Problem Sheet 4

I. ADIABATIC DEMAGNETIZATION AND EXPANSION

Adiabatic demagnetization is a technique to reach very low temperatures. Thereby a magnetic salt, which was brought to a temperature T in the presence of a field H, is demagnetized adiabatically (H is switched without allowing the system to exchange heat). As we will show here, this lowers the temperature of the system.

Make the qualitative connection between adiabatic demagnetization of a magnetic salt and adiabatic expansion of a gas: Show that magnetic field H and volume V play equivalent roles in the two processes: $S(H)$ and $S(V)$ are always monotonic functions, at any fixed T. Explain qualitatively why the temperature is lowered in an adiabatic process when V is increased, or when H is decreased, respectively.

Give now also a formal mathematical argument why T is lowered when H is decreased adiabatically by rewriting the adiabatic temperature change,

$$\frac{\partial T}{\partial H}|_S = ..., \tag{1.1}$$

by using techniques you have learnt in the lecture, and using that $\partial S/\partial H|_T < 0$.

A. Entropy of spins in a magnetic field

A simple model for adiabatic demagnetization is given by a collection of independent, that is, non-interacting spins (or magnetic moments). Those spins point either up or down along a fixed axis (so-called Ising spins). Such spins arise naturally in certain transition metal compounds. Ising spins are essentially classical objects, quantum mechanics playing no role for thermodynamics.

The energy of such a system of N spins is simply

$$E(\{s_i\}) = -\mu H \sum_{i=1}^{N} s_i, \tag{1.2}$$

where μ is called the magnetic moment of the spins. The possible spin configurations (the phase space) are just all the 2^N combinations of $s_i = \pm 1$. Argue that the statistical distribution function (in the canonical ensemble) for this system is

$$\rho(\{s_i\}) = \frac{e^{-\beta E(\{s_i\})}}{\mathcal{N}(\beta)} = \prod_{i=1}^{N} \frac{e^{\beta\mu H s_i}}{2\cosh(\beta\mu H)} \tag{1.3}$$

Calculate the entropy

$$S = -\langle \log(\rho(\{s_i\})\rangle \equiv -\sum_{\{s_i\}} \rho(\{s_i\}) \log(\rho(\{s_i\})) \tag{1.4}$$

as a function of $T = \beta^{-1}$ and H. [Convince yourself that this calculation is much more difficult if the spins interacted with each other!]

Verify that at strong fields (which fields should be considered "strong"?) the entropy becomes very small, even at finite T, and that S monotonically decreases with increasing H, as claimed in the lecture.

Discuss the behavior of $S(T, H)$ as T tends to zero. Nernst's theorem seems to be violated at $H = 0$! The reason here is different from the pathology of classical systems (where a $log(T)$ divergent piece in the entropy comes from the neglect of quantum constraints on the kinetic energy). Here, the ground state of the system is not unique, but extensively degenerate (all configurations have the same energy).

Convince your self and explain, that this is a pathology which arises, because we neglected to include the interactions between spins. Interactions will lift the degeneracy and restore $S(T \to 0) = 0$ even at $H = 0$. Consider thus a coupled Ising spin chain with energy

$$E(\{s_i\}) = -\mu H \sum_{i=1}^{N} s_i - J \sum_{i=1}^{N-1} s_i s_{i+1}. \tag{1.5}$$

with a small coupling J.

Below what temperature scale do you expect $S(T, H = 0)$ to deviate from the noninteracting result, and to tend to zero?

Bonus question (challenge! - The first to solve it correctly gets an offered drink in the coffee bar): Calculate the entropy of the spin chain and show explicitly that $S(T \to 0, H = 0) \to 0$! Hint: assume you know the partition function $Z_M(s_M)$ for a chain with M spins where the last spin has fixed orientation $s_M = \pm 1$. Derive a recursion relation between $Z_{M+1}(s_{M+1})$ and $Z_M(s_M)$, and in this way compute Z_N! (This is called the transfer matrix technique.)