Problem Sheet 1

I. PROBABILITY DISTRIBUTIONS

Definitions: $P(x)$ is a probability distribution for the variable $x \in \Omega$ if $P(x) \geq 0$ for all x, and $\int_\Omega P(x)dx = 1$. The space of possible events Ω can be *continuous* (e.g. the position of a particle in 3d, $\Omega = \mathbf{R}^3$), or *discrete* (e.g., the spin projections of an electron with spin 1/2, $\Omega = \{-1/2, 1/2\}$, or the outcome of throwing a dice, $\Omega = \{1, 2, 3, 4, 5, 6\}$). In the latter case the integral $\int_\Omega dx$ is to be interpreted as a sum over the possible discrete values.

The expectation value (or mean) is defined as

$$\mathbf{E}(x) \equiv \overline{x} := \int_\Omega dx P(x) x. \tag{1.1}$$

The variance is defined as

$$\mathbf{V}(x) := \int_\Omega dx P(x) x^2 - \overline{x}^2 = \overline{x^2} - \overline{x}^2. \tag{1.2}$$

The standard deviation or root mean square (RMS) is the square root of the variance, $\sigma(x) \equiv \mathbf{V}(x)^{1/2}$. It characterizes the strength of fluctuations of the random variable x.

Two random variables $x_{1,2}$ are called statistically independent if their joint probability distribution factorizes:

$$P(x_1, x_2) = P_1(x_1) P_2(x_2) \tag{1.3}$$

A. The Gaussian distribution

The Gaussian distribution is defined by

$$P_G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-(x - \overline{x})^2/2\sigma^2) \tag{1.4}$$

- Show that this is indeed a probability distribution (normalization!), and that its mean and standard deviation are given by $\overline{x}$ and σ.

- Assume $\overline{x} = 0$, and calculate the higher moments explicitly, defined in general as

$$\overline{x^m} = \int_\Omega P(x) x^m dx \tag{1.5}$$

- Compute the so-called generating function

$$\chi(k) := \overline{\exp[ixk]} = \int_\Omega P(x) \exp[ixk] dx \tag{1.6}$$

B. The central limit theorem

- Show that for a sum of independent variables x_i, $X = \sum_{i=1}^N x_i$ the expectation value and variance are given by

$$\overline{X} = \sum_{i=1}^N \overline{x_i}$$
$$\mathbf{V}(X) = \overline{X^2} - \overline{X}^2 = \sum_{i=1}^N \mathbf{V}(x_i) \tag{1.7}$$

- Now assume that the maximal value of x_i is bounded, and that $\overline{x_i} = 0$, for simplicity. The central limit theorem states that for N large the above sum X will be distributed according to a Gaussian law. Let us now discover this in steps, Choose either of the two routes A or B below:

- **A:** Compute the higher moments of X, and show that in the limit of a large number N of terms x_i in the sum this m'th moment is of order $N^{m/2}$ for m even, and 0 otherwise.

- Convince yourself that to this leading order in N, all higher moments are powers of the second moment. Compute the numerical proportionality factor in $\overline{X^m} = c_m \overline{X^2}^{m/2}$. Show or argue that $c_m = m!/2^{m/2}(m/2)! = (m-1)(m-3)...3 \cdot 1$.

- Compare with your results from above and conclude the asymptotic validity of the law of large numbers: X as defined in (1a) has approximately a Gaussian distribution, if it is a sum of a large number N of *independent* terms!

- **B:** Write $\chi(k) \equiv \exp\left[\sum_{n\geq 1} \frac{1}{n!}(ik)^n c_n\right]$ (expansion in k!). The c_n are called the *cumulants* of the random variable x.

- Show that the cumulants of a sum of independent terms, $\sum_{i=1}^N x_i$, are the sums of the cumulants of x_i.

- Show the cumulants of x/λ are $c_n' = c_n/\lambda^n$.

- Put these two facts together to obtain the characteristic function of $y = \frac{1}{N}\sum_{i=1}^N x_i$ in the limit of large N, and conclude that y is Gaussian distributed!

- What does this law mean for the fluctuations of macroscopic observables of thermodynamic systems.

- Assume that the internal energy in cubes of size 1 nm within a sample can be considered as independent from that of neighboring cubes. How large are the relative fluctuations of the total internal energy of a cube of linear size 1cm of this material?

- Do you know physical situations where such fluctuations become larger?

II. RANDOM WALK AND DIFFUSION

Imagine a walker who takes randomly a step to the right or to the left with step size a (time between steps: $\tau = 1$), starting from $x = 0$ at $t = 0$. The probability to take a step to the left or right be either equal $p_l = p_r = 1/2$ or biased $1 > p_l > 1/2$, $p_r = 1 - p_l$.

- Compute the probability that the walker is at position na at time $t = N\tau$.
- Find an asymptotic exact approximation to this formula in the limit of large $t \gg 1$, using the Stirling formula for factorials.
- Show that the position after time t is approximately a Gaussian, and determine the two parameters of the Gaussian as a function of $t, p_{l,r}$. Do it directly! Then use the result from problem 1 to check!
- Determine the discrete evolution equation for $P(t, x)$:

$$P(t + 1, x) = F[P(t, \{x \pm a\})], \tag{2.1}$$

  where F is a simple function.
- The above is a discrete random walk. If one coarse-grains space (over regions $\gg a$) and time (over time intervals $\gg \tau$), the process looks like a continuous process (the so-called Brownian motion).
- Derive the differential equation (diffusion equation) which the probability distribution $P(t, x)$ satisfies in this continuous limit!

$$\partial P/\partial t = ...? \tag{2.2}$$

  (This is the continuous version of the above discrete evolution).
- Check that your Gaussian found above indeed satisfies this diffusion equation!
- Determine the "diffusion constant" as a function of a and the time step (τ). What are the units of the diffusion constant?
- You will often hear that random walks and diffusion at large scales looks self-similar (after proper re-scaling of time and space you get the same phenomenon back). How do you have to rescale time if you rescale space by a factor $a \to a/\ell$?