Recent results on MHD Turbulence: weak MHD turbulence

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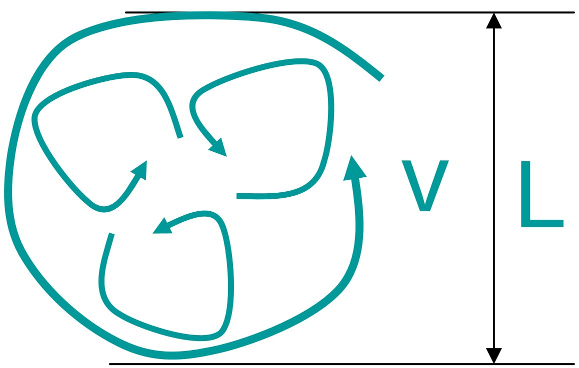
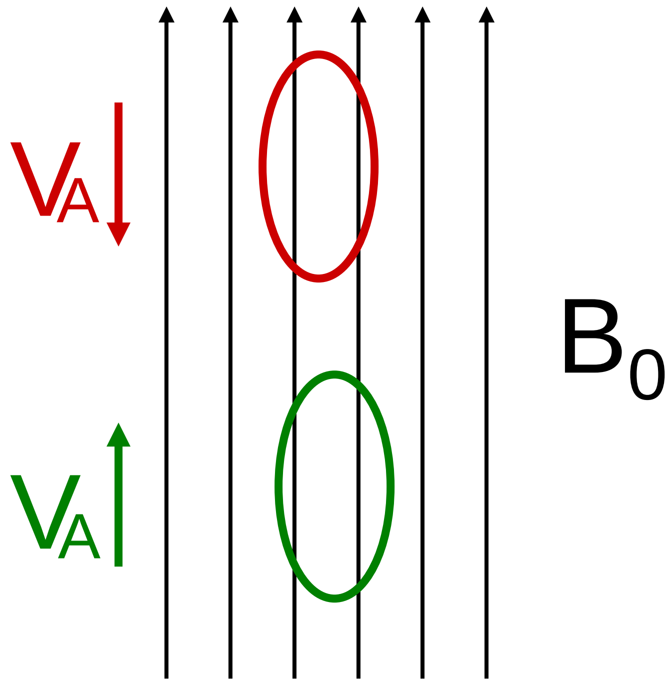
Hydrodynamic turbulence vs MHD turbulence

Incompressible HD turbulence: interaction of eddies

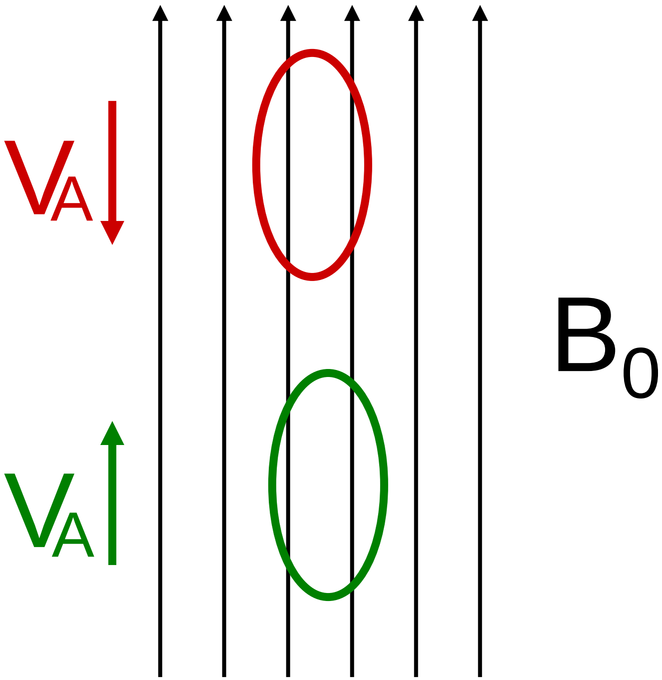


Incompressible MHD turbulence: interaction of wave packets moving with Alfven velocities

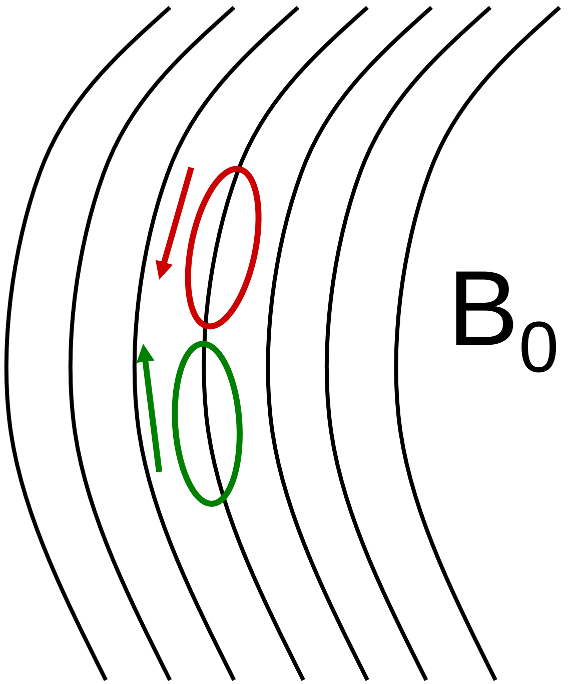


$$\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi\rho_0}$$

The nature of the guide field



B_0 imposed by
external sources



B_0 created by
large-scale eddies

Equations of Magnetohydrodynamics

$$\partial_t v + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \mathbf{j}\times\mathbf{B} + \mathrm{Re}^{-1}\nabla^2\mathbf{v} + \mathbf{f}$$

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \mathrm{Rm}^{-1}\nabla^2\mathbf{B},$$

$$\mathbf{j} = \nabla\times\mathbf{B}$$

Separate fluctuating part: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$

Use Elsasser variables: $\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{b}$

$$\partial_t \mathbf{z}^+ - (\mathbf{v}_A\cdot\nabla)\mathbf{z}^+ + (\mathbf{z}^-\cdot\nabla)\mathbf{z}^+ = -\nabla P$$

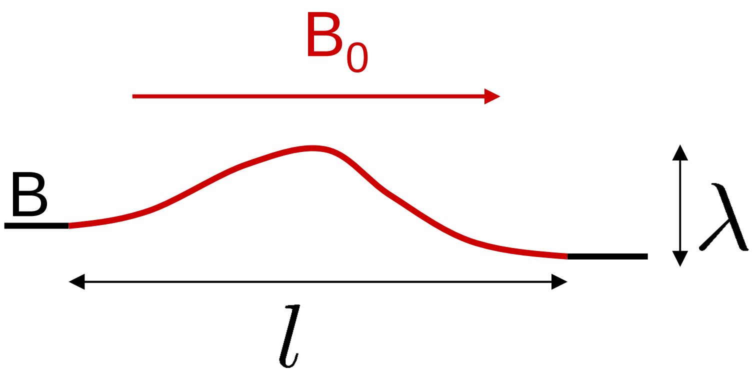
$$\partial_t \mathbf{z}^- + (\mathbf{v}_A\cdot\nabla)\mathbf{z}^- + (\mathbf{z}^+\cdot\nabla)\mathbf{z}^- = -\nabla P$$

When $\mathbf{z}^- \equiv 0$, $\mathbf{z}^+$ of any amplitude and shape propagates without dispersion against the guide field.

When $\mathbf{z}^+ \equiv 0$, $\mathbf{z}^-$ of any amplitude and shape propagates without dispersion along the guide field.

MHD turbulence

Anisotropy of “eddies”



Shear Alfvén waves
dominate the cascade:

$$\mathbf{z}^+_\lambda, \mathbf{z}^-_\lambda \perp \mathbf{B}_0$$

MHD equations in Elsasser variables

$$\mathbf{z}^\pm = \mathbf{v} \pm \mathbf{b}$$

$$\partial_t \mathbf{z}^+ - (\mathbf{v}_A \cdot \nabla)\mathbf{z}^+ + (\mathbf{z}^- \cdot \nabla)\mathbf{z}^+ = -\nabla P$$
$$\partial_t \mathbf{z}^- + (\mathbf{v}_A \cdot \nabla)\mathbf{z}^- + (\mathbf{z}^+ \cdot \nabla)\mathbf{z}^- = -\nabla P$$

$$V_A/l \gg b_\lambda/\lambda$$

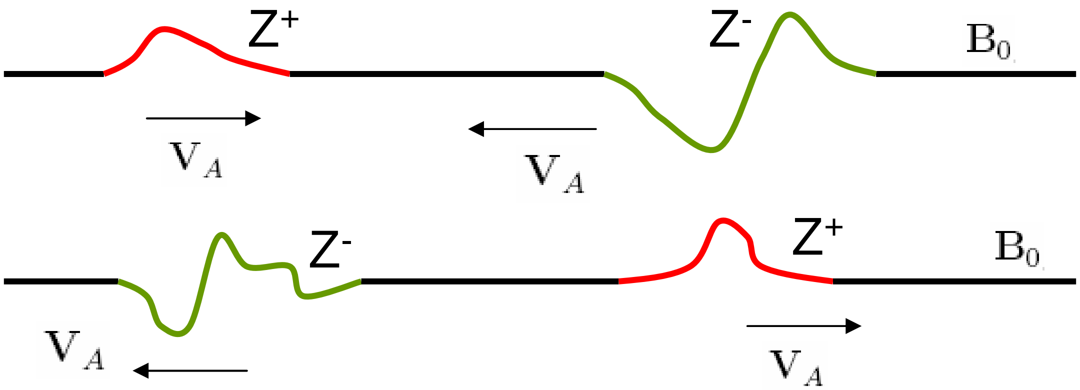
Weak turbulence

Alfvenic turbulence

$$\partial \mathbf{z}^{\pm} \mp (\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

Ideal system conserves the Elsasser energies

$$E^+ = \int (\mathbf{z}^+)^2 \, d^3x \qquad \qquad E = \frac{1}{2}\int (v^2 + b^2) d^3x$$
$$\equiv$$
$$E^- = \int (\mathbf{z}^-)^2 \, d^3x \qquad \qquad H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x$$



After interaction, shape of each packet changes, but energy does not.

Weak MHD turbulence: Phenomenology

Three-wave interaction of shear-Alfven waves

$$\omega(k) = |k_z| v_A$$

$$\begin{cases} \omega(k) = \omega(k_1) + \omega(k_2) \\ \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \end{cases}$$

Only counter-propagating waves interact, therefore, k_{1z} and k_{2z} should have opposite signs.

$$\Downarrow$$

Either $k_{1z} = 0$ or $k_{2z} = 0$

Wave interactions change $k_\perp$ but not k_z

At large $k_\perp$: $E(k_z, k_\perp) \propto g(k_z) k_\perp^{-\beta}$

Analytical framework

[Galtier, Nazarenko, Newell, Pouquet, 2000]

In the zeroth approximation, waves are not interacting and they have random phases:

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^+(\mathbf{k}') \rangle = e^+(k_z, k_\perp)\delta(\mathbf{k} + \mathbf{k}')$$

$$\langle \mathbf{z}^-(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = e^-(k_z, k_\perp)\delta(\mathbf{k} + \mathbf{k}')$$

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = 0$$

When the interaction is switched on, the energies slowly change with time: $e^\pm(k_z, k_\perp, t)$

$$\partial_t \mathbf{z}^\pm - (\mathbf{v}_A \cdot \nabla)\mathbf{z}^\pm + (\mathbf{z}^\mp \cdot \nabla)\mathbf{z}^\pm = -\nabla P$$

$$\partial_t \langle z^+ z^+ \rangle = ...\langle z^- z^+ z^+ \rangle + \langle z^+ z^- z^+ \rangle ...$$

$$\partial_t \langle z^- z^+ z^+ \rangle = ...\underbrace{\langle z^+ z^- z^+ z^+ \rangle} + \underbrace{\langle z^- z^- z^+ z^+ \rangle} + \underbrace{\langle z^- z^+ z^- z^+ \rangle}...$$

split into pair-wise correlators using Gaussian rule

Analytical framework

[Galtier, Nazarenko, Newell, Pouquet, 2000]

$$\partial_t \langle z^+ z^+ \rangle = \ldots \langle z^- z^+ z^+ \rangle + \langle z^+ z^- z^+ \rangle \ldots$$

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split into pair-wise correlators using Gaussian rule

$$\partial_t e^{\pm}(k_z, k_\perp) = \int M_{k,pq} e^{\mp}(0, q_\perp) \left[e^{\pm}(k_z, k_\perp) - e^{\pm}(k_z, p_\perp) \right] \delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p d^2q$$

$$M_{k,pq} = \frac{\pi}{v_A} \frac{(\mathbf{k}_\perp \times \mathbf{q}_\perp)^2 (\mathbf{k}_\perp \cdot \mathbf{p}_\perp)^2}{k_\perp^2 p_\perp^2 q_\perp^2}$$

This kinetic equation has all the properties discussed in the phenomenology: it is scale invariant, $z^{\pm}$ interact only with $z^{\mp}$, k_z does not change during interactions.

Analytical framework

[Galtier, Nazarenko, Newell, Pouquet, 2000]

$$\partial_t e^{\pm}(k_z, k_\perp) = \int M_{k,pq} e^{\mp}(0, q_\perp) \left[e^{\pm}(k_z, k_\perp) - e^{\pm}(k_z, p_\perp)\right] \delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p d^2q$$

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Consider statistically balanced case: $e^+ = e^-$

The general balanced solution of the Galtier et al Eqs is:

$$e^+(k_z, k_\perp) = e^-(k_z, k_\perp) = g(k_z) k_\perp^{-3}$$

where $g(k_z)$ is an arbitrary function smooth at k_z=0.

The spectrum of weak balanced MHD turbulence is therefore:

$$E^{\pm}(k_z, k_\perp) = e^{\pm}(k_z, k_\perp) 2\pi k_\perp \propto k_\perp^{-2}$$

Unbalanced MHD turbulence

(non-balanced, imbalanced, cross-helical…)

Imbalance means that cross-helicity is nonzero:

$$H^C = \int (\mathbf{v}\cdot\mathbf{b})d^3x = \frac{1}{4}(E^+ - E^-) \neq 0$$

Or, equivalently, the energies of waves traveling in opposite directions along the guide field are not equal. This is a very common situation in nature:

• Solar wind: more Alfven waves travel out of the sun than toward the sun

• Interstellar medium: MHD turbulence is driven by spatially localized sources

• Even when balanced overall, MHD turbulence is always locally unbalanced– it creates patches of positive and negative cross-helicity.

[Lithwick & Goldreich (2003); Ng et al (2003); Rappazzo et al (2007); Chandran (2008); Beresnyak & Lazarian (2008); Matthaeus et al (2008); Perez & SB (2009)]

Unbalanced weak MHD turbulence

(where problems begin)

$$\partial_t e^{\pm}(k_z,k_\perp) = \int M_{k,pq} e^{\mp}(0,q_\perp)\left[e^{\pm}(k_z,k_\perp) - e^{\pm}(k_z,p_\perp)\right]\delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p\, d^2q$$

$$M_{k,pq} = \frac{\pi}{v_A}\frac{(\mathbf{k}_\perp \times \mathbf{q}_\perp)^2(\mathbf{k}_\perp \cdot \mathbf{p}_\perp)^2}{k_\perp^2 p_\perp^2 q_\perp^2}$$

The kinetic equation has a one-parameter family of solutions:

$$e^{+}(k_z,k_\perp) = g^{+}(k_z)k_\perp^{-3-\alpha}$$
$$e^{-}(k_z,k_\perp) = g^{-}(k_z)k_\perp^{-3+\alpha}$$

with $-1 < \alpha < 1$

What do these solutions mean? Hint: calculate energy fluxes!

The solution with *steeper (shallower)* spectrum corresponds

to *larger (smaller)* energy flux toward large $k_\perp$.

[Galtier et al 2000]

Assume that e+ has the steeper spectrum and

denote the energy fluxes ϵ^{+} and ϵ^{-} : $\epsilon^{+} > \epsilon^{-}$

Then: $\boxed{\alpha = f(\epsilon^{+}/\epsilon^{-})}$

Unbalanced weak MHD turbulence

(where problems begin)

$$\partial_t e^{\pm}(k_z,k_\perp) = \int M_{k,pq} e^{\mp}(0,q_\perp)\left[e^{\pm}(k_z,k_\perp) - e^{\pm}(k_z,p_\perp)\right]\delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p\, d^2q$$

$$M_{k,pq} = \frac{\pi}{v_A}\frac{(\mathbf{k}_\perp \times \mathbf{q}_\perp)^2(\mathbf{k}_\perp \cdot \mathbf{p}_\perp)^2}{k_\perp^2 p_\perp^2 q_\perp^2}$$

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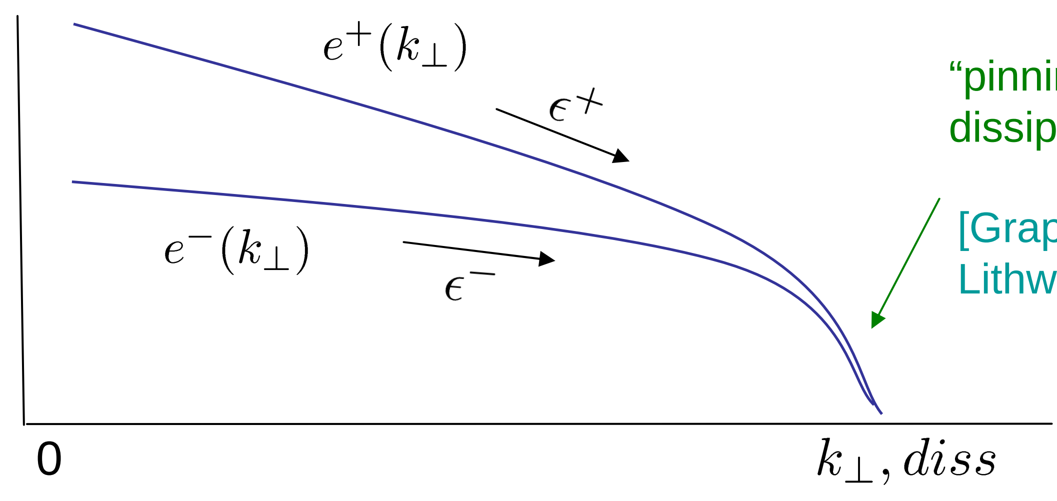
The energy spectra (log-log plot)



"pinning" at the dissipation scale

[Grappin et al 1983, Lithwick & Goldreich 2003]

Unbalanced weak MHD turbulence

(where problems begin)

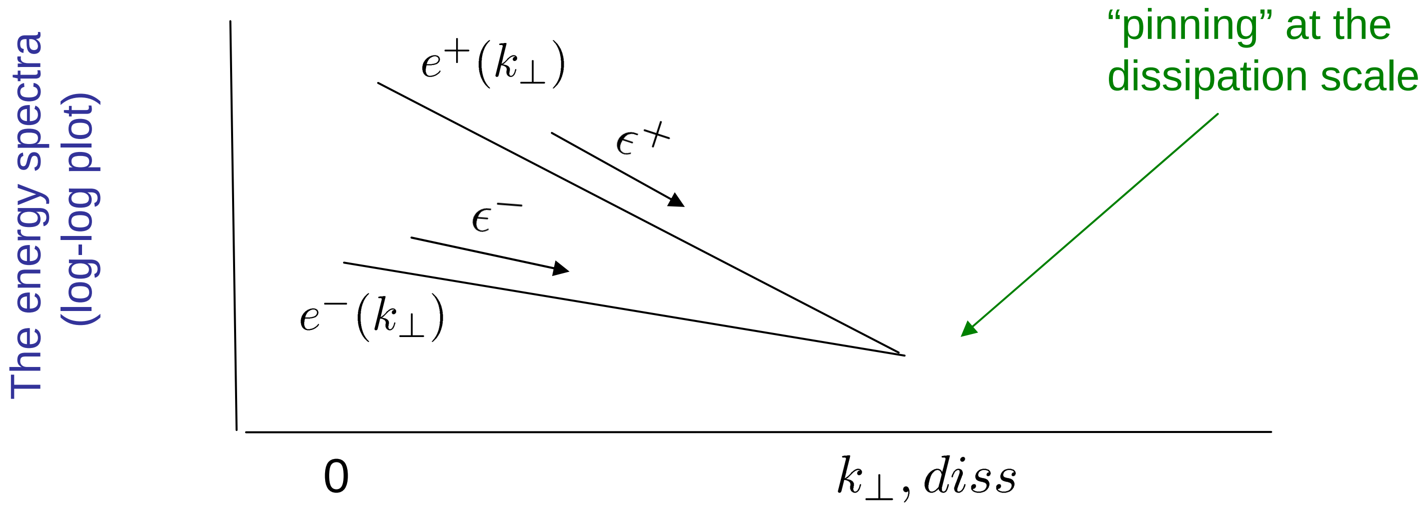
$$e^+(k_z, k_\perp) = g^+(k_z) k_\perp^{-3-\alpha}$$
$$e^-(k_z, k_\perp) = g^-(k_z) k_\perp^{-3+\alpha}$$

$-1 < \alpha < 1$

$\alpha = f(\epsilon^+/\epsilon^-)$

The spectra are “pinned” at the dissipation scale.

- If the ratio of the energy fluxes is specified, then the slopes are specified, but the amplitudes depend on the dissipation scale, or on the Re number.



Unbalanced weak MHD turbulence

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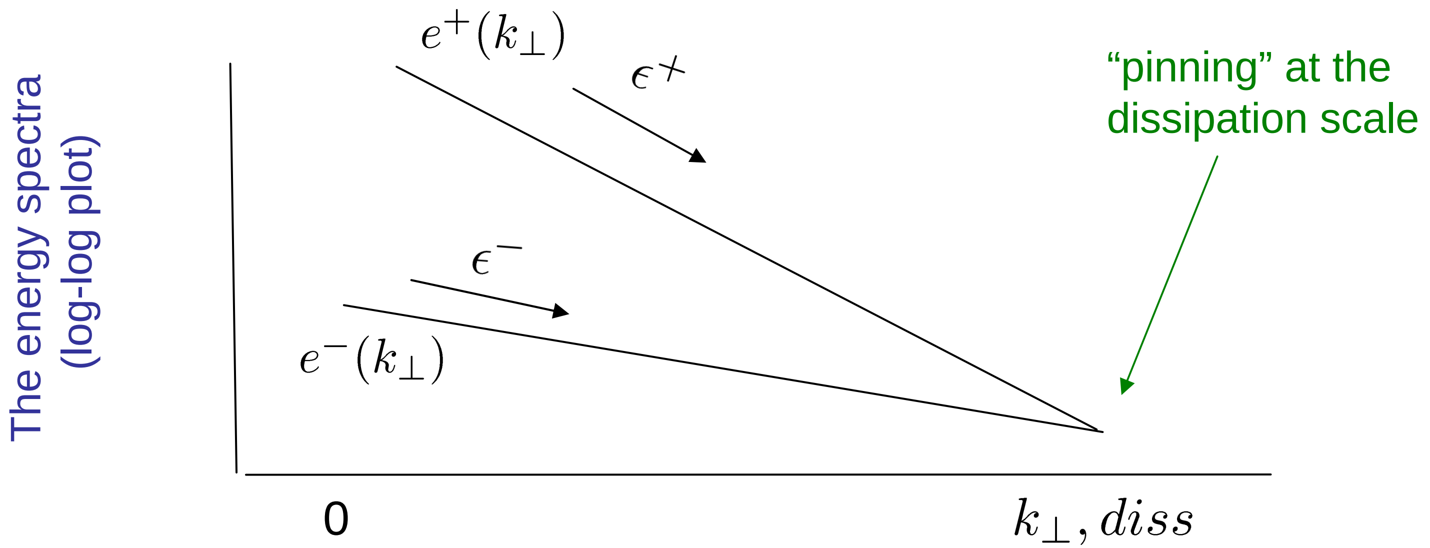

Unbalanced weak MHD turbulence

(where problems begin)

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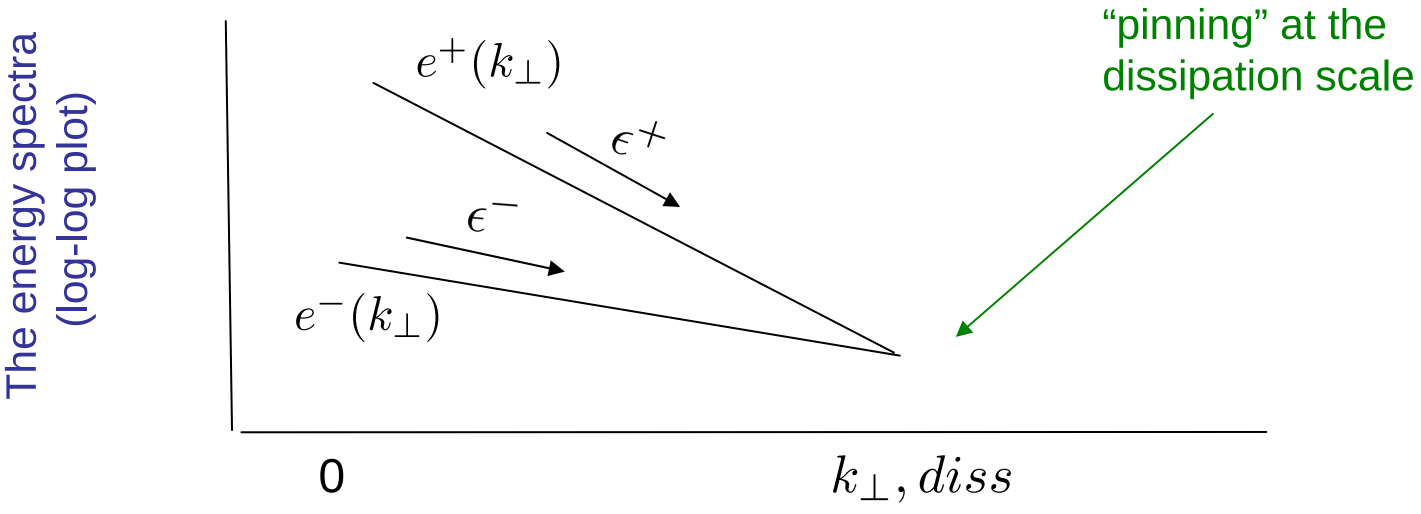
$-1 < \alpha < 1$

$$\alpha = f(\epsilon^+/\epsilon^-)$$

The spectra are “pinned” at the dissipation scale.

- If the amplitudes at $k_\perp = 0$ are specified, then slopes and fluxes depend on the Re number.



Unbalanced weak MHD turbulence

(where problems begin)

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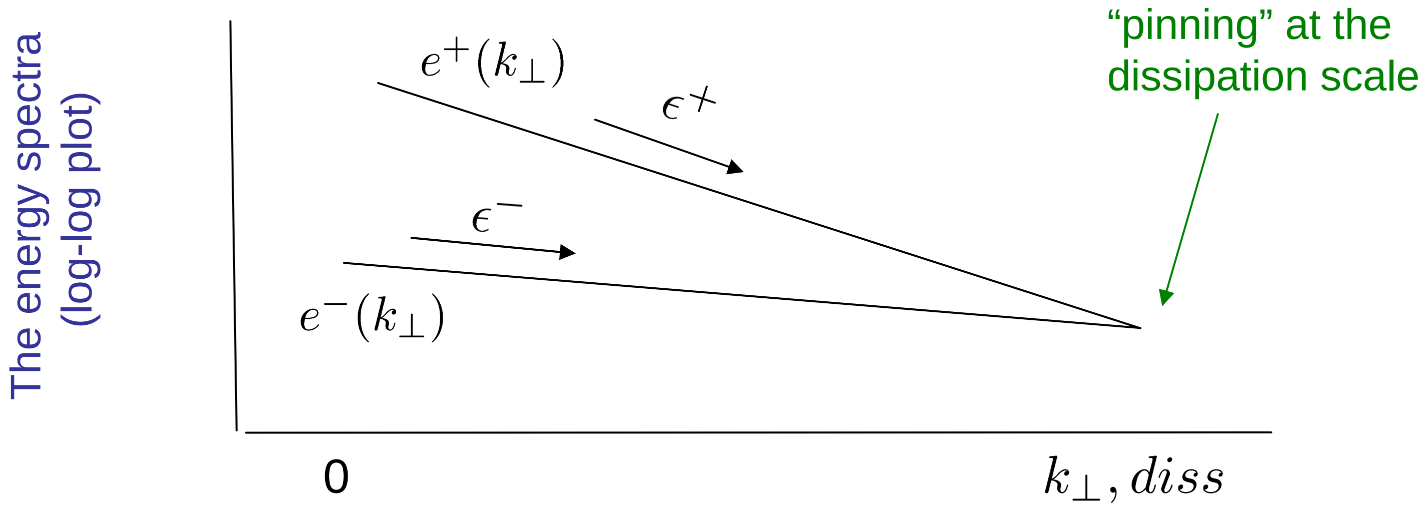


Unbalanced weak MHD turbulence

(contradiction)

$$e^+(k_z, k_\perp) = g^+(k_z) k_\perp^{-3-\alpha}$$
$$e^-(k_z, k_\perp) = g^-(k_z) k_\perp^{-3+\alpha}$$

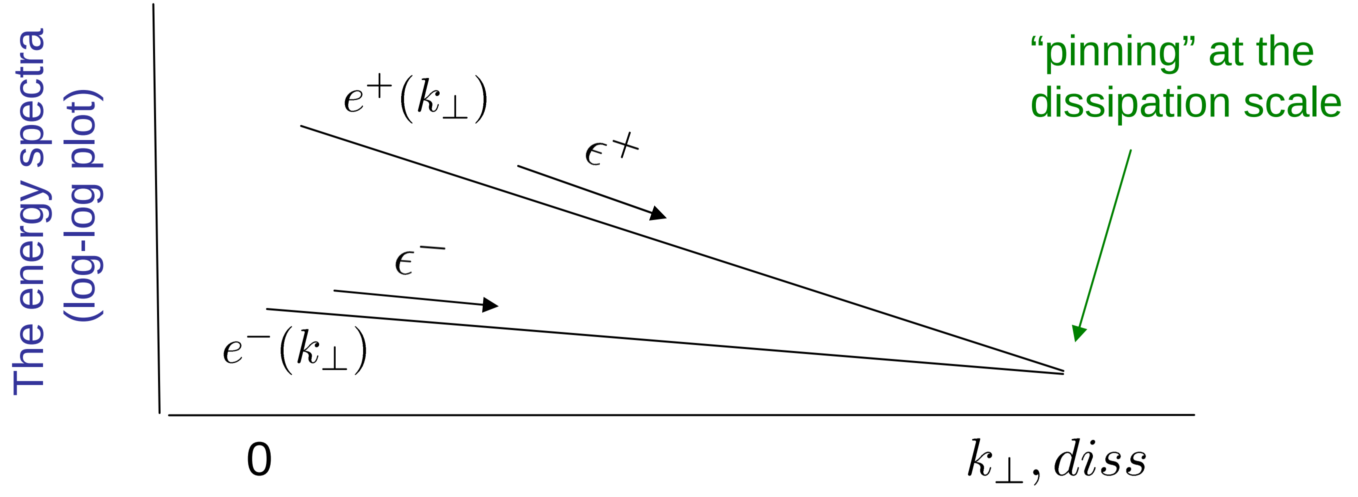
$-1 < \alpha < 1$

$\alpha = f(\epsilon^+/\epsilon^-)$

The spectra are “pinned” at the dissipation scale.

- If the amplitudes at $k_\perp=0$ are specified, then slopes and fluxes depend on the Re number.
- If the fluxes are specified, then the amplitudes depend on the Re number

Both possibilities seem to contradict the physical intuition that large-scale properties of turbulence are independent of the dissipation scale.



Unbalanced weak MHD turbulence

Revise the basic assumptions:

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^+(\mathbf{k}') \rangle = e^+(k_z, k_\perp)\delta(\mathbf{k}+\mathbf{k}')$$ OK

$$\langle \mathbf{z}^-(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = e^-(k_z, k_\perp)\delta(\mathbf{k}+\mathbf{k}')$$ OK

Different K -- different phases

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = 0$$?

$\langle \mathbf{z}^+ \cdot \mathbf{z}^- \rangle = \langle v^2 \rangle - \langle b^2 \rangle = 0$ since in Alfven waves v=$\pm$ b

However, at $k_z = 0$, the fluctuations are NOT waves: $\omega = |k_z| v_A$

One can introduce the "condensate":

$$\boxed{\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = \Delta(k_z) e_0(k_\perp)\delta(\mathbf{k}+\mathbf{k}')}$$

where $\Delta(k_z)$ is concentrated at k_z=0.

What is the physical meaning and the role of the condensate?

[S.B. & Perez (2009)]

A model for weak unbalanced MHD turbulence

(the role of the condensate)

Introduce weak condensate

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = \Delta(k_z) e_0(k_\perp) \delta(\mathbf{k} + \mathbf{k}') \qquad \Delta(k_z) e_0(k) \ll e^\pm(k_z, k_\perp)$$

How will it change the kinetic equations?

$$\partial_t e^\pm(k_z, k_\perp) = \int M_{k,pq} e^\mp(0, q_\perp) \left[e^\pm(k_z, k_\perp) - e^\pm(k_z, p_\perp) \right] \delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p d^2q$$
$$+\tilde{\Delta}(k_z) \int R_{k,pq} \left[e^\pm(k_\perp, k_z) e_0(q_\perp) + e^\pm(k_z, q_\perp) e_0(k_\perp) \right] \delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p d^2q$$

The first integral in the right hand side is degenerate;
it is zero for a one parameter family of solutions:

$$e^\pm(k_z, k_\perp) = g^\pm(k_z) k_\perp^{-3\mp\alpha}$$

The second integral (interaction with the condensate) is zero ONLY if

$$e^\pm(k_\perp) \propto k_\perp^{-3} \qquad \text{AND} \qquad e_0(k_\perp) \propto k_\perp^{-3}$$

Condensate lifts the degeneracy. The only universal solution is $e^\pm \propto e_0 \propto k_\perp^{-3}$

[S.B. & Perez (2009)]

A model for weak unbalanced MHD turbulence

(the physical meaning of the condensate)

Unbalanced MHD turbulence is not mirror invariant, since, e.g.,

$$H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x \neq 0$$

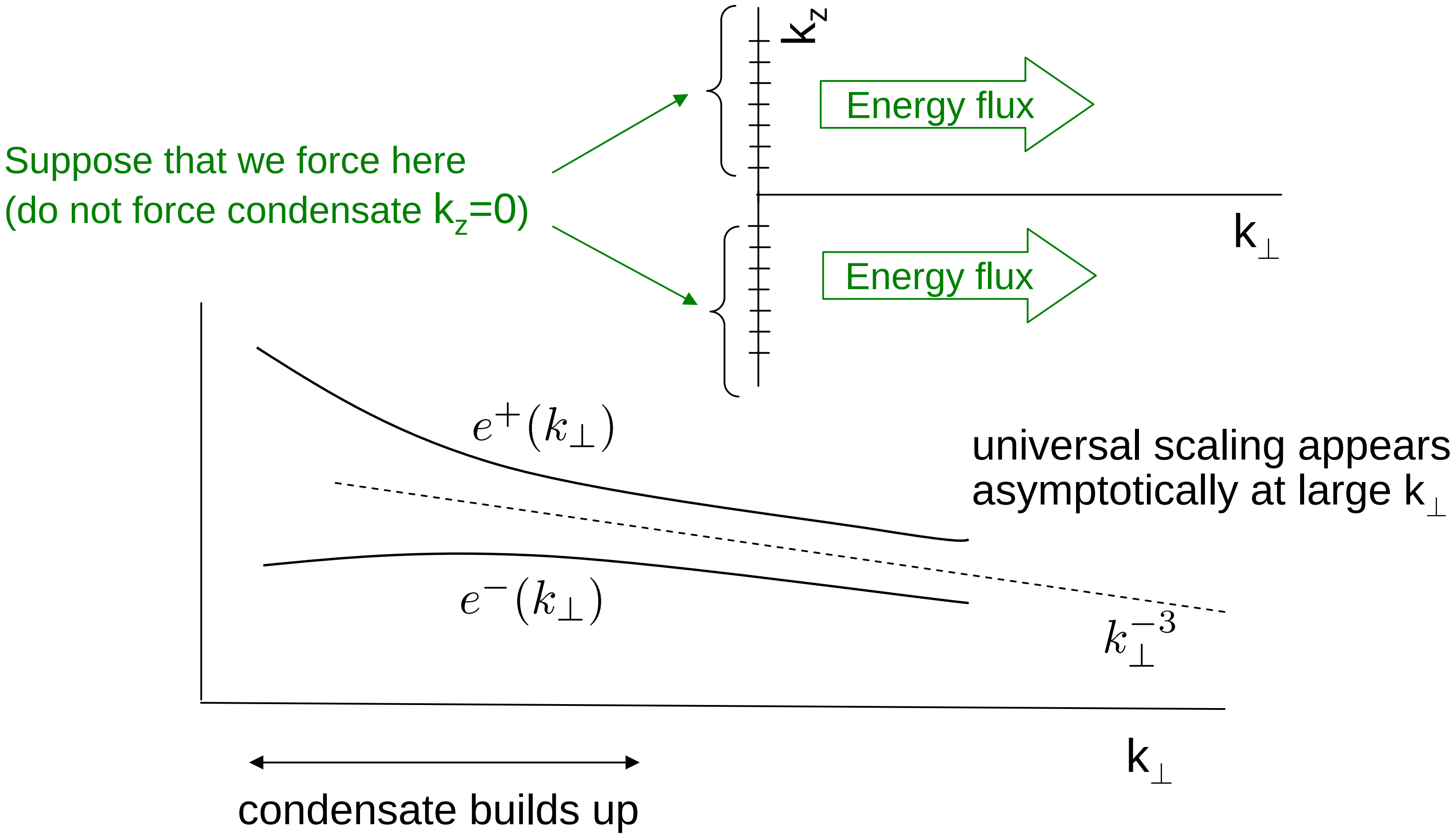
Magnetic helicity, another ideal invariant, cascades toward large scales. Magnetic field can pile up at k_z=0, where it is not in equipartition with the velocity field:

$$-\langle \mathbf{z}^+ \cdot \mathbf{z}^- \rangle_{k_\parallel = 0} = \langle b^2 - v^2 \rangle > 0$$

[S.B. & Perez (2009)]

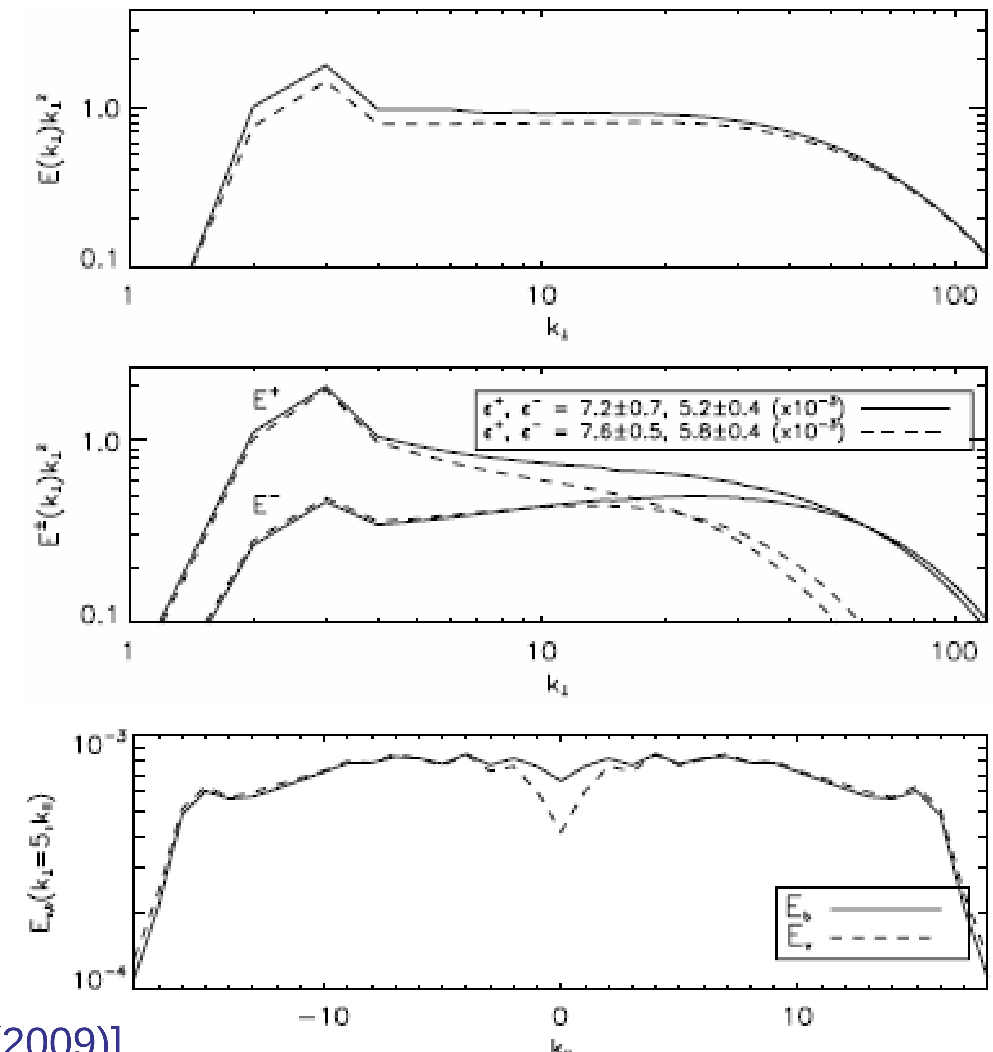
A model for weak unbalanced MHD turbulence

(the physical meaning of the condensate)



[S.B. & Perez (2009)]

Numerical results (1024^2x256)



Balanced

Unbalanced.
Fixed ϵ^+, ϵ^-
Two Re cases
are shown

Condensate
(note the Log scale)

[S.B. & Perez (2009)]

Conclusions

- Weak MHD turbulence spontaneously generates a condensate of the residual energy $E_b - E_v$ at small $k_{||}$
  The condensate is the consequence of mirror symmetry breaking in unbalanced turbulence.

- When turbulence is balanced, its spectrum is

$$E^{\pm}(k_\perp) \propto k_\perp^{-2}$$

- When turbulence is unbalanced, the interaction with the condensate becomes essential, and the universal spectrum $\propto k_\perp^{-2}$ is established asymptotically at large $k_\perp$