Turbulent Mixing and Beyond
Advanced School

COMPRESSIBILITY EFFECTS IN RAYLEIGH-TAYLOR INSTABILITIES INDUCED FLOWS

S. Gauthier

CEA/Bruyères-le-Châtel, France

Outline

- The Kovàsznay modes (vorticity, acoustics and entropic modes)
- The Navier-Stokes equations and the quasi-incompressible limits.
- Some stability results about stratified flows
  - local instability criteria for perfect fluids;
  - Scannapieco's approach;
- The linear Rayleigh-Taylor instability for miscible fluids;
  - RTI: exact results for the linear regime for perfect fluids;
  - RTI for viscous fluids
- The nonlinear regime of the Rayleigh-Taylor instability;
- Isotropic compressible turbulence: Chandrasekhar's invariant;
- Conclusion.

Kovàsznay modes

- Small perturbations of a compressible fluid produces vorticity, acoustics and entropic (Kovàsznay 1952, Monin and Yaglom 1971).

  - Nonlinear regime: interaction of the three basic modes.

- Definition of a compressible flow: one of the two modes acoustics and entropic are excited, with a non-zero velocity.

- Flows with variable density are not necessarily "compressible" (e.g. mixing of two incompressible flows, etc.)

Linear Kovásznay modes

Continuity equation $$\frac{\partial \rho}{\partial t} + \frac{\partial \rho\, u_i}{\partial x_i} = 0$$

Velocity equation $$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_i} \right) = - \frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j}$$

Energy equation $$\rho\, T \left(\frac{\partial s}{\partial t} + u_j \frac{\partial s}{\partial x_i} \right) = \sigma_{ij}\, D_{ij} + \frac{\partial}{\partial x_i} \left(\kappa\, \frac{\partial T}{\partial x_i} \right)$$

EOS $\quad P = R\, \rho\, T \quad$ with $\quad \sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_\ell}{\partial x_\ell} \right) \quad D_{ij} = \frac{1}{2} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right),$

Perturbative expansion $\alpha \sim \frac{|T'|}{T_o}, \frac{|\rho'|}{\rho_o}, \frac{|p'|}{p_o} \ll 1, \quad$ and $\quad M = \frac{|u_i'|}{c_o} \ll 1$

The Kovásznay modes

The Navier-Stokes equations may be rewritten as a system for the vorticity, the divergence, the pressure and the entropy:

$$\frac{\partial \omega_k}{\partial t} = \nu \, \frac{\partial^2 \omega_k}{\partial x_i \partial x_i}.$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u_i'}{\partial x_i}\right) = -\frac{c_o^2}{\gamma\, p_o}\,\frac{\partial^2 p'}{\partial x_i \partial x_i} + \tfrac{4}{3}\,\nu\,\frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{\partial u_i'}{\partial x_i}\right).$$

$$\frac{\partial p'}{\partial t} = \frac{\gamma\, p_o}{c_p}\,\chi\,\frac{\partial^2 s'}{\partial x_i \partial x_i} + (\gamma - 1)\,\chi\,\frac{\partial^2 p'}{\partial x_i \partial x_i} - \gamma\, p_o\, D,$$

$$\rho_o\,\frac{\partial s'}{\partial t} = \frac{\partial}{\partial x_i}\left(\frac{\kappa}{c_p}\,\frac{\partial s'}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(\frac{\kappa}{p_o}\,\frac{\gamma - 1}{\gamma}\,\frac{\partial p'}{\partial x_i}\right).$$

Linear Kovásznay modes

Dimensionless form: k^{-1} (length), $(c_o\,k)^{-1}$ (velocity) $\tau = c_o\,k\,t$ (time)

$$\omega(\mathbf{x},\tau) = \widetilde{\omega}(\tau)\,e^{\,\mathrm{i}\,\mathbf{n}\cdot\mathbf{x}},\quad D(\mathbf{x},\tau) = \widetilde{D}(\tau)\,e^{\,\mathrm{i}\,\mathbf{n}\cdot\mathbf{x}},\quad p'(\mathbf{x},\tau) = \tilde{p}(\tau)\,e^{\,\mathrm{i}\,\mathbf{n}\cdot\mathbf{x}},\quad s'(\mathbf{x},\tau) = \tilde{s}(\tau)\,e^{\,\mathrm{i}\,\mathbf{n}\cdot\mathbf{x}}$$

$$\begin{aligned}
\frac{d\widetilde{D}}{d\tau} &= +\widetilde{p} - \tfrac{4}{3}\,\frac{\nu\,k}{c_o}\,\widetilde{D},\\
\frac{d\widetilde{s}}{d\tau} &= -\frac{\chi\,k}{c_o}\,\widetilde{s} - (\gamma-1)\,\frac{\chi\,k}{c_o}\,\widetilde{p},\\
\frac{d\widetilde{\omega}}{d\tau} &= -\frac{\nu\,k}{c_o}\,\widetilde{\omega},\\
\frac{d\widetilde{p}}{d\tau} &= -\frac{\chi\,k}{c_o}\,\widetilde{s} - (\gamma-1)\,\frac{\chi\,k}{c_o}\,\widetilde{p} - \widetilde{D}.
\end{aligned}$$

Kovásznay modes: Vorticity, Acoustics and Entropy

$$
\begin{aligned}
D(\tau) &= D_0^{(1)} e^{\lambda_1 \tau} + D_0^{(2)} e^{\lambda_2 \tau} - \frac{\varepsilon}{Pr}\, s_0\, e^{\lambda_3 \tau}, \\
p(\tau) &= D_0^{(1)} \left[\mathrm{i} + \tfrac{\varepsilon}{2} \left(\tfrac{4}{3} - \frac{\gamma - 1}{Pr} \right) \right] e^{\lambda_1 \tau} + D_0^{(2)} \left[-\,\mathrm{i} + \tfrac{\varepsilon}{2} \left(\tfrac{4}{3} - \frac{\gamma - 1}{Pr} \right) \right] e^{\lambda_2 \tau}, \\
s(\tau) &= -\frac{\gamma - 1}{Pr}\, \varepsilon \left[D_0^{(1)} e^{\lambda_1 \tau} + D_0^{(2)} e^{\lambda_2 \tau} \right] + s_0\, e^{\lambda_3 \tau}, \\
\omega(\tau) &= \omega(0)\, e^{-\varepsilon \tau}.
\end{aligned}
\tag{1}
$$

with $\lambda_{1,2} = \pm\,\mathrm{i} - \dfrac{\varepsilon}{2} \left(\tfrac{4}{3} + \dfrac{\gamma - 1}{Pr} \right)$, $\quad \lambda_3 = -\dfrac{\varepsilon}{Pr}$ $\quad$ and $\quad \varepsilon = \dfrac{\nu\, k}{c_o} = \dfrac{M}{Re}$.

Pressure waves of the form: $p(\mathbf{x}, \tau) = e^{\,\mathrm{i}\,(\mathbf{k}\cdot\mathbf{x} \pm \mathrm{i}\, c_o\, k\, t)} e^{-\frac{1}{2}\,(\nu + (\gamma - 1)\chi)\, k^2\, t}$.

Nonlinear interactions of Kovásznay modes

Variables are expanded as: $\rho = \rho_o + \rho^{(1)} + \rho^{(2)} + \dots$

$$\begin{aligned}
\frac{\partial \rho^{(n)}}{\partial t} + \rho_o \nabla \cdot \mathbf{v}^{(n)} &= \mathbf{F}_1^{(n)}, \\
\frac{\partial \mathbf{v}^{(n)}}{\partial t} + \nabla p^{(n)} - \mu_o \nabla^2 \mathbf{v}^{(n)} - \frac{1}{3}\mu_o \nabla \left(\nabla \cdot \mathbf{v}^{(n)}\right) &= \mathbf{F}_2^{(n)}, \\
\frac{p_o}{R}\frac{\partial \mathbf{v}^{(n)}}{\partial t} - \kappa_o \nabla^2 T^{(n)} &= \mathbf{F}_3^{(n)}, \\
\frac{p^{(n)}}{p_o} - \frac{\rho^{(n)}}{\rho_o} - \frac{T^{(n)}}{T_o} &= \mathbf{F}_4^{(n)}, \\
\frac{E^{(n)}}{R} - \frac{\gamma}{\gamma - 1}\frac{T^{(n)}}{T_o} + \frac{p^{(n)}}{p_o} &= \mathbf{F}_5^{(n)},
\end{aligned}$$

$$\mathbf{v}^{(1)} = \mathbf{v}_\Omega^{(1)} + \mathbf{v}_p^{(1)} + \mathbf{v}_s^{(1)}.$$

6 interactions VE; VA; EA; VV; EE; AA.

Nonlinear interactions of Kovásznay modes

	Sound source	Vorticity source	Entropy source
Sound-Sound	"steepening" and "self-scattering" $\frac{\partial^2 v_{pi}\, v_{pj}}{\partial x_i \partial x_j} +$ $+ c_o^2\, \nabla^2\, P_p^2$ $\frac{\partial^2 v_{pi}\, v_{pj}}{\partial x_i \partial x_j} +$	$O(\alpha^2\, \varepsilon)$	$O(\alpha^2\, \varepsilon)$
Vorticity-Vorticity	"scattering" $\frac{\partial^2 v_{\Omega i}\, v_{\Omega j}}{\partial x_i \partial x_j}$	"self-convection" $-v_{pj}\, \frac{\partial \Omega_i}{\partial x_j} + \Omega_i\, \frac{\partial \Omega_{pi}}{\partial x_i}$	$O(\alpha^2\, \varepsilon)$
Entropy-Entropy	$O(\alpha^2\, \varepsilon)$	$O(\alpha^2\, \varepsilon)$	$O(\alpha^2\, \varepsilon)$
Sound-Vorticity	"scattering" $\frac{\partial^2 v_{\Omega i}\, v_{\Omega j}}{\partial x_i \partial x_j} +$	"vorticity-convection" $-v_{pj}\, \frac{\partial \Omega_i}{\partial x_j} + \Omega_i\, \frac{\partial \Omega_{pi}}{\partial x_i}$	$O(\alpha^2\, \varepsilon)$
Sound-entropy	"scattering" $\frac{\partial^2}{\partial t \partial x_i}(S_s\, v_{pi})$	"generation" $-\alpha_o^2\, \nabla S_s \times \nabla P_p$	"heat convection" $-v_{pi}\, \frac{\partial S_s}{\partial x_i}$
Vorticity-entropy	$O(\alpha^2\, \varepsilon)$	$O(\alpha^2\, \varepsilon)$	"heat convection" $-v_{\Omega i}\, \frac{\partial S_s}{\partial x_i}$

Approximations for low-Mach number flows

• Kovàsznay modes $\rightarrow$ vorticity, acoustics and entropy;

• acoustics $\rightarrow$ CFL condition for the numerics $\Delta t = \dfrac{\Delta x}{u + c_s} \ll 1$;

• When c_s is large $\Delta t \rightarrow 0 \Rightarrow$ Approximations of Navier-Stokes equations.

• Compressibility effects: stratification and dynamic compressibility,
$\Rightarrow$ various approximations available: low-Mach number, anelastic approximation, etc.

Approximations for low-Mach number flows

The Navier-Stokes equations are written under the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\rho u_i = 0,$$

$$\frac{\partial}{\partial t}\rho u_i + \frac{\partial}{\partial x_j}\rho u_i u_j = -\frac{1}{\gamma\, Ma^2}\frac{\partial P}{\partial x_i} + \frac{1}{Re}\frac{\partial \sigma_{ij}}{\partial x_j} + \frac{1}{Fr}\rho\delta_{i2},$$

$$\rho\left(\frac{\partial T}{\partial t} + u_j\frac{\partial T}{\partial x_j}\right) = \frac{\gamma - 1}{\gamma}\left(\frac{\partial P}{\partial t} + u_j\frac{\partial P}{\partial x_j}\right) + \frac{1}{RePr}\frac{\partial^2 T}{\partial x_i \partial x_i} + \frac{Ma^2(\gamma - 1)}{Re}\sigma_{ij}D_{ij},$$

with the EOS $P = \rho T$.

and the dimensionless numbers:

$$Re = \frac{\rho_* v L}{\mu}, \quad Ma^2 = \frac{v^2}{\gamma R_* T_*}, \quad Fr = \frac{v^2}{g\, L} \quad \text{and} \quad Pr = \frac{\mu C_p}{\kappa}$$

Approximations of the Navier-Stokes equations

Full Navier-Stokes equations

$$\downarrow$$

$$\gamma\, Ma^2 \ll 1$$

$$Fr \ll 1, \quad \frac{\gamma\, Ma^2}{Fr} \sim 1 \qquad\qquad Fr \sim 1, \quad \frac{\gamma\, Ma^2}{Fr} \ll 1$$

$$\swarrow \qquad\qquad\qquad\qquad \searrow$$

Anelastic approximation | Low Mach number Approximation

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

high gravity | low gravity

weak vertical extension | weak temperature gradient

$$\searrow \qquad\qquad\qquad\qquad \swarrow$$

Boussinesq Approximation

The Low-Mach number approximation (or quasi-isobaric) (I)

One uses the Mach number for an asymptotic expansion:

$$\rho = \rho^{(0)} + \gamma\, Ma^2 \rho^{(1)} + \left(\gamma\, Ma^2\right)^2 \rho^{(2)} + \ldots$$

$$P = P^{(0)} + \gamma\, Ma^2 P^{(1)} + \left(\gamma\, Ma^2\right)^2 P^{(2)} + \ldots$$

$$u = u^{(0)} + \gamma\, Ma^2 u^{(1)} + \left(\gamma\, Ma^2\right)^2 u^{(2)} + \ldots$$

Order $1/\gamma\, Ma^2$:

one obtains $\dfrac{\partial P^{(0)}}{\partial x_i} = 0, \Rightarrow P^{(0)} = P^{(0)}(t)$

provided that $\dfrac{\gamma\, Ma^2}{Fr} \ll 1$ and $\dfrac{\gamma\, Ma^2}{Re} \ll 1$

However $\dfrac{\gamma\, Ma^2}{Fr} = \dfrac{g\, L}{R\, T} = \gamma \dfrac{g\, L}{c_s^2}$: stratification of the column of height L.

The Low-Mach number approximation (II)

zeroth order:

$$\frac{\partial \rho^{(0)}}{\partial t} + \frac{\partial}{\partial x_i}\rho^{(0)} u_i^{(0)} = 0,$$

$$\frac{\partial}{\partial t}\rho^{(0)} u_i^{(0)} + \frac{\partial}{\partial x_j}\rho^{(0)} u_i^{(0)} u_j^{(0)} = -\frac{\partial P^{(1)}}{\partial x_i} + \frac{1}{Fr}\rho^{(0)}\delta_{i2} + \frac{1}{Re}\frac{\partial \sigma_{ij}^{(0)}}{\partial x_j},$$

$$\frac{\partial}{\partial t}\rho^{(0)} T^{(0)} + \frac{\partial}{\partial x_i}\rho^{(0)} u_i^{(0)} T^{(0)} - \frac{\gamma - 1}{\gamma}\frac{dP^{(0)}}{dt} = \frac{1}{RePr}\frac{\partial^2 T^{(0)}}{\partial x_j \partial x_j},$$

$$P^{(0)} = \rho^{(0)} T^{(0)}.$$

The dissipation function $\sigma_{ij} D_{ij}$ has disappeared from the energy equation.

The Low-Mach number approximation (III)

- Examples of calculation with LM approximation (Fröhlich and Peyret, 1990; Fröhlich et al., 1992; Le Quéré et al., 1992; Sameen et al., 2008, etc.);
- Comparison with the full NS equations:
  - small density differences (but larger than Boussinesq approximation!)
  - numerical methods close to "incompressible" ones (same cost)
  - time steps much larger (one order of magnitude at least)

The anelastic approximation (I)

When the stratification of the fluid is large, one has:

$$\frac{\gamma\, Ma^2}{Fr} = \frac{g\, L}{R\, T} = \gamma \frac{g\, L}{c_s^2} \sim O(1)$$

Order $1/\gamma\, Ma^2$: $$\frac{1}{\gamma\, Ma^2} \frac{\partial P^{(0)}}{\partial x_i} = \frac{1}{Fr} \rho^{(0)}\, \delta_{i2}$$

The anelastic approximation (II)

zeroth order:

$$\frac{\partial}{\partial x_i}\rho^{(0)}u_i^{(0)} = 0,$$

$$\frac{\partial}{\partial t}\rho^{(0)}u_i^{(0)} + \frac{\partial}{\partial x_j}\rho^{(0)}u_i^{(0)}u_j^{(0)} = -\frac{\partial P^{(1)}}{\partial x_i} + \frac{1}{Fr}\rho^{(1)}\delta_{i2} + \frac{1}{Re}\frac{\partial \sigma_{ij}^{(0)}}{\partial x_j},$$

$$\frac{\partial}{\partial t}\rho^{(0)}T^{(1)} + \frac{\partial}{\partial x_i}\rho^{(0)}u_i^{(0)}T^{(1)} + \rho^{(1)}u_i^{(0)}\frac{\partial}{\partial x_i}T^{(0)}$$

$$-\frac{\gamma-1}{\gamma}\left(\frac{\partial P^{(1)}}{\partial t} + u_i^{(0)}\frac{\partial P^{(1)}}{\partial x_i}\right) = \frac{1}{RePr}\Delta T^{(1)} + \frac{\gamma-1}{\gamma}\frac{1}{Re}\sigma_{ij}^{(0)}D_{ij}^{(0)},$$

$$\frac{\rho^{(1)}}{\rho^{(0)}} = \frac{P^{(1)}}{P^{(0)}} - \frac{T^{(1)}}{T^{(0)}}$$

Examples of anelastic applications

- Examples of anelastic applications: Astrophysics and Geophysics

Astrophysics : Gough, J. Atmos. Sc. 1968; Dintrans and Rieutord, Mon. Not. R. Astron. Soc., 2000; Rogers and Glatzmaier ApJ, 2005; Bannon *et al.*, Mon. Weather Rev., 2006, etc.

Boussinesq approximation

• The Boussinesq approximation are obtained from the low Mach numbers approximations for layers of small thickness: Spiegel and Veronis, 1960, Gray and Giorgini, 1976.

Other approximation of the Navier-Stokes equations (I)

- The previous approximations are single-length, single-time-scale.
- In some cases, there are several length-scales and a single-time-scale.

  Example: interaction of quasi-incompressible, small scale turbulent flow with with acoustics waves of same time-scale (Klein *et al.*, J. Eng. Math., 2001 and JCP, 1995).

  The ratio of the length-scales are given by:

$$\frac{1}{M} = \frac{L_a\,T_a}{L_f\,T_f} = \frac{L_a}{L_f}; \quad \text{one uses two scales} \quad x \quad \text{and} \quad M\,x.$$

  One uses the Mach number for an asymptotic expansion:

$$P(x,t) = P^{(0)}(x, M\,x, t) + \; M P^{(1)}(x, M\,x, t) + M^2\,P^{(2)}(x, M\,x, t) + o(M^2)$$

$$u(x,t) = u^{(0)}(x, M\,x, t) + M\,u^{(1)}(x, M\,x, t) + o(M)$$

Other approximation of the Navier-Stokes equations (II)

one gets:

$$p^{(0)} = p^{(0)}(t), \qquad p^{(1)} = p^{(1)}(M\,x, t), \qquad p^{(2)} = p^{(2)}(x, M\,x, t)$$

- The pressure $p^{(0)}$ is uniform (equivalent to the thermodynaamic pressure),

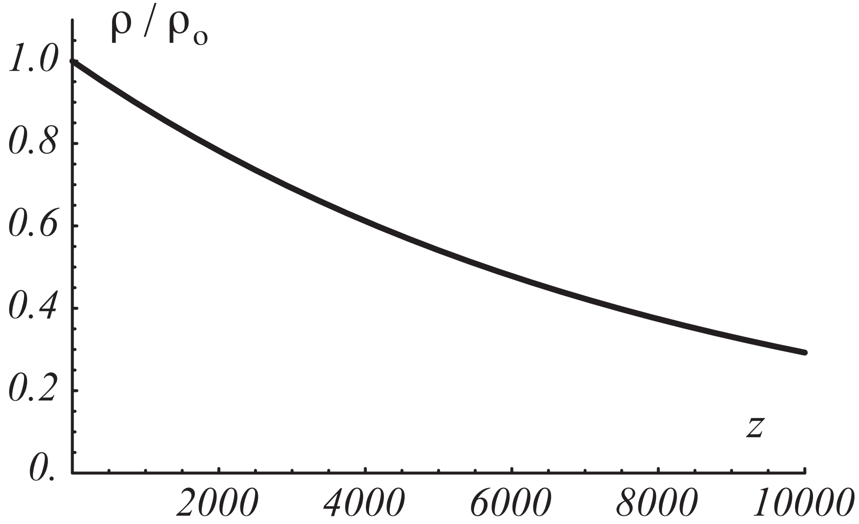
- $p^{(1)}$ depends on $M\,x$, and acts as a large scale driving force,

- $p^{(2)}$ allows us to satisfy the divergence constraint.

• Example: baroclinic vorticity generation by a long-wave acoustics:

the initial condition is stratified density profile and a right-running acoustic pulse in the horizontal direction.

Stratified flows



Hydrostatic equilibrium

$$\frac{dp}{dz} + \rho\, g = 0 \text{ with } p = \rho\, R\, T$$

Isotherm

$$\frac{\rho}{\rho_o} = \frac{p}{p_o} = e^{-\dfrac{g\, z}{R\, T_o}}$$

Isentropic

$$\frac{\rho(z)}{\rho(0)} = \left(1 - \frac{\gamma - 1}{\gamma}\frac{\rho(0)}{p(0)} g\, z\right)^{1/(\gamma-1)}$$

Thermodynamical assumptions

- Isentropic or homentropic:
  - no diffusion process (Reynolds and Péclet numbers $\to \infty$)
  - large diffusion time scales
  - $p = p(\rho) \sim \rho^{\gamma}$
- Isothermal or homothermal:
  - very large heat conduction Péclet number $\to 0$
  - $p = p(\rho) = R\,T_o\,\rho = c_T^2\,\rho \sim \rho$

Polytrop $p = p(\rho) \sim \rho^{\Gamma}$ $\Gamma = \gamma$, isentropic, $\Gamma = 1$, isothermal. Γ polytropic index.

Stability of stratified flows: heuristic criterion

Landau and Lifshitz: The equilibrium is unstable if a particle of fluid which is moved up is lighter than the surroundings fluid:

Unstability criterion: $V(p', s') - V(p', s) < 0$

$$V(p', s') - V(p', s) \approx \left(\frac{\partial V}{\partial s}\right)_p (s' - s)$$

or

$$\left(\frac{\partial V}{\partial s}\right)_p \frac{ds}{dz} < 0 \qquad \text{for most of the fluids} \qquad \frac{ds}{dz} < 0$$

since $\left(\frac{\partial V}{\partial s}\right)_p = \left(\frac{\partial V}{\partial T}\right)_p \frac{T}{C_p} > 0$

Stability of stratified flows: equations of motion

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho\, u_j}{\partial x_j} = 0,$$

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} - g\, \rho\, \delta_{i2},$$

$$\frac{\partial}{\partial t} (p\, \rho^{\gamma}) + u_j \frac{\partial}{\partial x_j} (p\, \rho^{\gamma}) = \dot{S} \qquad \text{with the EOS} \qquad p = p(\rho, F).$$

the equilibrium state satisfies $\dfrac{1}{\rho_o} \dfrac{dp_o}{dz} = -g$

$$\frac{dp_o}{dz} = \frac{1}{\rho_o} \left(\frac{\partial p_o}{\partial \rho_o} \right)_S \frac{d\rho_o}{dz} + \frac{1}{\rho_o} \left(\frac{\partial p_o}{\partial S_o} \right)_\rho \frac{dS_o}{dz} \equiv -c_F^{(o)\,2} \chi_\rho - \Sigma^{(o)}$$

$$\chi_\rho \equiv \frac{1}{\rho^{(o)}} \frac{d\rho^{(o)}}{dz} \quad \text{and} \quad \Sigma^{(o)} \equiv \frac{1}{\rho^{(o)}} \left(\frac{\partial p}{\partial F} \right)_\rho^{(o)} \frac{dF^{(o)}}{dz},$$

Rigorous approach

Normal mode analysis:

For linear system of differential equations with constant coefficients, we seek solutions under the form:

$$\Psi(x,y,z,t) = \widehat{\Psi}(z)\, e^{\sigma\, t}\, e^{\,\mathrm{i}\,(k_x x + k_y y)},$$

where $n = \sigma + \mathrm{i}\,\omega$ and $k = (k_x^2 + k_y^2)^{1/2}$.

Initial boundary value problem $\rightarrow$ boundary value problem

$$\frac{\partial V}{\partial t} + u_i \frac{\partial V}{\partial x_i} = S \qquad \text{with B.C.} \qquad \text{becomes} \qquad A\,V = \lambda\,V$$

where $V = (\rho\ u\ v\ T)^T$.
The matrix A depends on the mean field $V_o = (\rho_o\ u_o\ v_o\ T_o)^T$, and the derivative D.

Stability of stratified flows: Sturm-Liouville problem

Pressure perturbation $\widehat{p}$, for the mode k, satisfies a second-order differential equation [Gamaly et al. (1976); Sitt (1980); Gamaly et al. (1980)]

$$\frac{d}{dz}\left(\frac{1}{\rho^{(o)}}\frac{d\widehat{p}}{dz}\right) - \frac{1}{\rho^{(o)}}\left[k^2 + \left(\sigma^2 + \frac{g\,k^2\,\Sigma^{(o)}}{\sigma^2}\right)\frac{1}{c_F^{(o)\,2}}\right]\widehat{p} = 0.$$

The EOS iso-F for the perturbations of density and pressure is

$$\widehat{p} = c_F^{(o)\,2}\,\widehat{\rho}$$

$\rightarrow$ Sturm-Liouville problem

Stability of stratified flows

$\Rightarrow$ it admits two infinite, discrete sets of perturbation modes, the growth rate of which are given by the following expression

$$\sigma_n^{\pm\,2} = \frac{1}{2}\left[\zeta_n \pm \left(\zeta_n^2 - 4\,g\,k^2\,\Sigma^{(o)}\right)^{1/2}\right].$$

where $\zeta \equiv -\left(\sigma^2 + g\,k^2\,\Sigma^{(o)}/\sigma^2\right)$ and for $\Sigma^{(o)} = c^{st}$

- σ_n^+ convective-type modes. Unstable or marginally stable.
- $\sigma_n^{-\,2} \propto -k^2\,c_F^{(o)\,2}$ acoustic-type modes (Lamb's modes).

$$\text{Amplitude } \widehat{p} \propto e^{Re\left\{\sigma_n^-\right\}t}\, e^{\imath Im\left\{\sigma_n^-\right\}t}$$

Stability of stratified flows

$$\sigma_n^+ > 0 \quad \text{if} \quad g\,\Sigma^{(o)} < 0$$

This provides a necessary and sufficient condition for convective instability for a stratified perfect compressible fluid against iso-F perturbations

$$g \left(\frac{\partial p}{\partial F}\right)_\rho^{(o)} \frac{dF^{(o)}}{dz} < 0.$$

As particular cases, we have the following criteria for instability:

- the isothermal criterion $\nabla_i p^{(o)} \nabla_i T^{(o)} > 0$;
- the isentropic one $\nabla_i p^{(o)} \nabla_i S^{(o)} > 0$, or $\dfrac{dp^{(o)}}{dz}\left(\dfrac{1}{\gamma\, p^{(o)}}\dfrac{dp^{(o)}}{dz} - \dfrac{1}{\rho^{(o)}}\dfrac{d\rho^{(o)}}{dz}\right) > 0$;
- the incompressible one $\nabla_i p^{(o)} \nabla_i \rho^{(o)} < 0$, obtained by setting $\gamma \to \infty$.

Scannapieco's approach for adiabatic stratified flows (I)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho\, u_j}{\partial x_j} = 0,$$

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -Sr^{-1} \frac{\partial p}{\partial x_i} - \rho\, \delta_{i2},$$

$$\frac{\partial}{\partial t} (p\, \rho^{\gamma}) + u_j \frac{\partial}{\partial x_j} (p\, \rho^{\gamma}) = \dot{S} \qquad \text{with the EOS} \qquad p = \rho\, T.$$

Exponentially varying zeroth-order quantity:

$$\rho_o = \overline{\rho}_o \exp\left(z/H \right) \quad \text{where } H \text{ is the density-gradient length-scale} \quad H = \left(\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial z} \right)^{-1}.$$

Hypothesis: - we seek solutions over an interval d such that $d \ll H$,

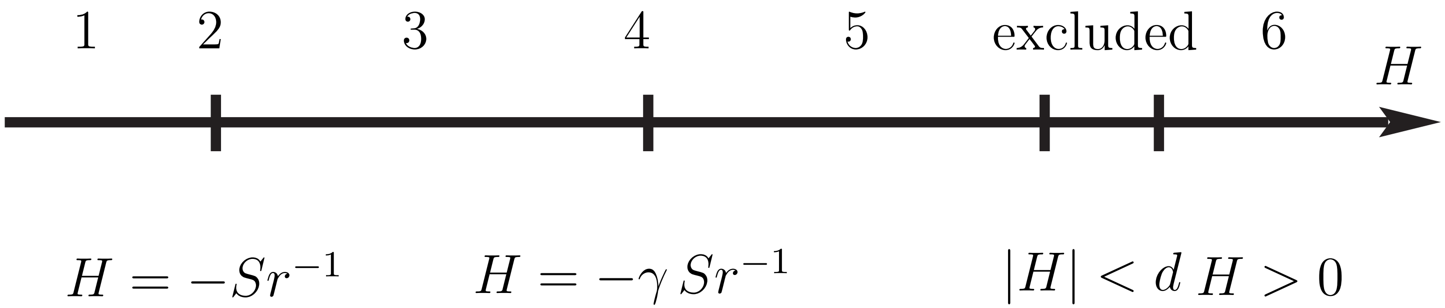
- the speed of sound c_s is supposed to be constant.

Scannapieco's approach for adiabatic stratified flows (II)

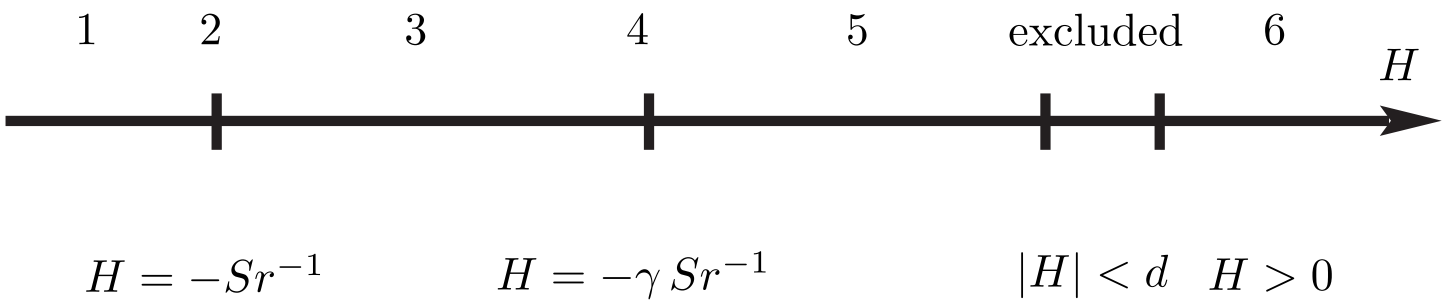
An equation for the momentum $\mu = \rho\, u_{1\,z}$ is written

$$D^2\mu + D\mu\left(-\frac{1}{H} - n^2\frac{H^{-1}+Sr}{k^2+Sr\,n^2}\right)$$
$$+\mu\left[-k^2 - Sr\,n^2 + \frac{k^2}{n^2}\left(\frac{1}{H}+\frac{Sr}{\gamma}\right) - \left(H^{-1}+Sr\right)\frac{Sr\,k^2 - n^2/H}{n^2+k^2/Sr}\right] = 0$$

The general solution is $\mu = A_1\,\exp\left(q_1\,z\right) + A_2\,\exp\left(q_2\,z\right)$

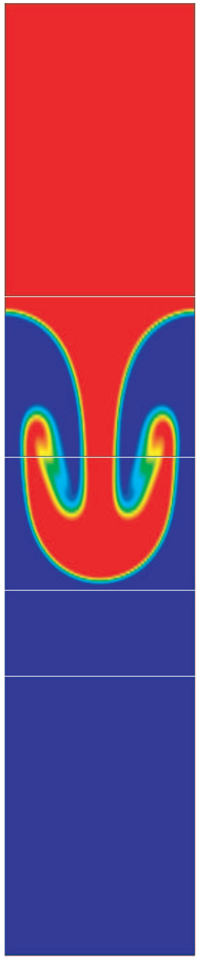


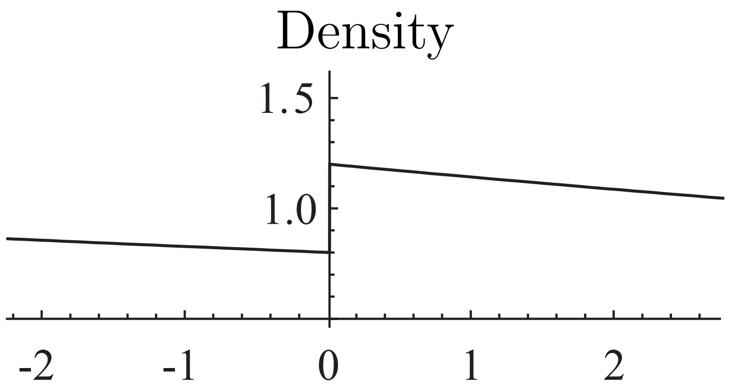
Scannapieco's approach for adiabatic stratified flows (IV)

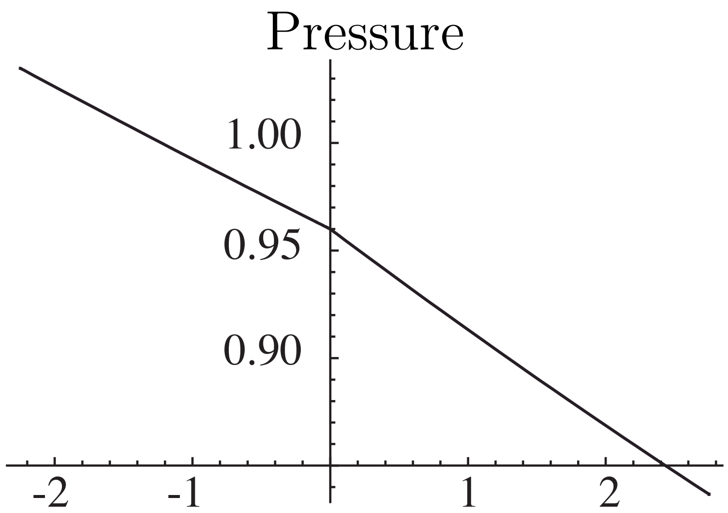


	Acoustic	Gravity	Lamb
$H > 0$	oscillatory	growing	growing-oscillatory
$H > -c_s^2/\gamma\, g = -1/\gamma Sr$	oscillatory	oscillatory	growing-oscillatory
$H = -c_s^2/\gamma\, g = -1/\gamma Sr$	oscillatory	oscillatory	oscillatory
$H < -c_s^2/\gamma\, g = -1/\gamma Sr$	oscillatory	oscillatory	growing-oscillatory
$H > -c_s^2/g = -1/Sr$	oscillatory	growing	growing-oscillatory

RTI for compressible fluids







RTI for compressible miscible fluids

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho\, u_j}{\partial x_j} = 0,$$

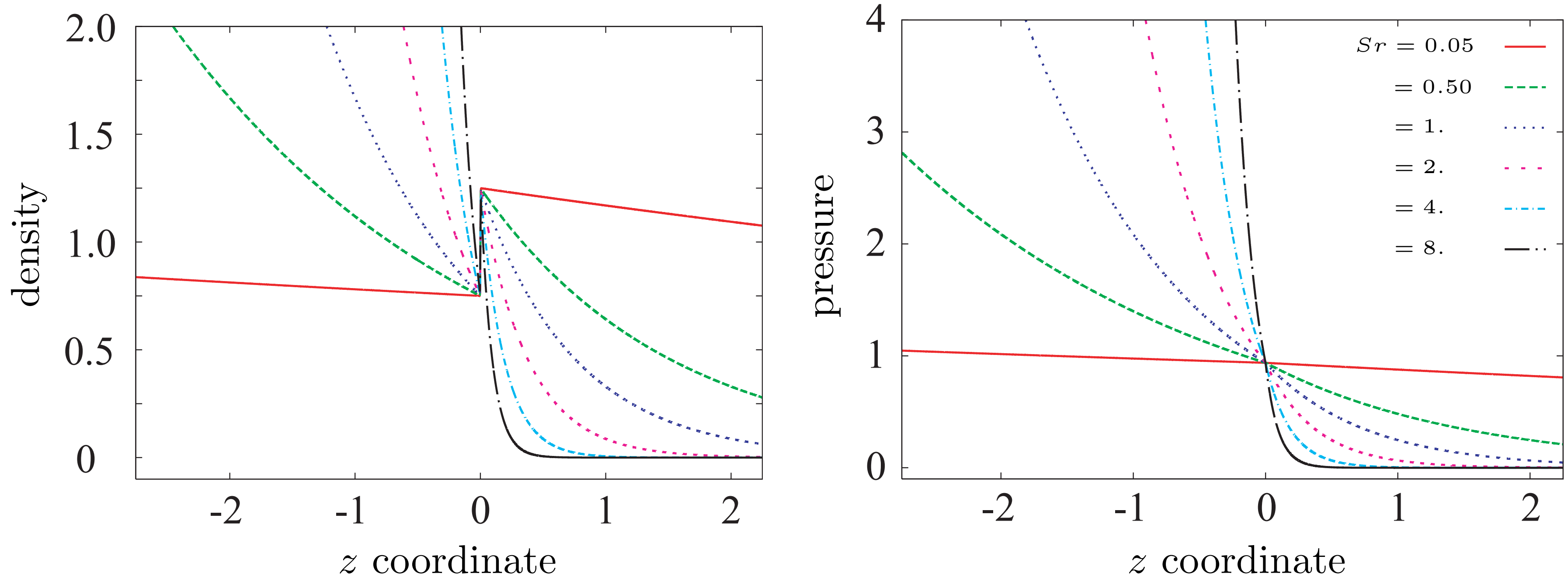
$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -Sr^{-1}\, \frac{\partial p}{\partial x_i} + Re^{-1}\, \frac{\partial \sigma_{ij}}{\partial x_j} - \rho\, \delta_{i2},$$

$$\begin{aligned} \rho\, C_{v;m} \left(\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} \right) &= (\gamma_r - 1) \left[Sr\, Re^{-1}\, \sigma_{ij}\, D_{ij} - p\, \frac{\partial u_j}{\partial x_j} \right] \\ &\quad + Re^{-1} \left[-Sc^{-1}\, T\, d_c C_{v;m}\, \frac{\partial^2 c}{\partial x_j \partial x_j} + Pr^{-1}\, \gamma_r\, \frac{\partial^2 T}{\partial x_j \partial x_j} \right], \end{aligned}$$

$$\frac{\partial c}{\partial t} + u_j\, \frac{\partial c}{\partial x_j} = (\rho\, Re\, Sc)^{-1}\, \frac{\partial^2 c}{\partial x_j \partial x_j},$$

$$p = \rho\, T\, (1 + A_t - 2\, A_t\, c)\,.$$

RTI: Equilibrium state



Density (left) and pressure (right) profiles for various values of the stratification parameter Sr, for an Atwood number $A_t = 0.25$.

Linear stability of RTI: solution method

- For perfects fluids

  - in each fluid, the second order ODE for the amplitude is solved

  - matching condition for the pressure at the interface $\Rightarrow$ dispersion relation.

- For viscous fluids

  - approximate equilibrium state in presence of diffusion (viscosity, thermal conduction and diffusion of species)

  - numerical methods to solve the linear eigenvalue problem:

$$A\,\widehat{\phi} = \lambda\,\widehat{\phi} \quad \text{where} \quad \widehat{\phi} = \left(\widehat{\rho}\ \widehat{u}_i\ \widehat{T}\ \widehat{c}\right)^T$$

RTI: linear stability

	Perturbation type		
Hydrostatic Equilibrium type	Isothermal (T) $Re\,Pr \to 0$, $Re \to \infty$	Isentropic (S) $Re \to \infty$	General (G)
Isothermal (T) $Re\,Pr \neq 0$	TT-case Blake (1972), Baker (1983) Mathews *et al.* (1977) Ribeyre *et al.* (2004) Depends on A_t, Sr, and $h_{H,L}$. γ-independent	TS-case Bernstein-Book (1983), Livescu (2004) Hoshoudy (2007) Livescu (2008) Depends on A_t, Sr, $\gamma_{H,L}$ and $h_{H,L}$	TG-case Lafay *et al.* (2007) Depends on A_t, Sr, $\gamma_{H,L}$, $h_{H,L}$, Re, Sc and Pr
Isentropic (S) $Re \to \infty$	Irrelevant case	SS-case Lezzi-Prosperetti (1989) Depends on A_t, Sr, $\gamma_{H,L}$ and $h_{H,L}$	Irrelevant case
General (G)	Has to be done	Has to be done	Has to be done

RTI: linear stability of perfect fluids

Two types of parameters: stratification and EOS (γ's)

General trends:

- the growth rate strongly decreases as the stratification increases;
- the growth rate increases as the adiabatic indices $\gamma_{H,L}$ increase;
- stratification and compressibility effects are more important at small wave numbers;
- growth rates are larger when the light fluid is more compressible than the heavy one ($\gamma_L < \gamma_H$);
- compressibility effects are larger at small Atwood numbers.

RTI: linear stability of viscous fluid

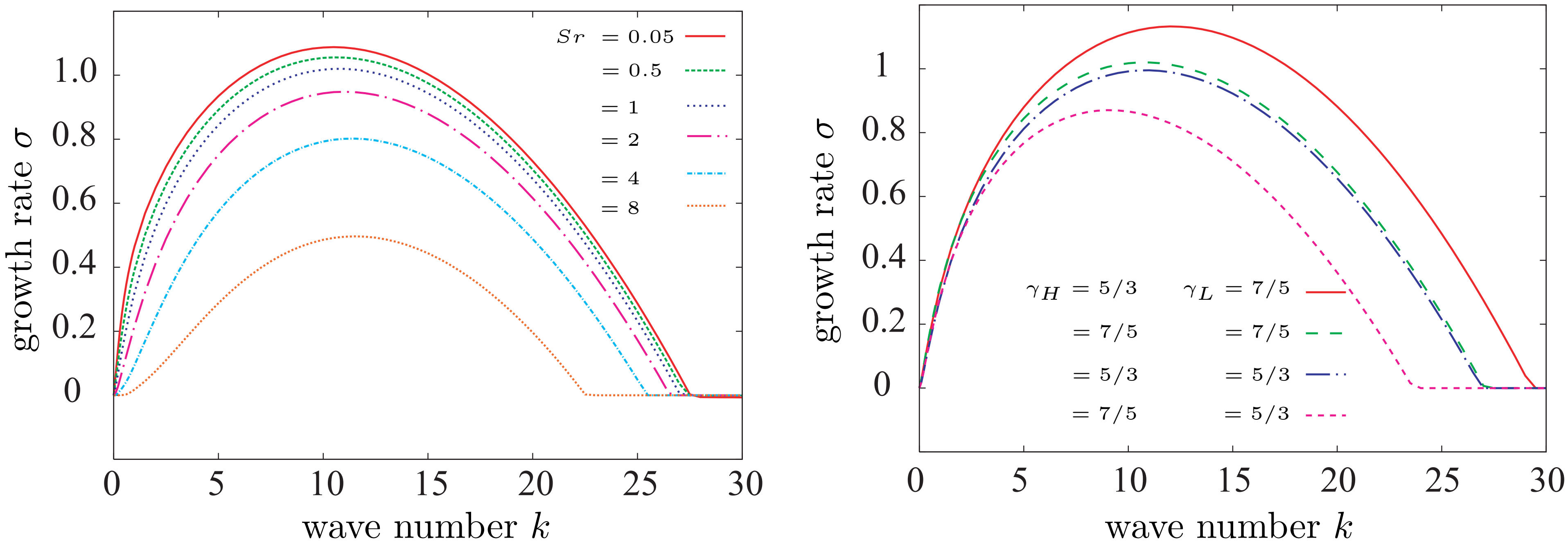


Figure 1: Dispersion curves. Stratification parameter Sr is ranging from $Sr = 0.05$ to $Sr = 8$. Dispersion curves for various values of the adiabatic indices γ, $\gamma_H \neq \gamma_L$.

RTI: Numerical Simulations

	DNS	LES
Euler equations	TREK 1^{st}-order FD method in space Sin'kova et al. (1999)	FLASH PPM-type method for miscible fluids 2^{nd}-order in space and time Fryxell et al. (2000)
	2D-MAH, MAH-3D FV method Volkov et al. (1999)	TURMOIL-3D explicit FD method FT method for non-miscible fluids Youngs (1991)
	FronTier 2^{nd}-order FD method in space 4^{th}-order Runge-Kutta in time FT method for non-miscible fluids Jin et al. (2005) George and Glimm (2005)	Dimonte et al. (2004) Simulation results comparison MILES approach with various numerical schemes
	LEEOR-3D ALE with FT method Hecht et al. (1995) Belotserkovskii and Oparin (1999) Inogamov and Oparin (1999) 2^{nd}-order scheme	

RTI: Numerical Simulations

	DNS	LES
Navier-Stokes equations	AMÉNOPHIS Pseudo-spectral adaptive method Le Creurer and Gauthier (2008)	Mellado et al. (2005) 6^{th}-order Padé in space 4^{th}-order Runge-Kutta in time.
		Cook (2007) 10^{th}-order compact scheme 4^{th}-order Runge-Kutta in time

RTI: Numerical Simulations

- Most of the simulations carried out with the Euler or the full NS equations;
- The initial equilibrium state is usually weakly stratified and the γ's are equals (5/3 or 7/5). $\Rightarrow$ compressibility effects are small;
- However:

  - Single mode two-dimensional simulations (Jin et al. (2005)).
  Stratification, $10^{-2} \leq M^2 = \lambda\, g/c_H^2 \leq 0.50$, and the γ's: $1.1 \leq \gamma_{H,L} \leq 4$;
  - George and Glimm (2005): mixing zones arising from stratified equilibrium states, self-similarity is lost ($L_{H,L}$).
  Time-dependent Atwood number:

  $$h = \alpha_{\mathit{eff}} \int_0^t \int_0^{s_1} 2\, A_t(s)\, g\; ds\, ds_1 \Rightarrow \text{self-similarity recovered.}$$

  - Olson and Cook (2007) example of a strong compressible behavior in the turbulent regime. In a compressible fluid, acoustics waves are generated $\rightarrow$ shocklets $\rightarrow$ coalesce into a shock wave.

Influence of initial conditions in turbulence

- It is well-known that in a nonlinear dissipative system, the initial conditions are forgotten;
- Mixing layers: how much time is it necessary to forget the initial conditions ?
- RT-mixing layers: Studied by several authors (Ramaprabhu et al. (2005) and Andrews (2009))
- In *compressible* flows, the nonlinear system of equations is partially parabolic.
- Chandrasekhar's observation (1951) on the fluctuations of density in isotropic compressible turbulence.

  "The largest structures in the density fluctuations are determined by the initial conditions and represent permanent features of the flow."

Isotropic (compressible) turbulence

- No real turbulent flow is isotropic or homogeneous in the large scales;
- But is it easy to develop analytical statistical theories;
- Homogeneous and isotropic turbulence

$$\overline{u_i}\left(x,t\right) = c^{cste} \qquad \text{and} \qquad \overline{\rho}\left(x,t\right) = c^{cste}$$

- Exemple of double correlation of velocities:

$$B_{ij}\left(r,t,t'\right) = \overline{u_i\left(x,t\right)u_j\left(x+r,t'\right)},$$

By using the properties of isotropic and homogeneity, one can show that this correlation may be written as

$$B_{ij}\left(r\right) = -\frac{1}{2r}\frac{\partial B_{LL}\left(r\right)}{\partial r}r_i r_j + \left(B_{LL} + \frac{r}{2}\frac{\partial B_{LL}}{\partial r}\right)\delta_{ij}.$$

where B_{LL} is the double correlation of longitudinal velocities.

Chandrasekhar's invariant

From $\dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u_i}{\partial x_i} = 0,$ one gets $\dfrac{\partial}{\partial t}\overline{\rho\,\rho'} + \dfrac{\partial}{\partial x_i}\overline{\rho'\,\rho\,u_i} + \dfrac{\partial}{\partial x_i}\overline{\rho\,\rho'\,u_i'} = 0.$

Correlation of density fluctuations $\delta\rho$ as

$$\varpi(r,t) = \overline{(\rho - \overline{\rho})\,(\rho' - \overline{\rho})} = \overline{\delta\rho'\,\delta\rho} \qquad \text{where} \qquad \overline{\rho} = c^{ste},$$

Correlation between mass flux and density is written as:

$$\overline{\rho'\,\rho\,u_i} = -\overline{\rho\,\rho'\,u_i'} \equiv L(r,t)\,\xi_i, \qquad \text{where} \qquad \xi_i = x_i - x_i'.$$

The previous equation writes $r^2\,\dfrac{\partial \varpi}{\partial t} = 2\,\dfrac{\partial}{\partial r}\left(r^3\,L\right) \quad \Rightarrow \dfrac{\partial}{\partial t}\displaystyle\int_0^r r^2\,\varpi\,dr = 2\,r^3\,L.$

If $L \to 0$ faster than r^{-3} [ASSUMPTION], then

$$I = \int_0^\infty r^2 \varpi(r,t)\,dr = c^{ste}.$$

Meaning of the Chandrasekhar's invariant ?

$$\frac{\partial}{\partial t}\overline{\rho\,\rho'} + \frac{\partial}{\partial x_i}\overline{\rho'\,\rho\,u_i} + \frac{\partial}{\partial x_i}\overline{\rho\,\rho'\,u_i'} = 0 \rightarrow \mathrm{TF} \rightarrow \frac{\partial \Pi(\mathbf{k})}{\partial t} = \mathrm{i}\;[\Lambda_j(\mathbf{k}) + \Lambda_j(-\mathbf{k})]\;k_j$$

$$\text{with}\quad \Pi(\mathbf{k}) = \frac{1}{(2\,\pi)^3}\int \overline{\delta\rho'\,\delta\rho}(\mathbf{r})\,e^{\,\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{r};\quad \Lambda_j(\mathbf{k}) = \frac{1}{(2\,\pi)^3}\int \overline{\rho'\,\rho\,u_i}(\mathbf{r})\,e^{\,\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{r}$$

If $\Pi(\mathbf{k})$ has an expansion of the form: $\Pi(\mathbf{k}) = \Pi_o + \Pi_j\,k_j + \Pi_{\ell m}\,k_\ell\,k_m + \ldots$ for $|\mathbf{k}| \rightarrow 0$

and $\Lambda_j(\mathbf{k})$ finite at $r = 0$, it follows that $\dfrac{d\Pi_o}{dt} = 0$

or $\Pi(\mathbf{0}) = \dfrac{1}{(2\,\pi)^3}\displaystyle\int \overline{\delta\rho'\,\delta\rho}(\mathbf{r})\;d\mathbf{r} = \dfrac{1}{2\,\pi^2}\displaystyle\int_0^\infty r^2\,\varpi(r,t)\,dr = c^{cste}$

It means that the spectrum of $\overline{\delta\rho'\,\delta\rho}$, for $|\mathbf{k}| \rightarrow 0$ does not depend on time.

Chandrasekhar's invariant

- "The largest structures in the density fluctuations are determined by the initial conditions and represent permanent features of the flow." (Chandrasekhar (1951)) Provided that the correlation $\overline{\rho' \rho u_i}$ goes to zero fast enough with the distance r.
- Analogy with the Loitszanskï invariant for HIT for an incompressible fluid.

Incompressible IHT: von Kármán - Howarth equation

- Navier-Stokes equations:

$$\frac{\partial u_i(x)}{\partial t}+\frac{\partial u_k(x)\,u_i(x)}{\partial x_k}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\frac{\partial^2 u_i(x)}{\partial x_k \partial x_k}.$$

$$\frac{\partial B_{ij}}{\partial t}(r,t)=\frac{\partial}{\partial r_k}\left[B_{ik,j}(r,t)-B_{i,jk}(r,t)\right]+\frac{1}{\rho}\left[\frac{\partial B_{pj}}{\partial r_i}(r,t)-\frac{\partial B_{ip}}{\partial r_j}(r,t)\right]+2\nu\frac{\partial^2 B_{ij}(r,t)}{\partial r_k \partial r_k}.$$

- no correlation between the velocity field and a scalar in a HIT.

- contraction over the indices i and j.

- von Kármán - Howarth equation

$$\frac{\partial B_{LL}}{\partial t}=\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 B_{LL,L}\right)+\frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial B_{LL}}{\partial r}\right).$$

Incompressible IHT: Loitszanskï integral

- This is an integral relationship on the double correlation B_{LL}.

$$\frac{\partial}{\partial t}\int_o^\infty r^4 B_{LL}\left(r,t\right) dr = \lim_{r\to\infty} r^4 B_{LL,L} + 2\nu \lim_{r\to\infty} r^4 \frac{\partial B_{LL}}{\partial r}.$$

- If the correlations $B_{LL,L}$ and B_{LL} go to zero fast enough, one has:

$$\int_o^\infty r^4 B_{LL}\left(r,t\right) dr = c^{ste}.$$

- Controversy over the last decades, numerical simulations and turbulence modeling (EDQNM) have shown that the Loitszanskï integral depends on time.

- Recent work (Ishida et al., 2006, JFM): numerical simulation $\rightarrow$ Loitszanskï integral is [indeed] a constant.

- Reason: suppression of long-range velocity correlations in isotropic turbulence, by a screening effect due to vorticity structure.

Concluding remarks

- We have review some compressibility effects in stratified flows and Rayleigh-Taylor induced flows.
- Two main effects: compressibilities (EOS) and stratification.
- Various approximations of Navier-Stokes, depending on the application.
- Compressibilities may induced various types of behaviors (*e.g.* Lamb's modes)

Open problems:

- Are there linear Lamb modes in viscous compressible RTI (see Barthélémy's talk) ?
- What about the nonlinear stability of Lamb's modes ?
- Compressible homogeneous isotropic turbulence: Is Chandrasekhar's quantity an invariant ? (dependence on initial conditions)

References

M. J. Andrews. Review on experiments on RTI (effect of initial conditions on late-time development of Rayleigh-Taylor mixing). *Phil. Trans. Roy. Soc.*, 2009.

L. Baker. Compressible Rayleigh-Taylor instability. *Phys. Fluids*, 26(4):950–952, 1983.

O. M. Belotserkovskii and A. M. Oparin. Computational analysis of the specific features of a spatial progress of Rayleigh-Taylor instability. In E. Meshkov, Y. V. Yanilkin, and V. Zhmailo, editors, *Proceedings of the 7th International Workshop on The Physics of Compressible Turbulent Mixing*, pages 379–384, St. Petersburg, 1999.

I. B. Bernstein and D. L. Book. Effect of compressibility on the Rayleigh-Taylor instability. *Phys. Fluids*, 26(2):453–458, 1983.

G. M. Blake. Fluid dynamic stability of double radio sources. *Mon. Not. R. Astron. Soc.*, 156:67, 1972.

S. Chandrasekhar. The fluctuations of density in isotropic turbulence. *Proc. Roy. Soc. Lond.*, 210:18–25, 1951.

A. W. Cook. Artificial fluid properties for large-eddy simulation of compressible turbulent mixing. *Phys. Fluid*, 19:055103, 2007.

G. Dimonte, D. L. Youngs, A. Dimits, S. Weber, M. Marinak, S. Wunsch, C. Garasi, A. Robinson, M. J. Andrews, P. Ramaprabhu, A. C. Calder, B. Fryxell, J. Biello, L. Dursi, P. MacNeice, K. Olson, P. Ricker, R. Rosner, F. Timmes, H. Tufo, Y. N. Young, and M. Zingale. Comparative study of the turbulent Rayleigh-Taylor instability using high-resolution simulations: The Alpha-Group collaboration. *Phys. Fluids*, 16:1668–1692, 2004.

J. Fröhlich and S. Gauthier. Numerical investigations from compressible to isobaric rayleigh-bénard convection in two dimensions. *Eur. J. Mech., B/Fluids*, 2:141–159, 1993.

J. Fröhlich and R. Peyret. Calculations of non-boussinesq convection by a pseudo-spectral method. *Comp. Meth. Appl. Mech. Eng.*, 80:425–433, 1990.

J. Fröhlich, P. Laure, and R. Peyret. Large departures from boussinesq approximation in the rayleigh-bénard problem. *Phys. Fluids*, A4:1355–1372, 1992.

B. Fryxell, K. Olson, P. Ricker, F. X. Timmes, M. Zingales, D. Q. Lamb, P. MacNeice, R. Rosner, J. W. Truman, and H. Tufo. FLASH: An adaptive mesh hydrodynamics code for modeling astrophysical thermonuclear flashes. *Astrophys. J. Suppl. Ser.*, 131:273–334, 2000.

E. G. Gamaly, V. E. Neuvazhaev, and L. I. Shibarshov. On the stability of an arbitrary adiabatic flow. In N. N. Yanenko, editor, *Chislennye metody mekhaniki sploshnoi sredy (Numerical Methods of Fluid Mechanics)*, volume 7, pages 47–56, Novosibirsk, 1976.

E. G. Gamaly, V. B. Rosanov, A. A. Samarsky, V. F. Tishkin, N. N. Tyurina, and A. P. Favorsky. Hydrodynamic stability of compression of spherical laser targets. *Sov. Phys. JETP*, 52(2):230–237, 1980.

E. George and J. Glimm. Self-similarity of Rayleigh-Taylor mixing rates. *Phys. Fluid*, 17: 054101, 2005.

J. Hecht, D. Ofer, U. Alon, D. Shvarts, S. A. Orszag, and R. L. McCrory. Nonlinear evolution of multimode Rayleigh-Taylor instability in two and three dimensions. *Laser Part. Beams*, 13:423–440, 1995.

G. A. Hoshoudy. Compressible effects on the Rayleigh-Taylor instability of three layers. *Phys. Fluids*, 19:096103–8, 2007.

N. A. Inogamov and A. M. Oparin. Comparison of evolution of 3D RT and RM hexagonal, square and triangular perturbations from linear stage to steady-state. In E. Meshkov, Y. V. Yanilkin, and V. Zhmailo, editors, *Proceedings of the 7th International Workshop on The Physics of Compressible Turbulent Mixing*, pages 259–265, St. Petersburg, 1999.

T. Ishida, P. A. Davidson, and Y. Kaneda. On the decay of isotropic turbulence. *J. Fluid Mech.*, 564:455–475, 2006.

H. Jin, X. F. Liu, T. Lu, B. Cheng, J. Glimm, and D. H. Sharp. Rayleigh-Taylor mixing rates for compressible flows. *Phys. Fluid*, 17:024104, 2005.

M.-A. Lafay, B. Le Creurer, and S. Gauthier. Compressibility effects on the Rayleigh-Taylor instability growth between miscible fluids. *Europhys. Letters*, 79:64002, 2007.

B. Le Creurer and S. Gauthier. A return toward equilibrium in a two-dimensional Rayleigh-Taylor instability for compressible miscible fluids. *Theor. Comput. Fluid Dyn.*, 22:125–144, 2008.

P. Le Quéré, R. Masson, and P. Perrot. A chebyshev collocation algorithm for 2d non-boussinesq convection. *J. Comput. Phys.*, 103:320–335, 1992.

A. M. Lezzi and A. Prosperetti. Rayleigh-Taylor instability for adiabatically stratified fluids. *Phys. Fluids A*, 1(11):1784–1795, 1989.

D. Livescu. Compressibility effects on the Rayleigh-Taylor instability growth between immiscible fluids. *Phys. Fluids*, 16:118–127, 2004.

D. Livescu. Comment on "Compressibility effects on the RayleighTaylor instability of three layer" [Phys. Fluids 19, 096103 (2007)]. *Phys. Fluids*, 20:029103, 2008.

W. G. Mathews and G. R. Blumenthal. Rayleigh-Taylor stability of compressible and incompressible radiation-supported surfaces and slabs: application to QSO clouds. *Ap. J.*, 214: 10–20, 1977.

J. P. Mellado, S. Sarkar, and Y. Zhou. Large-eddy simulation of Rayleigh-Taylor with compressible miscible fluids. *Phys. Fluid*, 17:076101, 2005.

B. J. Olson and A. W. Cook. Rayleigh-Taylor shock waves. *Phys. Fluid*, 19:128108, 2007.

P. Ramaprabhu, G. Dimonte, and M. J. Andrews. A numerical study of the influence of initial perturbations on the turbulent Rayleigh-Taylor instability. *J. Fluid Mech.*, 536:285–319, 2005.

X. Ribeyre, V. T. Tikhonchuk, and S. Bouquet. Compressible Rayleigh-Taylor instabilities in supernova remnants. *Phys. Fluids*, 16(12):4661–4670, 2004.

A. Sameen, R. Verzicco, and K. R. Sreenivasan. Non-boussinesq convection at moderate rayleigh numbers in low temperature gaseous helium. *Phys. Scr.*, T132:014053, 2008.

O. G. Sin'kova, V. N. Sofronov, A. L. Stadnik, V. P. Statsenko, and Y. V. Yanilkin. Numerical simulation of gravitational turbulent mixing of a light flat stratum with surrounding fluid. In E. Meshkov, Y. V. Yanilkin, and V. Zhmailo, editors, *Proceedings of the 7th International Workshop on the Physics of Compressible Turbulent Mixing*, pages 767–793, St. Petersburg, 1999.

B. Sitt. Sur les instabilités convectives linéaires dans un fluide compressible non dissipatif. *C. R. Acad. Sc. Paris*, t. 291(Série B):243–246, 1980.

V. I. Volkov, V. A. Gordeychuk, N. S. Es'kov, and O. M. Kozyrev. Numerical simulation of interfaces by MAH-3D code using an unstructured mesh of markers. In E. Meshkov, Y. V. Yanilkin, and V. Zhmailo, editors, *Proceedings of the 7th International Workshop on the Physics of Compressible Turbulent Mixing*, pages 338–344, St. Petersburg, 1999.

D. Youngs. Three-dimensional numerical simulation of turbulent mixing by Rayleigh-Taylor instability. *Phys. Fluids A*, 3(5):1312, 1991.