# GR HW 5

Due Dec 19 at 11am, in class

*Goals: Covariant stress-energy tensor, Conserved quantities on curved spacetime, Non-Cartesian coordinates*

1. We know that energy is conserved along the geodesic if the metric tensor does not depend on time. If the metric does not depend on $x^1$, is there a corresponding conservation law?

2. Find the energy momentum tensor of electromagnetic field by varying the action

$$S = -\frac{1}{4}\int d^4x\sqrt{-g}F^2, \qquad F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} \tag{1}$$

   with respect to the metric.

3. When calculating the energy momentum tensor of electromagnetic field by varying with respect to the metric, what should you keep fixed $A_\mu$ or $A^\mu$?

4. Calculate the total energy (gravitational field plus matter) for a Schwarzschild black hole. Schwarzschild metric is

$$ds^2 = -(1 - r_g/r)dt^2 + \frac{dr^2}{(1 - r_g/r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2}$$

5. For two-dimensional flat space, $ds^2 = dr^2 + r^2d\theta^2$, calculate Christoffel symbols $\Gamma^i_{jk}$ for all $i, j, k = r, \theta$. Using the obtained Christoffel symbols, calculate the divergence of a vector field $\nabla_i A^i$. Explain why your expression differs from the standard textbook expression
   $\nabla \cdot \boldsymbol{A} = \frac{1}{r}\partial_r(rA_r) + \frac{1}{r}\partial_\theta A_\theta$

6. Find the light cone (null hypersurface) with a vertex at $t = x = y = z = 0$ (the light cone is formed by all light rays passing through the vertex) in the gravitational field

$$h_{xx} = x^2/a^2, \qquad h_{yy} = y^2/b^2, \qquad h_{zz} = z^2/c^2, \tag{3}$$

   where $a, b, c$ are some constants.