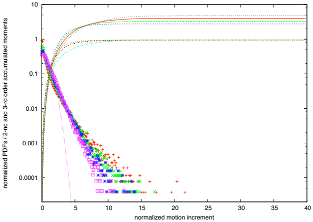
- Least squares using the 3-rd structure functions yields :

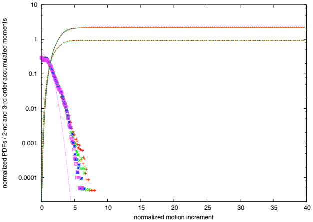
$$\begin{cases} \hat{\epsilon}_\omega^{mid} \simeq 2.58 \pm 0.78 \times 10^{-15} s^{-3} \\ \hat{\epsilon}_\omega^{low} \simeq 4.16 \pm 0.23 \times 10^{-15} s^{-3} \end{cases}$$

- Same order of magnitude as observed by *[Charney71, Lindborg01,Tung03]*
- Direct energy cascade observed only on one layer ...

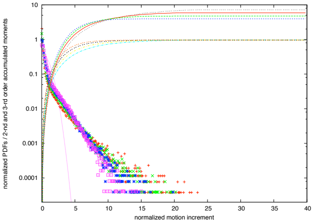
Sufficiently converged statistics ?

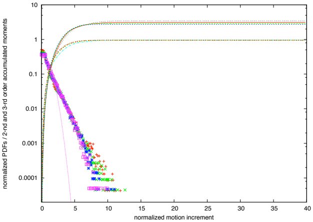
Lower layer : Small scales | Large scales





Intermediate layer : Small scales | Large scales





- Convergence of accumulated moments of second and third order $C_2(z,\ell)$, $C_3(z,\ell)$:

$$C_p(z,\ell) = \int_0^z \left| \frac{\delta v_{||}(\ell)}{\sigma_\ell} \right|^p \sigma_\ell p_\ell \left(\frac{\delta v_{||}(\ell)}{\sigma_\ell} \right) d\left(\frac{\delta v_{||}(\ell)}{\sigma_\ell} \right)$$