Quantum criticality and black holes

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Abstract
Many condensed matter experiments explore the finite temperature dynamics of systems near
quantum critical points. Often, there are no well-defined quasiparticle excitations, and so
quantum kinetic equations do not describe the transport properties completely. The theory
shows that the transport coefficients are not proportional to a mean free scattering time (as is the
case in the Boltzmann theory of quasiparticles), but are completely determined by the absolute
temperature and by equilibrium thermodynamic observables. Recently, explicit solutions of this
quantum critical dynamics have become possible via the anti-de Sitter/conformal field theory
duality discovered in string theory. This shows that the quantum critical theory provides a
holographic description of the quantum theory of black holes in a negatively curved anti-de
Sitter space, and relates its transport coefficients to properties of the Hawking radiation from the
black hole. We review how insights from this connection have led to new results for
experimental systems: (i) the vicinity of the superfluid–insulator transition in the presence of an
applied magnetic field, and its possible application to measurements of the Nernst effect in the
cuprates, (ii) the magnetohydrodynamics of the plasma of Dirac electrons in graphene and the
prediction of a hydrodynamic cyclotron resonance.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

A major focus of research in condensed matter physics is on
systems which are not described by the familiar paradigms
of order parameters and quasiparticles.   In states with
broken symmetry, we focus on the order parameter which
characterizes the broken symmetry, and work with effective
classical equations of motion for the order parameter.   In
states with long-lived quasiparticle excitations, we write down
quantum transport equations (e.g. via the Keldysh formulation)
for the quasiparticles and deduce a variety of transport
properties. However, it has become clear in recent years that
a number of interesting correlated electron materials do not
fit easily into either of these paradigms. One promising and
popular approach for describing these systems is to exploit
their frequent proximity to quantum phase transitions [1], and
so use a description in terms of 'quantum criticality'. This term
refers to dynamics and transport in the non-zero temperature
(T) quantum critical region which spreads out from the $T = 0$
quantum critical point or phase (see figure 2 below). Because
there are no well-defined quasiparticles, and the excitations are
strongly interacting, the description of quantum criticality is a
challenging problem.

It is useful to begin our discussion of quantum criticality
by recalling a well-understood example of non-quasiparticle
and non-order parameter dynamics: the Tomonaga–Luttinger
(TL) liquid [2].   This is a compressible quantum state of
many fermion or many bosons systems confined to move in
one spatial dimension. Electronic quasiparticles are not well
defined in this state, but there is a well-developed theory of
the transport properties of the TL liquid.   Historically, this
theory evolved over several decades of research on quantum
many body systems in one dimension. Key in the historical
development were exact solutions of model Hamiltonians via
the Bethe Ansatz.   Insights gained from the structure of
excitations in the Bethe Ansatz solutions led to a more general
understanding of the low energy excitations of a generic
Hamiltonian, and a universal low energy theory of the TL
liquid.   Thus while the exact solutions were restricted to
artificial models, they played a key role in the development
of the general theory. Of course, after the fact, with the general
theory of the TL liquid before us, we can justify it in its own
terms, and largely dispense with reference to the Bethe Ansatz
solutions.

While the TL liquid does describe quantum critical states
in one dimension, many of its characteristics do not generalize
to quantum criticality in higher spatial dimensions.   In
particular, it is not difficult to show [2–4] that the correlators
of all conserved currents in TL liquids show a characteristic
'free particle' ballistic behavior at all frequencies (ω) and T.
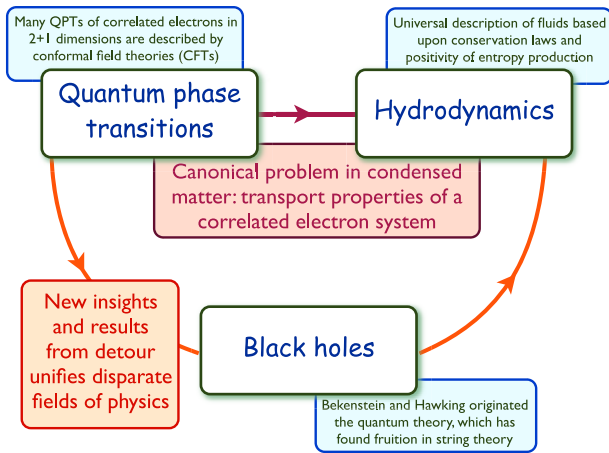
Figure 1. The connections between quantum criticality and black holes described in this paper.

Figure 2. Phase diagram of the Bose Hubbard model at integer filling. Dashed lines are crossovers, while the full line is the Kosterlitz–Thouless transition at T_{KT}.

This is because a free-field description of the conserved charges can usually be found; there are no collisions between the excitations and consequently no relaxation to collision-dominated hydrodynamic behavior. Hydrodynamics emerges only after including 'irrelevant' perturbations to the theory of the TL liquid. As we will review below, generic strongly-coupled quantum critical points in higher dimensions behave very differently: there are no known free-field formulations, even with variables which are highly non-local in terms of the microscopic degrees of freedom. Furthermore, relaxation to local thermal equilibrium and hydrodynamic behavior emerges already within the universal theory of the quantum critical point.

We will focus in this paper on strongly interacting quantum critical points in $2 + 1$ dimensions [1]. Prominent examples include the transition between antiferromagnetic and spin gap states in Mott insulators, and the superfluid–insulator transition in the lattice boson models. A theory for the quantum critical transport of such models has been developed [3, 5, 6], using insights from various weak-coupling perturbative renormalization group analyses. However, this approach was limited to a narrow range of physical parameters: for the case of the superfluid–insulator transition, the density of the particles was exactly equal to the commensurate density of the Mott insulator and there were no impurities. In this paper we will review how this field has received an impetus from an unexpected direction: the AdS/CFT duality of string theory [7]. This duality has provided a set of exact solutions [4, 8] of quantum critical transport in $2 + 1$ dimensions. Admittedly, the exact solutions are for rather artificial models, with a high degree of supersymmetry. However, just as was the case for the TL liquid, these exact solutions have provided an impetus to the general theory, and led to significant progress in formulating a more general theory of quantum critical transport, without the restrictions on the physical parameters noted above.

We have summarized the conceptual relationship between the developments in condensed matter and string theory in figure 1.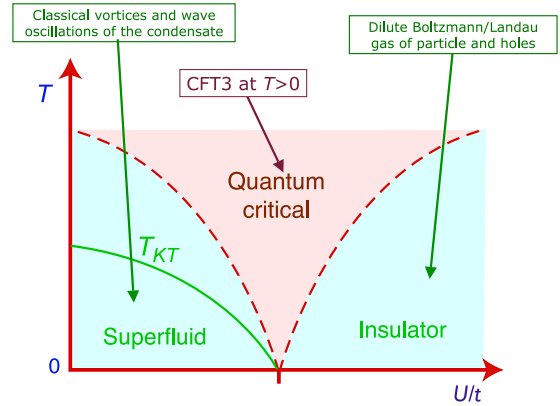 The remainder of the paper will expand on connections shown in this figure. In section 2, we will review the theory of quantum critical transport from a traditional condensed matter perspective. The modern developments in black holes and string theory will be summarized in section 3. Our new results for quantum critical transport in $2 + 1$ dimensions will be presented in section 4. Finally, in section 5 we briefly note emerging new directions of research in the application of the AdS/CFT duality to condensed matter physics.

2. Quantum critical transport

Here we review the ideas presented in [3] on transport near the superfluid–insulator transition in two spatial dimensions. For definiteness, we will consider the simplest model [9] which exhibits such a transition: the Bose Hubbard model at integer filling. As shown in figure 2, the transition is induced by varying the ratio U/t, where U is the on-site repulsion between the bosons, and t is the boson hopping matrix element. At low T, and generic low U/t, figure 2 shows that the system is in one of two possible regimes, both of which have effective classical theories of their dynamics and transport. At small U/t, in the superfluid region and across the Kosterlitz–Thouless transition, we can use the Gross–Pitaevskii equation to describe the spin-wave and vortex fluctuations. At large U/t, we have to consider the particle and hole excitations on the insulator, and these are dilute enough at low T to allow a classical gas description. However, our primary interest here is in the intermediate quantum critical regime, where neither classical description is possible. The theory of this region involves the dynamics of the quantum critical point, which is known to be described by a $2 + 1$ dimensional conformal field theory (CFT3) associated with the Wilson–Fisher fixed point [9].

The theory of the quantum critical region shows that the transport coefficients, and the relaxation time to local equilibrium, are not proportional to a mean free scattering time between the excitations, as is the case in the Boltzmann theory of quasiparticles. Such a time would typically depend upon the

interaction strength, $\sim U$, between the particles. Rather, the system behaves like a 'perfect fluid' in which the relaxation time is as short as possible, and is determined universally by the absolute temperature. Indeed, it was conjectured in [1] that the relaxation time generically obeys

$$\tau \geqslant \frac{\hbar}{k_{\mathrm{B}} T} \Phi_\tau \qquad (1)$$

where Φ_τ is a dimensionless constant of order unity determined by the precise definition of τ. The inequality in equation (1) is saturated in the quantum critical region, while the other regimes of figure 2 have significantly larger τ. Anticipating our discussion of black holes in section 3, closely related bounds on τ have appeared recently in considerations of the semiclassical theory of gravity-wave oscillations near black holes [10, 11].

The transport coefficients of this quantum critical perfect fluid also do not depend upon the interaction strength, and can be connected to the fundamental constants of nature. In particular, the electrical conductivity, σ, is given by

$$\sigma_Q = \frac{e^{*2}}{h} \Phi_\sigma, \qquad (2)$$

where Φ_σ is a universal dimensionless constant of order unity, and we have added the subscript Q to emphasize that this is the conductivity for the case at integer filling, no impurity, and at zero magnetic field. Here e^* is the charge of the carriers: for a superfluid–insulator transition of Cooper pairs, we have $e^* = 2e$, while in our application to graphene below we have $e^* = e$.

In addition to charge transport, we can also consider momentum transport. This was considered in the context of applications to the quark–gluon plasma [12]; application of the analysis of [3] shows that the viscosity, η, is given by

$$\frac{\eta}{s} = \frac{\hbar}{k_{\mathrm{B}}} \Phi_\eta, \qquad (3)$$

where s is the entropy density, and again Φ_η is a universal constant of order unity. The entropy density itself obeys [1]

$$s = \frac{k_{\mathrm{B}}^3 T^2}{\hbar^2 v^2} \Phi_s, \qquad (4)$$

where v is a characteristic velocity of the CFT3, much smaller than the physical velocity of light in condensed matter realizations. The universal constant Φ_s is the analog of the 'central charge' of CFT2s in $1 + 1$ dimension.

The values of the universal constants Φ_s, Φ_τ, and Φ_σ have been determined [1, 3] to leading order in expansions in $1/N$ (where N is the number of order parameter components) and in $3 - d$ (where d is the spatial dimensionality). However, both expansions for $\Phi_{\tau,\sigma}$ are neither straightforward nor rigorous, and require a physically motivated resummation of the bare perturbative expansion to all orders. It would therefore be valuable to have exact solutions of quantum critical transport where the above results can be tested, and we turn to such solutions in section 3.

3. Black holes and the AdS/CFT correspondence

We now turn to a seemingly distinct subject: the quantum theory of black holes.

Soon after Einstein's formulation of the theory of general relativity, Schwarzschild discovered its black hole solutions: a dense object of mass M was surrounded by a horizon of radius $R = 2GM/c^2$ which enclosed a region causally disconnected from the rest of the universe (here c is the actual velocity of light). The next result of relevance to us appeared in 1973, when Bekenstein [13] postulated a remarkable connection between the properties of black holes and the laws of thermodynamics. In particular, he argued that each black hole carried an entropy

$$\frac{S}{A} = \frac{k_{\mathrm{B}}}{4\ell_{\mathrm{P}}^2}, \qquad (5)$$

where $A = 4\pi R^2$ is the area of the horizon and $\ell_{\mathrm{P}} = \sqrt{G\hbar/c^3}$ is the Planck length (the precise numerical factor of 1/4 appeared in the subsequent work of Hawking); thus each Planck length square on the horizon has roughly one qubit degree of freedom. Bekenstein was partly motivated by the similarity between the law $dA \geqslant 0$, which is a consequence of the dynamics of the black holes in the theory of general relativity, and the second law of thermodynamics, $dS \geqslant 0$. He also presented physical arguments that without such a relation between S and A, we could violate the second law of thermodynamics simply by throwing matter into a black hole.

Bekenstein's proposal was put on a firmer footing by Hawking [14], who considered the quantum fluctuations of fields in the vicinity of the horizon of a black hole. He found the horizon also had a temperature, $T = \hbar^2/(8\pi k_{\mathrm{B}} M \ell_{\mathrm{P}}^2)$, and emitted black body radiation at this temperature. Using this temperature and the expression in equation (5), we can express S in terms of T (eliminating M and R), and find $S \sim 1/T^2$. Consequently, the specific heat of a black hole is negative, and it is thermodynamically unstable: its eventual fate is evaporation into Hawking radiation.

Recent advances in string theory have found a remarkable and explicit realization of the extension of the Bekenstein–Hawking results to black holes in an anti-de Sitter (AdS) universe. This is a homogeneous, negatively curved spacetime and black holes in such a universe can be thermodynamically stable (as we will see below). The complete solution of certain supersymmetric theories of quantum gravity in AdS spaces are now available. The central feature of the theory is the equivalence between the quantum theory of gravity in AdS4 and a CFT3 living in a flat $2 + 1$ dimensional spacetime: this is the famous AdS/CFT correspondence [7]. This correspondence allows computation of the black hole entropy via its identification with the entropy of the CFT3. In a number of cases, the result has been verified to be in precise agreement with the Bekenstein–Hawking result in the appropriate semiclassical limit, after the T of the CFT is identified with the Hawking temperature of the black hole.

One benefit of these advances is that Bekenstein's connection between black holes and thermodynamics can now be extended to dynamical properties [12]. We can

thus relate the dynamics of gravity and matter near a black hole to the dynamic correlations of a CFT3. In particular, transport coefficients of the CFT3, such as σ and η, can be related to properties of waves propagating in the presence of a black hole in an asymptotically AdS space (the AdS–Schwarzschild metric). The dissipation and irreversibility associated with these transport coefficients emerges naturally from the irreversibility of matter falling past the horizon of the black hole. Indeed, even a simple saddlepoint treatment of the gravity theory already yields a CFT dynamics which has non-zero and non-singular transport coefficients, has positive entropy production, and relaxes to local thermal equilibrium. These are the features which make the AdS/CFT correspondence so attractive: they are unprecedented in previous mean-field theories of quantum many body systems.

We now give a brief account of the correspondence between the gravity theory on AdS4 space and the CFT3, a topic which is extensively covered in the string theory literature. We consider physics on AdS4 space with metric

$$ds^2 = \frac{u^2}{L^2}(-c^2 dt^2 + dx^2 + dy^2) + L^2 \frac{du^2}{u^2}. \quad (6)$$

Here (t, x, y) define the $2+1$ dimensional spacetime on which the CFT3 will reside, u is the fourth coordinate of AdS4, and L is the radius of curvature. Note that here we are using the symbol c (and not v) for the characteristic velocity of the CFT3 to connect to the theories of black holes. This metric is a solution of Einstein's equations with a negative cosmological constant whose value is related to L. The central property of the metric in equation (6) is that it has a large five-dimensional isometry group $SO(3, 2)$. This group coincides with the group of conformal symmetries in $2 + 1$ dimensional spacetime, i.e. the conformal group of CFT3s, and this is a primary reason for the AdS/CFT correspondence. In particular, the scale invariance of the CFT3 is realized by an isometry of equation (6) under which

$$(t, x, y, u) \rightarrow (t/\lambda, x/\lambda, y/\lambda, \lambda u). \quad (7)$$

Note that while t, x, y have the usual co-ordinate scaling dimension -1, u has scaling dimension 1, and so scales like an energy. Indeed, we can think of u as an energy scale co-ordinate of the CFT3: $u \rightarrow \infty$ describes high energies, while $u \rightarrow 0$ describes low energies. Of course, the physics of the unperturbed CFT3, or the infinite AdS4 space, is scale invariant, and so the properties of all u are identical.

We are invariably interested in the CFT3 when it has been perturbed by some low energy scale, such as a temperature $T > 0$, a non-zero chemical potential, or an applied magnetic field. In the dual gravity theory, such perturbations will appear as deviations in the metric from the perfect AdS4 form at low energies i.e. at small u. We consider here the simplest case of a non-zero T: this corresponds to choosing a solution of Einstein's equations with a black hole so that the Hawking temperature of the black hole is T. Such black hole solutions are given by the AdS–Schwarzschild metric [15]:

$$ds^2 = -f(u)c^2 dt^2 + \frac{du^2}{f(u)} + \frac{u^2}{L^2}(dx^2 + dy^2), \quad (8)$$

where

$$f(u) = \frac{u^2}{L^2} - \frac{GM}{uc^2}, \quad (9)$$

and M is the mass of the black hole. Note that (8) reduces to (6) as $u \rightarrow \infty$ i.e. at high energies. The black hole horizon is at $u = u_0$ where $f(u_0) = 0$. Hawking's computation [14] gives a prescription for the Hawking temperature in terms of the asymptotic behavior of the metric near $u = u_0$:

$$k_B T = \frac{\hbar c f'(u_0)}{4\pi} = \frac{3\hbar (GMc)^{1/3}}{4\pi L^{4/3}}. \quad (10)$$

Note that unlike the flat space case considered by Hawking, the temperature of the AdS black hole $T \sim M^{1/3}$, and so T increases with increasing M; this will be important below for the thermodynamic stability of the black hole. Also, the metric (8) shows that the horizon of the AdS black hole is actually the entire flat $x-y$ plane at $u = u_0$; because of this the black hole is more properly called a 'black brane'. The area of the horizon is $A = A_{xy} u_0^2/L^2$, where A_{xy} is the (infinite) area of the flat $x-y$ plane. From equation (5), the Bekenstein–Hawking entropy of the AdS black hole/brane is

$$\frac{S}{A_{xy}} = \frac{k_B u_0^2}{4 \ell_P^2 L^2} = \frac{k_B G^{2/3} M^{2/3}}{4 c^{4/3} L^{2/3} \ell_P^2}. \quad (11)$$

Eliminating M between equations (10) and (11) we see that $S \sim T^2$; so the specific heat is positive and the black hole is thermodynamically stable. A key point in the AdS/CFT correspondence is that the black hole entropy in equation (11) should be identified with the entropy density of the CFT3 in equation (4) at a temperature given by equation (10). Comparing these expressions for the entropy we observe that they both scale as $S \sim T^2 \sim M^{2/3}$ and so are in agreement with each other. The numerical prefactors also agree if we have (after noting $v = c$)

$$\Phi_s = \left(\frac{2\pi L}{3 \ell_P}\right)^2. \quad (12)$$

We note that the Bekenstein–Hawking entropy in equation (5) only applies in the semiclassical gravity limit, and so the above value of Φ_s only applies in the limit $L \gg \ell_P$. For $L \sim \ell_P$, the entropy would have to be computed directly from the CFT3.

The AdS/CFT correspondence has the remarkable consequence that the spacetime itself, and its dimensionality is an 'emergent' property. From our discussion above, we may conclude that a CFT3 is better interpreted as 'living' in $2 + 1$ dimensions when $\Phi_s \sim 1$, while some CFT3s with $\Phi_s \gg 1$ are secretly theories of gravity in 3+1 dimensions with weak quantum fluctuations.

Other important perturbations to the AdS–Schwarzschild metric have been described in the literature: a chemical potential applied to the CFT3 corresponds to an electric charge on the black hole, while a magnetic field turns into a magnetic charge on the black hole [16]. We will not present these metrics here, but computations [8] on such 'dyonic' black holes led to the results presented in section 4.

We conclude this section by noting the recipe [7] by which dynamic correlators and transport coefficients of the CFT3 can

be computed. Each operator in the CFT3, $\mathcal{O}$, is 'dual' to some field, ϕ, in the AdS4 gravity theory. For our computations below, we note that the electrical current, J_μ, of the CFT3 is dual to a $U(1)$ gauge field, while the stress energy tensor, $T_{\mu\nu}$, is dual to fluctuations of the metric on AdS4. The central relation between the correlations of $\mathcal{O}$ and ϕ is [17, 18]

$$\left\langle \exp\left(\int dx\, dy\, dt\, \mathcal{O}(x, y, t)\phi_0(x, y, t)\right)\right\rangle_{\text{CFT3}}$$
$$= Z_{\text{AdS4}}\big[\phi(x, y, t, u \to \infty) = u^{\Delta-3}\phi_0(x, y, t)\big]. \quad (13)$$

Here $\phi_0(x, y, t)$ is an arbitrary source field, and Δ is the scaling dimension of $\mathcal{O}$. So the left-hand side of equation (13) is the generating function of correlators of $\mathcal{O}$ in the CFT3, a quantity of great interest to us. On the right-hand side we have the complete partition function of quantum gravity on AdS4 subject to a boundary condition on the fluctuating field ϕ as $u \to \infty$. Note the consistency of the power of u in the boundary condition with the identification of u as an energy scale: operators with smaller Δ have their dual field ϕ larger at smaller values of u i.e. they are more important in the infrared. For real time correlations at $T > 0$, there are some additional analytic continuation subtleties in applying equation (13): these were resolved in [19, 20].

For complete quantitative computations, the correspondence in equation (13) is only useful when the gravity theory on AdS4 is weakly coupled. This is known to be the case for a particular CFT3: the Yang–Mills gauge theory with $\mathcal{N} = 8$ supersymmetry and a $SU(N)$ gauge group. In the limit $N \to \infty$, the dual theory is classical supergravity on AdS4. From computations on the latter theory, we can obtain precise numerical results for the Yang–Mills CFT3: the numerical constants in equations (2)–(4) are given by [21, 12, 22, 4]:

$$\Phi_\sigma = \sqrt{\tfrac{2}{9}}N^{3/2}; \qquad \Phi_\eta = \frac{1}{4\pi}; \qquad \Phi_s = \frac{8\sqrt{2}\pi^2}{27}N^{3/2}. \quad (14)$$

These are the first exact results for a strongly-coupled CFT in $2 + 1$ dimensions.

4. The superfluid–insulator transition and graphene

Here we review recent new results [4, 8] for electrical and thermal transport in the quantum critical region of a generic CFT in $2 + 1$ dimensions. Thus we can envisage applications to the superfluid–insulator transition, and have presented scenarios under which such a framework can be used to interpret measurements of the Nernst effect in the cuprates [8, 23]. We have also described a separate set of applications to graphene [24–26]: while graphene is strictly not a CFT3, the Dirac spectrum of electrons leads to many similar results, especially in the inelastic collision-dominated regime associated with the quantum critical region. These results on graphene are reviewed in a separate paper [27], where explicit microscopic computations are also discussed.

The results presented below were obtained in two separate computations, associated with the methods described in sections 2 and 3:

- We solved the classical Einstein–Maxwell equations for linearized fluctuations about the metric of a dyonic black hole in a space which becomes AdS as $u \to \infty$. The results were used to obtain correlators of a CFT3 using equation (13). We have no detailed knowledge of the strongly-coupled quantum gravity theory which is dual to the CFT3 describing the superfluid–insulator transition in condensed matter systems. Nevertheless, it is a reasonable postulate that its low energy effective field theory essentially captured by the Einstein–Maxwell theory. Armed with this postulate, we can then obtain a powerful set of results for CFT3s. There is some loss of quantitative accuracy compared to the solvable supersymmetric theory noted above: we no longer have the precise quantitative values of L/ℓ_P and a few other couplings in the effective Einstein–Maxwell theory, and so cannot compute the values of universal numerical constants like $\Phi_{\sigma,\eta,s}$ of the CFT3.

- With the picture of relaxation to local equilibrium at frequencies $\hbar\omega \ll k_B T$ developed in [3], we postulate that the equations of relativistic magnetohydrodynamics should describe the low frequency transport. The basic principles involved in such a hydrodynamic computation go back to the nineteenth century: conservation of energy, momentum, and charge, and the constraint of the positivity of entropy production. Nevertheless, the required results were not obtained until our recent work [8]: the general case of a CFT3 in the presence of a chemical potential, magnetic field, and small density of impurities is very intricate, and the guidance provided by the dual gravity formulation was very helpful to us. In this approach, we do not have quantitative knowledge of a few transport coefficients, and this is complementary to our ignorance of the effective couplings in the Einstein–Maxwell theory noted above.

In the end, we obtained complete agreement between these two independent computations. This agreement demonstrates that the assumption of a low energy Einstein–Maxwell effective field theory for a strongly-coupled theory of quantum gravity is equivalent to the assumption of hydrodynamic transport for $\hbar\omega \ll k_B T$ in a strongly-coupled CFT3. See also [28], and references therein.

Finally, we turn to our explicit results for quantum critical transport with $\hbar\omega \ll k_B T$.

First, consider adding a chemical potential, μ, to the CFT3. This will induce a non-zero number density of carriers ρ. The value of ρ is defined so that the total charge density associated with ρ is $e^*\rho$. Note that we can choose e^* at our convenience, and there is no implicit assumption that the elementary excitations of the CFT3 carry charge e^*. Then the electrical conductivity at a frequency ω is

$$\sigma(\omega) = \frac{e^{*2}}{h}\Phi_\sigma + \frac{e^{*2}\rho^2 v^2}{(\varepsilon + P)}\frac{1}{(-\mathrm{i}\omega + 1/\tau_{\text{imp}})}. \quad (15)$$

In this section, we are again using the symbol v to denote the characteristic velocity of the CFT3 because we will need

c for the physical velocity of light below. Here ε is the energy density and P is the pressure of the CFT3. We have assumed a small density of impurities which lead to a momentum relaxation time τ_{imp}. In general, Φ_σ, ρ, ε, P, and $1/\tau_{\text{imp}}$ will be functions of $\mu/k_{\text{B}}T$ which cannot be computed by hydrodynamic considerations alone. However, apart from Φ_σ, these quantities are usually amenable to direct perturbative computations in the CFT3, or by quantum Monte Carlo studies. The physical interpretation of equation (15) should be evident: adding a charge density ρ leads to an additional Drude-like contribution to the conductivity. This extra current cannot be relaxed by collisions between the unequal density of particle and hole excitations, and so requires an impurity relaxation mechanism to yield a finite conductivity in the d.c. limit.

Now consider thermal transport in a CFT3 with a non-zero μ. The d.c. thermal conductivity, κ, is given by

$$\kappa = \Phi_\sigma \left(\frac{k_{\text{B}}^2 T}{h}\right)\left(\frac{\varepsilon + P}{k_{\text{B}}T\rho}\right)^2, \tag{16}$$

in the absence of impurity scattering, $1/\tau_{\text{imp}} \to 0$. This is a Wiedemann–Franz-like relation, connecting the thermal conductivity to the electrical conductivity in the $\mu = 0$ CFT. Note that κ diverges as $\rho \to 0$, and so the thermal conductivity of the $\mu = 0$ CFT is infinite.

Next, turn on a small magnetic field B; we assume that B is small enough that the spacing between the Landau levels is not as large as $k_{\text{B}}T$. The case of large Landau level spacing is also experimentally important, but cannot be addressed by the present analysis. Initially, consider the case $\mu = 0$. In this case, the result equation (16) for the thermal conductivity is replaced by

$$\kappa = \frac{1}{\Phi_\sigma}\left(\frac{k_{\text{B}}^2 T}{h}\right)\left(\frac{\varepsilon + P}{k_{\text{B}}T B/(hc/e^*)}\right)^2 \tag{17}$$

also in the absence of impurity scattering, $1/\tau_{\text{imp}} \to 0$. This result relates κ to the electrical *resistance* at criticality, and so can be viewed as Wiedemann–Franz-like relation for the vortices. A similar $1/B^2$ dependence of κ appeared in the Boltzmann equation analysis of [6], but our more general analysis applies in a wider and distinct regime [26], and relates the coefficient to other observables.

We have obtained a full set of results for the frequency-dependent thermoelectric response functions at non-zero B and μ. The results are lengthy and we refer the reader to [8] for explicit expressions. Here we only note that the characteristic feature [8, 29] of these results is a new *hydrodynamic cyclotron resonance*. The usual cyclotron resonance occurs at the classical cyclotron frequency, which is independent of the particle density and temperature; further, in a Galilean-invariant system this resonance is not broadened by electron–electron interactions alone, and requires impurities for non-zero damping. The situation for our hydrodynamic resonance is very different. It occurs in a collision-dominated regime, and its frequency depends on the density and temperature: the explicit expression for the resonance frequency is

$$\omega_{\text{c}} = \frac{e^* B \rho v^2}{c(\varepsilon + P)}. \tag{18}$$

Further, the cyclotron resonance involves particle and hole excitations moving in opposite directions, and collisions between them can damp the resonance even in the absence of impurities. Our expression for this intrinsic damping frequency is [8, 29]

$$\gamma = \frac{e^{*2}}{h}\Phi_\sigma\frac{B^2 v^2}{c^2(\varepsilon + P)}, \tag{19}$$

relating it to the quantum critical conductivity as a measure of collisions between counter-propagating particles and holes. We refer the reader to a separate discussion [24] of the experimental conditions under which this hydrodynamic cyclotron resonance may be observed.

5. Recent developments

In this concluding section we mention a number of new research directions that have emerged since our work in [4, 8]. These works are generally aimed in two directions: to move out of the quantum critical regime of figure 2 into the low-temperature regimes, and to address a wider class of quantum critical points of interest in condensed matter physics.

We see from figure 2 that one possible fate of the quantum critical system at low T is to undergo a phase transition to a superfluid state. Gubser and collaborators [30–33], Hartnoll, Herzog, and Horowitz [15, 34–36] and others [37–41] have argued that the AdS4 dual of such a transition would involve condensation of a charged scalar field in the background an electrically charged black hole. Their solutions of such models have yielded fascinating results for the conductivity of the superfluid state, with intriguing connections to, and differences from, conventional weak-coupling results in the condensed matter literature.

Lee [42] has recently studied the fermionic counterpart, by examining a non-zero density of charged fermions in the background of an electrically charged black hole. The Green's functions of the boundary theory appear to display an unusual non-Fermi liquid singularity over a 'Fermi ball'.

Extensions which break parity and time-reversal, or relevance to the quantum Hall effect, have also been studied [43, 44].

A separate set of developments involve looking at quantum critical theories whose symmetry group is not the relativistic conformal group considered in the present paper. The most prominent example of such a quantum critical point is the three-dimensional non-relativistic Fermi gas near a Feshbach resonance. The phase diagram of this model is efficiently described by a scale-invariant RG fixed point at zero density [45]; this fixed point was shown [46] to have a non-relativistic conformal symmetry (the Schrödinger group). A number of investigators [47–59] have realized the Schrödinger group as the group of isometries of a gravity theory, and obtained interesting information on the low-temperature properties of the boundary critical theory dual to this gravity theory.

Finally, we mention studies [60, 59, 61, 62] of gravity duals of quantum critical theories with a general value of the dynamic critical exponent, z.

Acknowledgments

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