Problem Sheet 8

I. BOSONS

A. Bose Einstein condensation

Bose condensates in various dimensions: Consider bosons in a box with a quadratic dispersion $\epsilon_{\mathbf{k}} = k^2/2m$. Does Bose-Einstein condensation occur in low dimensions ($d = 1, 2$)? Why not? What would change if the dispersion were linear, $\epsilon_{\mathbf{k}} = c|k|$, as it is for massless bosons? What about condensation in higher dimensions $d > 3$?

Determine how the chemical potential approaches zero as the temperature tends to $T = 0$, both for $d = 1$ and $d = 2$. How does the average number of bosons occupying the lowest state $n_{\mathbf{k}=\mathbf{0}}(T)$ behave as a function of T?

Bose condensates in 3d: Denoting $N_{\mathbf{k}=\mathbf{0}}(T)$ the occupation umber of the lowest energy level, $f_c = N_{\mathbf{k}=\mathbf{0}}(T)/N_{\text{tot}}$ is called the "condensate fraction". It is finite for $T < T_c$. Determine how f_c grows from zero as T is lowered a little bit below T_c. What is the exponent in the power law $f_c(T) \sim (T_c - T)^\gamma$?

How does the pressure of the Bose condensed system behave as T goes to zero? (Use $\Omega = -PV$, for example.) Contrast this to the Fermi systems and notice how very different the systems behave!

B. Photons

Explain why the chemical potential for photons is always $\mu = 0$!

Do photons Bose-Einstein condense? Why not?

Blackbody radiation: Use the Stefan-Boltzmann law to calculate the radiation energy influx per square meter (in kiloWatt per square meter) coming from the sun. (This is called the solar constant.) Assume that the sun has a surface temperature $T_S = 5780K$, a radius $R_S = 7 \cdot 10^8 m$ and is at a distance $D = 1.5 \cdot 10^{11} m$ from the Earth.

Calculate the pressure of a photon gas in a closed volume V at temperature T! Use that the grandcanonical potential is $\Omega = -PV$. Using a partial integration, show that

$$P = \int \frac{d^d k}{(2\pi)^d}\, \hbar k_x\, c \frac{k_x}{k} n_k \tag{1.1}$$

where n_k is the Bose distribution function. Give this formula a kinetic meaning! (Compare to the derivation of the ideal gas formula).

What is the energy density ϵ of such a photon gas? Show that $\epsilon = dP$!

Remark for specialists: Check that this means that the stress-energy tensor $T^{\mu\nu}$ of the electromagnetic field is traceless, $\sum_{\nu=0}^{3} T^{\nu}_{\nu} = 0$! This is a consequence of the scale invariance of the photon spectrum: rescaling length and time equally does not change the dispersion law $E_{\mathbf{k}} = c|\mathbf{k}|$, and this implies the tracelessness of the stress-energy tensor. (More precisely, it is the conformal invariance of free electromagnetism which implies the tracelessness).