Infinite Randomness Fixed Point of the Superconductor-Metal Quantum Phase Transition

Adrian Del Maestro, Bernd Rosenow, Markus Müller, and Subir Sachdev
Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA
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We examine the influence of quenched disorder on the superconductor-metal transition, as described by a theory of overdamped Cooper pairs which repel each other. The self-consistent pairing eigenmodes of a quasi-one-dimensional wire are determined numerically. Our results support the recent proposal by Hoyos *et al.* [Phys. Rev. Lett. **99**, 230601 (2007)] that the transition is characterized by the same strong-disorder fixed point describing the onset of ferromagnetism in the random quantum Ising chain in a transverse field.

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Numerous recent experiments [1–6] have measured the electrical transport properties of quasi-one-dimensional nanowires. While thicker wires have vanishing resistance in the low temperature (T) limit, thinner wires do not display superconductivity even at the lowest T. The superconducting wires display clear signatures of thermal phase fluctuations of the Cooper pair order parameter, Ψ, at low T. Quantum fluctuations of the phase and amplitude of Ψ increase with decreasing wire thickness, leading to a transition to a nonsuperconducting state.

Recent work [7,8] has proposed that these experiments should be described by a quantum superconductor-metal transition (SMT) in the pair-breaking universality class. Arguments based upon microscopic BCS theory suggest a model of Ψ fluctuations damped by decay into unpaired electrons [9–13]. In this Letter, we study the influence of quenched disorder on this model. Its role near the quantum SMT is of considerable interest, as disorder correlations are of infinite range in the imaginary time direction and can lead to unusual critical phenomena [14].

In a renormalization group (RG) analysis of this overdamped Cooper pair model, Hoyos *et al.* [15] have recently argued that the SMT is described by an infinite randomness fixed point (IRFP). The latter exhibits activated dynamic scaling where the logarithm of characteristic frequencies of Ψ fluctuations grows as a power of their characteristic length scale. The IRFP was further argued to be in the same universality class as the one describing the onset of ferromagnetism in the random transverse field Ising model (RTFIM) in 1D. Many exact results obtained by Fisher [16] for this fixed point appeared to carry through to the case of the SMT [15].

Using a powerful numerical technique, we find convincing evidence that the strong disorder RG [15] indeed applies to the SMT. In addition, we confirm activated dynamical scaling in the frequency domain and an IRFP characterized by exponents fully consistent with Fisher's RTFIM values. This is a nontrivial result, as the RTFIM contains *no dissipation* and possesses a *discrete symmetry*. However, due to the dissipative dynamics at the SMT, order parameter fluctuations are so strongly suppressed that the physics of a discrete Ising symmetry is recovered. Technically, in both cases, this behavior is due to the marginal dynamics of finite size clusters [14].

Our analysis was carried out on a lattice discretization of the disordered overdamped Cooper pair model of Ref. [15] at $T = 0$. The degrees of freedom, $\Psi_j(\tau)$, are complex functions of imaginary time, τ, on the sites, j, of a one-dimensional chain with action

$$\mathcal{S} = \sum_j \int d\tau \left[D_j |\Psi_j - \Psi_{j+1}|^2 + \alpha_j |\Psi_j|^2 + \frac{u_j}{2} |\Psi_j|^4 \right] + \int \frac{d\omega}{2\pi} \sum_j \gamma_j |\omega| |\Psi_j(\omega)|^2, \quad (1)$$

where $\Psi_j(\omega)$ is the Fourier transform of $\Psi_j(\tau)$, and the couplings in $\mathcal{S}$ are all random functions of j. The quartic coefficients $u_j > 0$ ensure stability and repulsion between Cooper pairs, and the dissipation into the metallic bath is represented by $\gamma_j > 0$, due to causality. Finally, we can choose a gauge such that $D_j > 0$. A more careful analysis and suitable rescalings [17] allow us to reduce the randomness to the spatial dependence of D_j (uniformly distributed on $(0, 1]$) and α_j (taken to be Gaussian), while setting $u_j = u$ and $\gamma_j = 1$. At zero temperature, the SMT is tuned by reducing the mean of the α_j distribution, $\bar{\alpha}$, while keeping its variance constant at 0.25 in units of γ^2. The presence of pair-breaking, possibly due to magnetic moments on the wire's surface [6], ensures that any Josephson coupling is short ranged.

We could also work in a lattice model of fluctuating phases with $\Psi_j(\tau) = e^{i\theta_j(\tau)}$ of unit magnitude [7,12]; this should have the same properties as $\mathcal{S}$, but our analysis is facilitated by also allowing for amplitude fluctuations.

While $\mathcal{S}$ is suitable to describe the influence of disorder on the fluctuating Cooper pairs, we also have to consider its effects on the single electron states. We have estimated the latter within weak-coupling BCS theory: at criticality, on a scale parametrically smaller than the single electron localization length, the gain in condensation energy can offset the cost in elastic energy when order parameter fluctua-

tions take advantage of randomness in α_j. This justifies our focus on the influence of disorder in a purely bosonic overdamped Cooper pair theory [11].

The RG analysis [15] was carried out in a model with an N-component order parameter, and it was found that flows had only an irrelevant dependence on the value of N [18]. Thus the exact critical properties can be obtained by studying the model in the large N limit. This is equivalent to approximating $\mathcal{S}$ by the Gaussian action

$$\mathcal{S}_0 = \sum_j \int \frac{d\omega}{2\pi} [D_j |\Psi_j - \Psi_{j+1}|^2 + (r_j + |\omega|)|\Psi_j|^2], \tag{2}$$

where the r_j are determined self-consistently by solving

$$r_j = \alpha_j + u\langle |\Psi_j(\tau)|^2 \rangle_{\mathcal{S}_0}. \tag{3}$$

We set $u = 1$ to reach a strong coupling regime and use fixed but random boundary conditions [7]. Solving the innocuous looking Eq. (3) for a large number of disorder realizations and large system sizes was the primary time-consuming step in obtaining the results of this study. Similar numerical large-N methods have been used previously for disordered systems with conventional (power law) dynamic scaling [19,20] but the presence of activated scaling leads to sluggish dynamics and the necessity to properly include spurious disorder configurations that, although exponentially rare, can make large contributions to thermodynamic properties. The numerical solution is facilitated via the implementation of a method which we have dubbed the *solve-join-patch* (SJP) procedure. We begin by generating a realization of disorder for $L \gg 1$ sites. Near the critical point (where the correlation length $\xi \sim L$), the direct iterative solution of Eq. (3) is computationally costly. This is because the eigenmodes of $\mathcal{S}_0$ begin to delocalize, acquiring a characteristic energy scale that is exponentially small in the distance from criticality and thus the masses, r_j must be computed with exponentially increasing precision.

To cope, the L-site system is broken up into a group of smaller subsystems with adjusted boundary conditions. The subsystems are *solved*, then *joined* together in groups of two. The grouped subsystems are now close to satisfying Eq. (3) and they can be quickly brought into accordance by *patching*, which involves resolving a mini-system around the joint consisting of a small number of sites. The joined and patched subsystem is easily solved and the SJP procedure is iterated until a full solution to Eq. (3) is obtained for the complete chain of L sites. We have considered up to 3000 realizations of disorder for system sizes $L = 16$, 32, 64, and 128.

Fisher's remarkable solution of the RTFIM [16] includes asymptotically exact results for the exponents and correlation functions at the IRFP, and many directly translate to the RG calculations by Hoyos *et al.* [15] for the dissipative model considered here. In particular, one expects activated dynamic scaling with $\ln(1/\Omega) \sim L^{\psi}$ where Ω is a characteristic energy scale and $\psi = 1/2$ is a tunneling exponent. This reflects the fact that at an IRFP, the dynamical critical exponent z is formally infinite. The RG approach defines a real space decimation procedure that either creates or destroys *clusters* or *bonds* as the energy scale is reduced. The typical moment of a surviving cluster scales like $\mu \sim \ln^{\phi}(1/\Omega)$ at criticality, where $\phi = (1 + \sqrt{5})/2 \simeq 1.62$ is the golden mean. Average correlations are described by a correlation length which diverges as $\xi \sim |\delta|^{-\nu}$ with $\nu = 2$ and δ a measure of the distance from criticality. From Ref. [15], δ is expected to be proportional to $\overline{\ln(r_i/r_c)}$ where r_c is some critical value. Our numerical study reveals that close to criticality, this quantity is linearly related to the detuning of the average $\bar{\alpha}$ from its quantum critical value, $\bar{\alpha}_c$ (to be determined below) and it further demonstrates that correlations among the r_i due to their self-consistency do not affect the strong randomness RG flow.

The remainder of this Letter presents a numerical confirmation of the results of Ref. [15] by providing compelling evidence for dynamically activated scaling at the quantum SMT, characterized by exponents ν, ψ, and ϕ taking on their RTFIM values. We begin by studying the disorder averaged equal-time correlation function $\bar{C}(x) = \overline{\langle \Psi_x^*(\tau)\Psi_0(\tau)\rangle_{\mathcal{S}_0}}$, which can be computed from the quadratic effective action $\mathcal{S}_0$ once the full set of masses $\{r_j\}$ has been obtained. In the disordered phase, where $\delta \equiv \bar{\alpha} - \bar{\alpha}_c > 0$, the asymptotic form of $\bar{C}(x)$ for the RTFIM has been predicted to describe both exponential as well as stretched exponential decay in addition to power law behavior [16]

$$\bar{C}(x) \sim \frac{\exp[-(x/\xi) - (27\pi^2/4)^{1/3}(x/\xi)^{1/3}]}{(x/\xi)^{5/6}}. \tag{4}$$

Using $\bar{C}(x)$ to define the correlation length ξ, we can perform fits for each value of L and various $\bar{\alpha}$ to extract $\xi(L, \bar{\alpha})$ as is seen in Fig. 1 for $L = 64$. We find remarkable agreement (solid lines) with Eq. (4) over 6 orders of magnitude for all system sizes considered. The full scaling form of Eq. (4) involves the cluster size exponent ϕ, and we have confirmed that very good data collapse is obtained using the numerical value determined below.

As mentioned above, the length scale which describes average correlations is expected to diverge like $\xi \sim |\delta|^{-\nu}$ as the critical point is approached. We have employed this result to perform a log-log fit to the finite size scaled correlation length (data extrapolated to $L \to \infty$) as a function of δ, as is shown in the inset of Fig. 1. The value of $\bar{\alpha}_c$ was found from the mean of the critical α_j distribution which minimizes the least square error of power law fits involving $\delta = \bar{\alpha} - \bar{\alpha}_c$. This leads to a value of $\bar{\alpha}_c = -0.93(3)$ for the critical point and $\nu = 1.9(2)$ for the correlation length exponent with the number in brackets

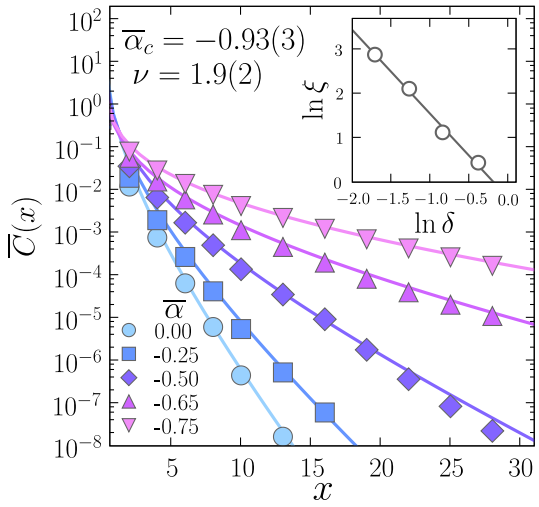

FIG. 1 (color online). The equal-time disorder averaged correlation functions for $L = 64$ and five values of the mean of α_j, $\bar{\alpha}$. The solid lines are fits to Eq. (4) via ξ and an overall scale parameter. The inset displays a fit to the power law form of the finite size scaled correlation length, providing an estimate for the location of the critical point $\bar{\alpha}_c = -0.93(3)$ and the correlation length exponent $\nu = 1.9(2)$.

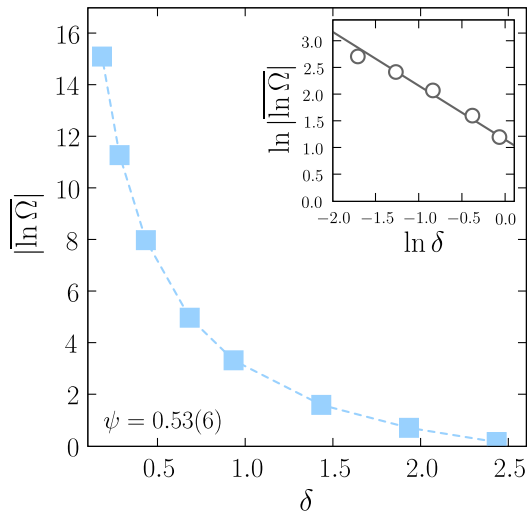

FIG. 2 (color online). The finite size scaled value of the disorder averaged logarithm of the minimum excitation energy plotted against the distance from the critical point δ. We observe divergence consistent with the scaling form $|\overline{\ln \Omega}| \sim \delta^{-\nu\psi}$, and using the value of $\bar{\alpha}_c$ and ν found above, we determine $\psi = 0.53(6)$ from a log-log linear fit (inset).

indicating the uncertainty in the last digit. The obtained exponent is in accord with $\nu = 2$ predicted for the RTFIM. We could also have defined the correlation length via the exponential tail of $\bar{C}(x)$ at large arguments. This yields compatible values for $\bar{\alpha}_c$ and ν.

For each realization of disorder and each value of $\bar{\alpha}$, we define the gap $\Omega(L)$ to be the smallest excitation energy in the system, which in general corresponds to the most delocalized mode of $\mathcal{S}_0$. Rare disorder configurations cause clusters to behave as if they were much more critical than the global value of δ would suggest. These clusters dominate the critical modes and exhibit abnormally small gaps that make large contributions to disorder averages of $\ln\Omega$, leading to the highly anisotropic scaling relationship between space and time that is the hallmark of IRFPs. An analysis of the probability distribution for the logarithm of the energy gap in the RTFIM was carried out by Young and Rieger [21] where they found cogent evidence for $z = \infty$. We have performed a similar analysis here, with the same result. Fisher and Young [22] found that the distribution of $\ln\Omega$ is Gaussian at criticality, while we find that in the quantum disordered phase, it is Gumbel as was previously shown [23]. If activated dynamic scaling is indeed present, the disorder averaged value of the logarithm of the gap should scale like $|\overline{\ln\Omega}| \sim \xi^{\psi} \sim \delta^{-\nu\psi}$ where we have used the scaling form of the correlation length. Such divergent behavior for the finite size scaled value of $|\overline{\ln\Omega}|$ is demonstrated in Fig. 2. The possibility of conventional scaling was considered but ultimately excluded through the examination of the maximum likelihood estimator for a wide range of power law fits. Using the previously determined values of $\bar{\alpha}_c$ and ν, the tunneling exponent can be extracted from a log-log linear fit of the average logarithmic spectrum as shown in the inset of Fig. 2, producing $\psi = 0.53(6)$ which is consistent with the RTFIM prediction of $1/2$.

To confirm full agreement with the universality class of the RTFIM, we must finally determine the exponent ϕ which controls the average moment, $\mu \sim |\ln\omega|^{\phi}$, of a cluster fluctuating with frequency ω. This is accomplished by investigating the imaginary part of the disorder averaged dynamical order parameter susceptibilities after analytical continuation to real frequencies. We study the average and local susceptibilities defined by

$$\mathrm{Im}\,\bar{\chi}(\omega) = \mathrm{Im}\frac{1}{L}\sum_j \overline{\langle \Psi_j^*(i\omega)\Psi_0(i\omega)\rangle}_{\mathcal{S}_0}|_{i\omega\to\omega+i\epsilon} \qquad (5)$$

$$\mathrm{Im}\,\overline{\chi_{\mathrm{loc}}}(\omega) = \mathrm{Im}\overline{\langle \Psi_j^*(i\omega)\Psi_j(i\omega)\rangle}_{\mathcal{S}_0}|_{i\omega\to\omega+i\epsilon} \qquad (6)$$

where $\langle\cdots\rangle_{\mathcal{S}_0}$ indicates an average over the large-N action [Eq. (2)] as well as a site average. Note that ω is now a real frequency, and our facile access to such dynamical quantities is one of the perquisites of the numerical approach we have taken. All frequencies are measured with respect to an ultraviolet cutoff which is required for convergence when computing the set of solutions to Eq. (3). At criticality, the average cluster moment will be given by the ratio of the average and local susceptibility due to the extra sum over sites in Eq. (5). We thus define

$$R(\omega) = \frac{\mathrm{Im}\bar{\chi}(\omega)}{\mathrm{Im}\overline{\chi_{\mathrm{loc}}}(\omega)} = |\ln\omega|^{\phi}\mathcal{F}(\delta^{\nu\psi}|\ln\omega|), \qquad (7)$$

and expect that the scaling function $\mathcal{F}$ approaches a con-

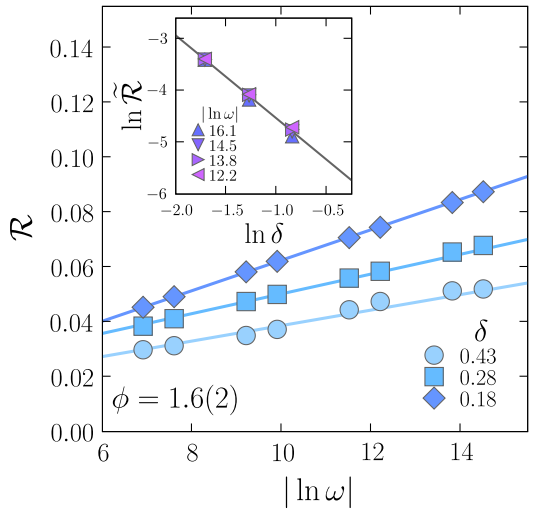

FIG. 3 (color online). The real frequency dependence of the finite size scaled value of the disorder averaged susceptibility ratio defined in Eq. (7) for three values of the $\delta = \bar{\alpha} - \bar{\alpha}_c$. We observe the predicted $|\ln\omega|$ behavior. After a suitable rescaling described in the text, we find that $\tilde{R}$ does not depend on frequency as $\omega \to 0$ (inset), and a log-log linear fit gives the value of the cluster exponent to be $\phi = 1.6(2)$.

stant when the dimensionless variable $\delta^{\nu\psi}|\ln\omega| \ll 1$. In the quantum disordered phase with $\delta^{\nu\psi}|\ln\omega| \gg 1$, a scaling analysis predicts $\mathcal{F}(x) \sim x^{1-\phi}$ [24], and hence $R \sim \delta^{\nu\psi(1-\phi)}|\ln\omega|$ [16,25]. To determine the value of ϕ, it is useful to consider a rescaled value of the susceptibility ratio $\tilde{R}(\delta) = R(\omega)/(\delta^{\nu\psi}|\ln\omega|)$ which should be frequency independent according to the predicted scaling form for $R(\omega)$ as $\omega \to 0$. We plot the finite size scaled susceptibility ratio in Fig. 3 for the three smallest values of δ, and indeed find confirmation of its linear $|\ln\omega|$ dependence. The inset of Fig. 3 confirms the frequency independence of $\tilde{R}$ and by determining the best linear fit of $\ln\tilde{R}$ to $\ln\delta$ for $\omega \leq 10^{-3}$ with $\nu\psi = 1.0(1)$, we find a cluster exponent $\phi = 1.6(2)$ which is very close to the predicted RTFIM value of $(1 + \sqrt{5})/2$.

The results of the above analysis, as highlighted in Figs. 1–3, provide compelling evidence for the applicability of the real space RG analysis of Ref. [15] and reproduces a number of results of Ref. [16] to unexpected accuracy. This confirms that the considered model for overdamped repulsive Cooperon fluctuations in the presence of quenched disorder near a SMT exhibits dynamically activated scaling and is controlled by an IRFP in the same universality class as the RTFIM. The transition is characterized by the numerically computed critical exponents $(\nu, \psi, \phi) \simeq (1.9, 0.53, 1.6)$ which are entirely consistent with those of the one-dimensional random quantum Ising model in a transverse field $[2, 1/2, (1 + \sqrt{5})/2]$.

In closing, we note that while our discussion has been framed in the context of the SMT, models similar to $\mathcal{S}$ describe the onset of a wide variety of orders in metallic systems [14]. Furthermore, the flow to the strong-disorder RTFIM fixed point is expected to also hold in higher dimensions [15]. We thus propose that our results provide strong support for the applicability of the RTFIM physics to many experiments involving the onset of spin- and charge-density wave orders in metals.

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